DENJOY EQUALITY AND INFINITE BINARY SEQUENCES ASSOCIATED WITH CIRCLE HOMEOMORPHISMS

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Content

Introduction

At the present time in the world one of the interesting problems of the theory of dynamical systems are the problems associated with the maps of the circle. Fundamental results in the theory of circle homeomorphisms were obtained in the works of such outstanding mathematicians as A. Poincaré, A. Denjoy, A.N. Kolmogorov, V.I. Arnold, J. Moser, M. Erman, J.C. Yokkoz, J.G. Sinai, K.M. Khanin, D. Ornstein, J. Katznelson and others. A natural generalization of circle diffeomorphisms are critical maps and piecewise smooth homeomorphisms with break points. In the last 20 years such kind of circle maps with singularity points intensively studied by many authors (M. Herman, J. Yoccoz, G. Swiatek, A.Avila, K.Khanin, D. Mayer, A. Dzhalilov and others).

Many problems of dynamical systems, information theory, geometry, cardiac problems[1], biology and practice reduced to study symbolic dynamical systems. To study the complexity of infinite symbolic sequences is one of classical problems of symbolic dynamical systems. Infinite binary words associated by the irrational rotation of the circle such called Sturmian words are very important in information theory.

[1] Courtemanche M. A circle map in a human heart. North-Holland, Amsterdam, Physica D 40 (1989).

Let A be a finite set(called **alphabet**). Denote by A^* the set of concatenation of the elements of A, which is called the **language** over A, any element $\omega \in A^*$ is called a **finite word** if the length $|\omega|$ of the word ω is finite and denote by ε the empty word. Denote by A^N the set of right infinite words (simply **infinite words**) i.e.

$$A^N = \{\omega = (\omega_1\omega_2....\omega_n...) : \omega_i \in A,\ i \geq 1\}.$$

A word ω is called a **binary word** over A if A has only two elements.

Definition 1.1.2.[2]

An integer $p \geq 1$ is a **period** of a word $\omega = \omega_1\omega_2...\omega_n$ where $\omega_i \in A$ for $1 \leq i \leq n$ if $\omega_i = \omega_{i+p}$ for $i = 1, ..., n-p$. The smallest period of $\omega = \omega_1\omega_2...\omega_n$ is called the period of ω.

Definition 1.1.3.

An infinite word ω is called **ultimately periodic** if there exist $N, T \in \mathbb{N}$ such that $\omega_{n+T} = \omega_n$ for each $n \geq N$. Otherwise ω is called aperiodic.

[2] Lothaire M. Combinatorics of Words. Encyclopedia of Mathematics and its Applications, 17, London, Addison-Wesley Publishing Company, (1983)

Definition 1.1.4.

The **factor complexity** of an infinite word ω is the function $p_\omega(n)$ counting the number of its factors of length n which satisfies the following properties:

- non-decreasing function, i.e., $p_\omega(n+1) \geq p_\omega(n)$
- $p(1) = \#A$
- for a periodic word ω there exists $C \in \mathbb{N}$ such that $p_\omega(n) \leq C$ for all $n \in \mathbb{N}$.

Definition 1.1.5.

An infinite binary word s is called **Sturmian** if $p_s(n) = n + 1$ for every natural n.

Definition 1.1.7.

An infinite binary word ω over the finite set A is called **balanced** if for any $n-$words u and v of ω the following inequality holds: $||u|_a - |v|_a| \leq 1$.

where, we denote by $|u|_a$ the number of $a \in A$ of the word u.

Consider the irrational rotation $f_\rho(x) = \{x + \rho\}$. Iterating f_ρ, one gets

$$f_\rho^n(x) = \{x + n\rho\}\,.$$

Let the intervals $I_0 = [0, 1-\rho)$ and $I_1 = [1-\rho, 1)$ be a partition of $[0, 1)$. Then, for each $n \geq 0$, we define binary infinite word $\omega(x) = \{\omega_1, \omega_2, ...\}$ by the coding

$$\omega_n(x) := \begin{cases} 1, & \text{if } f_\rho^n(x) = \{x + n\rho\} \in I_1, \\ 0, & \text{if } f_\rho^n(x) = \{x + n\rho\} \in I_0. \end{cases}$$

The infinite word $\omega(x) = \{\omega_1, \omega_2, ...\}$ is called **irrational mechanical word**.

Theorem 1.1.2.[3]

Let ω be an infinite binary word. The followings are equivalent:
(1) ω is Sturmian.
(2) ω is irrational mechanical.
(3) ω is balanced and aperiodic.

[3] Lothaire M. Combinatorics of Words. Encyclopedia of Mathematics and its Applications, 17, London, Addison-Wesley Publishing Company, (1983)

Definition 1.2.6.

A point $x_{cr} \in S^1$ is called **critical point** of order $(2d+1)$, $d \geq 1$ for homeomorphism f, if for some ε- neighborhood $U_\varepsilon(x_{cr}) = (x_{cr} - \varepsilon, x_{cr} + \varepsilon)$, the function f belongs to $C^{2d+1}(U_\varepsilon(x_{cr}))$ and

$$\frac{df}{dx}(x_{cr}) = \frac{d^2 f}{dx^2}(x_{cr}) = ... = \frac{d^{2d} f}{dx^{2d}}(x_{cr}) = 0, \quad \frac{d^{2d+1} f}{dx^{2d+1}}(x_{cr}) \neq 0.$$

Definition 1.2.7.

The map f is called **critical map**, if it has unique critical point of the odd order.

We consider the critical circle homeomorphism $f \in Cr(\bar{\rho})$ with $\bar{\rho} = \frac{\sqrt{5}-1}{2}$ and one critical point $x_0 := x_{cr}$. For $n \geq 1$, we write $p_n/q_n = [1, 1, ..., 1]$ the $n-$th convergent of $\bar{\rho}$, their denominators q_n satisfy the recurrent relation $q_{n+1} = q_n + q_{n-1}$ with initial conditions $q_0 = 1$, $q_1 = 1$. The forward orbit $O_f^+(x_0) = \{x_i = f^i(x_0), i = 0, 1, 2...\}$ of critical point defines a sequence of natural partitions of the circle[4].

[4]Cornfeld I.P., Fomin S.V. and Sinai Ya.G., Ergodic Theory, Springer Verlag, Berlin, (1982).

Indeed, denote by $I_0^{(n)} := I_0^{(n)}(x_0)$ the closed interval in S^1 with endpoints x_0 and $x_{q_n} = f^{q_n}(x_0)$. Notice that x_{q_n} lies to the left of x_0 if n is odd, and to right of x_0 if n is even. If $I_i^{(n)} = f^i(I_0^{(n)}),\ i \geq 1$, denote the iterates of the interval $I_0^{(n)}$ under f, it is well known, that the set $\mathcal{P}_n := \mathcal{P}_n(x_0)$ of intervals with mutually disjoint interiors, defined as

$$\mathcal{P}_n = \{I_i^{(n)}, 0 \leq i < q_{n+1}\} \cup \{I_j^{(n+1)}, 0 \leq j < q_n\}$$

determines a partition $\mathcal{P}_n$ of the circle for any n. The partition $\mathcal{P}_n$ is called the n**-th dynamical partition** of S^1 determined by the point x_0 and the map f.

We construct symbolic coding on the sequence of partitions $\mathcal{P}_n(x_c)$ for critical circle map f. Let $x \notin \{f^i(x_{cr}); i \geq 0\}$ be any point in S^1 and uniquely define the infinite word $(a_1, a_2, ...a_n, ...)$ by

$$a_n = a \ \text{ for } x \in I_i^n, \ 0 \leq i < q_{n-1},$$

$$a_n = 1 \ \text{ for } x \in I_{i+q_{n-1}}^{(n)}, \ 0 \leq i < q_n,$$

$$a_n = 0 \text{ for } x \in I_i^{(n+1)}, \ 0 \leq i < q_n.$$

We have constructed one-to-one correspondence

$$\phi : S^1 \setminus \{x_i, i \geq 0\} \leftrightarrow \{(a_1, ...a_n, ...); \ a_i = a, 0, 1; \ a_{i+1} = a$$

$$\text{iff } a_i = 0, i \geq 1\} = X_+$$

Notice that every interval $I^{(n)}$ of the dynamical partition $\mathcal{P}_n$ corresponds to the unique finite word $(a_0, a_2, ..., a_n)$ of length $n + 1$.

Consider the linear rotation $f_\rho(x) = x + \rho \, mod 1, \; x \in [0, 1)$ to the irrational angle $\rho \in (0, 1)$. Let $A \subset S^1$ be a measurable subset with $\ell(A) > 0$.
Define the **first return time** $R_A : A \to N$ as

$$R_A(x) = inf\{i \geq 1 : f_\rho^i(x) \in A\}.$$

Fix a point $z \in S^1$ and consider the interval $I_\varepsilon(z) = [z, z + \varepsilon] \subset S^1$. Consider the first hitting time to the interval $I_\varepsilon(z)$

$$R_\varepsilon(x) = inf\{i \geq 1 : f^i(x) \in I_\varepsilon(z)\}.$$

Fix $\theta \in (0, 1)$. Let $q_n, \; n \geq 1$ be the first return times for f_ρ . For every $n \geq 1$, we define the points $c_n(\theta)$ by:

$$\mu([x_0, c_n(\theta)]) = \theta \cdot \mu([x_0, {f_\rho}^{q_n}(x_0)]),$$

where $\mu(A)$ is the invariant probability measure on S^1.

Consider the hitting times $R_{n,\theta},\ n \geq 1$ to the intervals $[x_0, c_n(\theta))$ and **rescaled hitting times**

$$E^{(1)}_{n,\theta}(x) := \mu\left([x_0, c_n(\theta)\right) R_{n,\theta}(x).$$

There are two natural measures on the circle: invariant probability measure μ and Lebesgue measure ℓ. We define the distribution functions of $E^{(1)}_{n,\theta}(x)$:

$$F_{n,\theta}(t) = \mu\left(x \in S^1 : E^{(1)}_{n,\theta}(x) \leq t\right),\ t \in S^1;$$

$$\Phi_{n,\theta}(t) = \ell\left(x \in S^1 : E^{(1)}_{n,\theta}(x) \leq t\right),\ t \in S^1.$$

The class of P-homeomorphisms consists of orientation preserving circle homeomorphisms f which are differentiable except at a finite or countable number of break points x_b, at which the one-sided positive derivatives Df_- and Df_+ exist, which do not coincide and for which there exist constants $0 < c_1 < c_2 < \infty$, such that

- $c_1 < Df_-(x_b) < c_2$ and $c_1 < Df_+(x_b) < c_2$ for all $x_b \in BP(f)$, the set of break points of f in S^1;
- $c_1 < Df(x) < c_2$ for all $x \in S^1 \backslash BP(f)$;
- $\log Df$ has finite total variation in S^1.

The ratio $\sigma_f(x_b) = \frac{Df_-(x_b)}{Df_+(x_b)}$ is called the **jump ratio** of f at x_b.

Theorem 1.4.2.[5]

Let f be a P-homeomorphism with irrational rotation number ρ_f. Then for any x with $f^s(x) \notin BP(f), 0 \leq s < q_n$ the following inequality holds:

$$e^{-V} \leq Df^{q_n}(x) \leq e^{V}.$$

[5] Herman M.: Sur la conjugaison differentiable des diffeomorphismes du cercle a des rotations, Inst. Hautes Etudes Sci. Publ. Math. 49 ,1979, p. 5-233. Ergodic Theory Dynamic. Systems 9, 1989.

Definition 1.4.1.

Two P-homeomorphisms f_1 and f_2 of the circle are said to be **topologically equivalent**, if there exists a homeomorphism $\varphi : S^1 \to S^1$ such that

$$\varphi(f_1(x)) = f_2(\varphi(x)), \quad \text{for any } x \in S^1.$$

The homeomorphism φ is called a **conjugacy**. Now, we formulate the following generalization of the classical **Denjoy's theorem**.

Theorem 1.4.3.[6]

Suppose that a homeomorphism f satisfies the conditions of theorem 1.4.2. Then the homeomorphism f is topologically conjugate to the linear rotation f_ρ.

Herman considered P-homeomorphism with two break points(see Fig. 1.4.1).

[6] Herman M.: Sur la conjugaison differentiable des diffeomorphismes du cercle a des rotations, Inst. Hautes Etudes Sci. Publ. Math. 49 ,1979, p. 5-233. Ergodic Theory Dynamic. Systems 9, 1989.

1.4. Circle maps with breaks

Let us given two real numbers $\lambda > 1$ and $\beta > 0$, he defined for $x \in [0,1]$ the piecewise linear map $H_{\beta,\lambda} : [0,1] \to [0,1]$ as

$$H_{\beta,\lambda}(x) = \begin{cases} \lambda x, \quad if \;\; 0 \le x \le c, \\ \lambda^{-\beta}(x-1)+1, \;\; if \;\; c \le x \le 1, \end{cases}$$

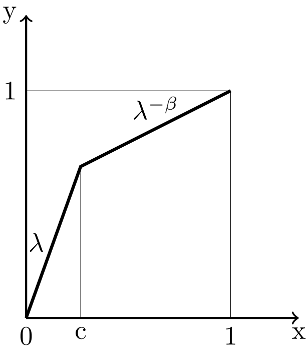


Fig. 1.4.1. Herman's maps $H_{\beta,\lambda}(x)$.

2.1. Main properties of infinite binary words associated with irrational rotations

We consider an irrational rotation $f_\rho(x) = x + \rho \; mod 1$ on the interval $[0, 1)$. Fix a point $b \in S^1$. Consider the partition $P = \{[0, b), [b, 1)\}$ of the circle. Put $P_1 := P$. In fact,

$$P_n := P \vee f_\rho^{-1}(P) \vee ... \vee f_\rho^{-n+1}(P)\}$$

Next using the sequence of partitions $P_n, \; n \geq 0$ we construct some special kind of symbolic dynamics. Put $I_1 := [0, b)$ and $I_0 := [b, 1)$.
Define, the coding function $\nu_b : S^1 \to \{0, 1\}$: for all $i \geq 0$

$$\nu_b(f_\rho^i(x)) := \begin{cases} 0, & \text{if } x \in f_\rho^{-i}(I_0), \\ 1, & \text{if } x \in f_\rho^{-i}(I_1). \end{cases}$$

2.1. Main properties of infinite binary words associated with irrational rotations

Take any $x \in S^1$. The corresponding infinite sequence $\underline{\omega} := \underline{\omega}(x)$ of zeros and ones we define as

$$\underline{\omega} = (\omega_0 \omega_1 ... \omega_n ...) := (\nu_b(x)\nu_b(f_\rho(x))...\nu_b(f_\rho^n(x))...) \qquad (2.1.3)$$

Recall, that an infinite sequence $\underline{\omega}$ is called **uniformly recurrent** if any its factor appears infinitely many times in $\underline{\omega}$ with bounded gaps.
Now, we formulate our main result of the section 2.1.

Theorem 2.1.1.[7]

Let $x, y \in S^1$ and $\underline{\omega}(x)$, $\underline{\omega}(y)$ be their infinite coding sequences defined by (2.1.3). Then followings are hold
1. For all $n \geq 0$, we have $W_n(\underline{\omega}(x)) = W_n(\underline{\omega}(y))$.
2. $\underline{\omega}(x)$ is uniformly recurrent.
3. $\underline{\omega}(x)$ is aperiodic for any $x \in S^1$.

[7] Jalilov A.A. The cardinality of factors of infinite binary words for irrational circle rotations// Uzbek mathematical journal, V 4, 2020 p. 52-63.

From theorem 1.1.2 follows that,
If $b \neq \{\rho, 1-\rho\}$, then $p_{\underline{\omega}}(n) \geq n+1$ for all $n \geq 1$. Denote by $r(n)$ the **number of right special factors** (r.s.f) of length n and let
$k_0 = min\{n \geq 1 : r(n) = 2\}$.
The $r(n)$ is used to determine the complexity function:

$$p_{\underline{\omega}}(n+1) = p_{\underline{\omega}}(n) + r(n).$$

First, we consider the case when the point b is not in the orbit of 0.

Theorem 2.2.1.[8]

Let f_ρ be the linear rotation to the irrational angle ρ, $\rho \in (0,1)$ and let $b + n\rho \notin Z$. Then the followings are hold
1. If $0 < \rho < b$, then $p_{\underline{\omega}}(n) = 2n$ for all $n \geq 1$
2. If $0 < b < \rho$, then

$$p_{\underline{\omega}}(n) := \begin{cases} n+1, & \text{if } n < k_0, \\ 2n - k_0 + 1, & \text{if } n \geq k_0. \end{cases}$$

[8] Jalilov A.A. The cardinality of factors of infinite binary words for irrational circle rotations// Uzbek mathematical journal, V 4, 2020 p. 52-63.

Next, we consider the case when b lies in the orbit of 0.

Theorem 2.2.2.[9]

Let f_ρ be same as theorem 2.2.1. and $b + d\rho \in Z$, for some $d \in Z \setminus \{0\}$. Then the followings are hold

1. If $0 < \rho < b$ then

$$p_{\underline{\omega}}(n) := \begin{cases} 2n, & \text{if } n < |d|, \\ n + d, & \text{if } n \geq |d|. \end{cases}$$

2. If $0 < b < \rho$, then

$$p_{\underline{\omega}}(n) := \begin{cases} n + 1, \text{ if } n < k_0, \\ 2n - k_0 + 1, \text{ if } k_0 \leq n < |d| \\ n + d - k_0 - 1, \text{ if } n \geq |d|. \end{cases}$$

[9] Jalilov A.A. The cardinality of factors of infinite binary words for irrational circle rotations// Uzbek mathematical journal, V 4, 2020 p. 52-63.

Let, $f \in Cr(\bar{\rho})$ be a critical circle maps with a single critical point of third order and irrational rotation number $\rho = \rho_f$. Let $\mu = \mu_f$ be the unique invariant probability measure of f. Consider the rescaled hitting time function

$$E_{n,\theta}^{(1)}(x) := E_n^{(1)}(x) = \frac{1}{q_{n+3}} R_{n,\theta}(x)$$

is a random variable taking values in $[0, 1]$ and distribution functions of it

$$\Phi_{n,\theta}(t) = \ell \left(x \in S^1 : E_n^{(1)}(x) \leq t \right), \ t \in S^1.$$

We formulate the following theorem.

Theorem 2.3.1. [10]

The distribution function of the rescaled hitting time function $E_n^{(1)}(x)$ has the following form:

i) if $t < 1/q_{n+3}$, then $\Phi_{n,\theta}(t) = 0$,

ii) if $m/q_{n+3} \le t \le (m+1)/q_{n+3}$, $1 \le m \le q_{n+1}$, then

$$\Phi_{n,\theta}(t) = \sum_{i=q_{n+1}-m}^{q_{n+1}-1} |B_i^{(n)}| + \sum_{j=q_{n+2}-m}^{q_{n+2}-1} |A_j^{(n)}| + \sum_{k=q_{n+3}-m}^{q_{n+3}-1} |C_k^{(n)}|,$$

iii) if $m/q_{n+3} \le t \le (m+1)/q_{n+3}$, $q_{n+1} \le m \le q_{n+2}$, then

$$\Phi_{n,\theta}(t) = \sum_{i=0}^{q_{n+1}-1} |B_i^{(n)}| + \sum_{j=q_{n+2}-m}^{q_{n+2}-1} |A_j^{(n)}| + \sum_{k=q_{n+3}-m}^{q_{n+3}-1} |C_k^{(n)}|,$$

[10] Ayupov Sh.A., Jalilov A.A. Asymptotic distribution of hitting times for critical maps of the circle// Vestnik Udmurskogo Universiteta, V3, 2021. (3. Scopus, IF=1.0).

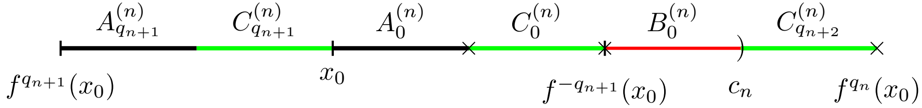
Theorem 2.3.1.(continue)

iv) if $m/q_{n+3} \le t \le (m+1)/q_{n+3}$, $q_{n+2} \le m \le q_{n+3}$, then

$$\Phi_{n,\theta}(t) = \sum_{i=0}^{q_{n+1}-1} |B_i^{(n)}| + \sum_{j=0}^{q_{n+2}-1} |A_j^{(n)}| + \sum_{k=q_{n+3}-m}^{q_{n+3}-1} |C_k^{(n)}|,$$

v) if $t \ge 1$, then $\Phi_{n,\theta}(t) = 1$,

where $|L_i^{(n)}|$ is Lebesgue measure of L_i^n. The intervals $A_j^{(n)}$, $B_j^{(n)}$ and $C_j^{(n)}$ are defined as (1.3.5). (see figure below)



We formulate the main result of this section.

Theorem 2.3.2.[11]

Let $\bar{\rho} = \frac{\sqrt{5}-1}{2}$ and let $f \in Cr(\bar{\rho})$ be critical circle map. Consider for $\theta \in (0,1)$ the sequence of distribution functions $\{\Phi_{n,\theta}(t)\}_{n=1}^{\infty}$ with respect to Lebesgue measure on circle corresponding to the first rescaled hitting times $E_{n,\theta}^{(1)}(x)$ to interval $[x_c, c_n(\theta)]$. Then
1) for all $t \in S^1$ there exists the finite limit

$$\lim_{n\to\infty} \Phi_{n,\theta}(t) = \Phi_{\theta}(t),$$

where $\Phi_{\theta}(t) = 0$, if $t \leq 0$, and $\Phi_{\theta}(t) = 1$, if $t > 1$;
2) the limit function $\Phi_{\theta}(t)$ is a strictly increasing on [0,1] and continuous distribution function on S^1;
3) $\Phi_{\theta}(t)$ is singular on [0,1] i.e. $\frac{d\Phi_{\theta}(t)}{dt} = 0$ a.e. with respect to Lebesgue measure ℓ on the circle.

[11] Ayupov Sh.A., Jalilov A.A. Asymptotic distribution of hitting times for critical maps of the circle// Vestnik Udmurskogo Universiteta, V3, 2021. (3. Scopus, IF=1.0).

Consider, an arbitrary P-homeomorphism f with irrational rotation number ρ_f and two break points a_0 and c_0, which are not on the same orbit. Obviously, the map f^{q_n} has $2q_n$ break points denoted by

$$BP_f^n := BP_f^n(a_0) \cup BP_f^n(c_0)$$

with

$$BP_f^n(a_0) := \{a_0^*, a_{-1}^*, ..., a_{-q_n+1}^*\},$$

respectively

$$BP_f^n(c_0) := \{c_0^*, c_{-1}^*, ..., c_{-q_n+1}^*\}.$$

It is clear, that these break points of the map f^{q_n} define a partition $B_n(f)$ of the circle S^1 into $2q_n$ intervals with pairwise non-intersecting interior.

Let $\mathcal{P}_n(a_0^*)$ be the n-th dynamical partition determined by the break point $a_0^* = a_0$ with respect to the map f. Assume that breaks points c_0^* and a_0^* are in different orbits, then the following three cases hold for the second break point c_0^*:

Case I. $c_0^* \in I_{i_0}^{(n)}(a_0)$, for some $0 \le i_0 < q_{n-1}$;

Case II. $c_0^* \in f^{i_0}((a_0, a_{-q_n}])$, for some $0 \le i_0 < q_n$;

Case III. $c_0^* \in f^{i_0}((a_{-q_n}, a_{q_{n-1}}))$, for some $0 \le i_0 < q_n$;

Also it is possible the case when both breaks points c_0^* and a_0^* are in the same orbit i.e.

Case IV. $c_0^* = f^{i_0}(a_0^*)$, for some $0 \le i_0 < q_n$.

In section 3.1. we consider the cases I and III.
The following lemma shows the location of break points of f^{q_n} for the Case I.

Lemma 3.1.1.

Assume $c_0^* \in I_{i_0}^{(n)}(a_0^*)$ for some i_0 with $0 \le i_0 < q_{n-1}$. Then the break points $a_{-i}^*, c_{-i}^*,\ 0 \le i \le q_n - 1$ of f^{q_n} belong to the following elements of the dynamical partition $\mathcal{P}_n(a_0^*)$ (see also Fig 3.1.1):

- $a_0^* \in I_0^{(n)}(a_0^*)$;
- $c_{-i_0+s}^* = f^s(c_{-i_0}^*) \in I_s^{(n)}(a_0^*), 0 \le s \le i_0$;
- $a_{-q_n+s}^* = f^s(a_{-q_n}) \in f^s(a_0^*, a_{-q_n}]) \subset I_s^{(n-1)}(a_0^*), 1 \le s \le i_0$;
- $a_{-q_n+s}^*,\ c_{-q_n-i_0+s}^* = f^s(c_{-q_n-i_0}) \in f^s((a_0^*, a_{-q_n}]) \subset I_s^{(n-1)}(a_0^*),\ \ i_0 + 1 \le s \le q_n - 1$

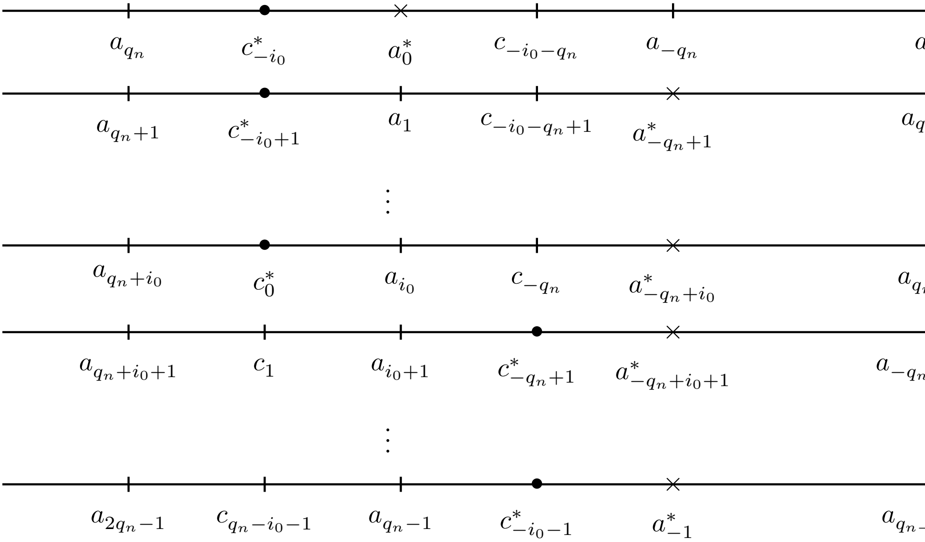


Fig. 3.1.1. $c^*_0 \in I^{(n)}_{i_0}(a^*_0)$.

In the next lemma, we can see the locations of all break points of f^{q_n} for the case III.

Lemma 3.1.2.

If $c_0^* \in f^{i_0}((a_{-q_n}, a_{q_{n-1}}])$ for some i_0 with$0 \leq i_0 < q_n$, the break points of f^{q_n} are located in the following elements of the dynamical partition $\mathcal{P}_n(a^*_{-q_n+1})$ of the break point $a^*_{-q_n+1}$ (see also Fig 3.1.2):

- $a^*_{-q_n+1+s} = f^s(a^*_{-q_n+1}), c^*_{-i_0+1+s} = f^s(c^*_{-i_0+1}) \in I_s^{(n-1)}(a^*_{-q_n+1}), 0 \leq s \leq i_0 - 1;$
- $a^*_{-q_n+i_0+1+s} = f^s(a^*_{-q_n+i_0+1}), c^*_{-q_n+1+s} = f^s(c^*_{-q_n+1}) \in I_{i_0+s}^{(n-1)}(a^*_{-q_n+1}), 0 \leq s \leq q_n - i_0 - 1.$

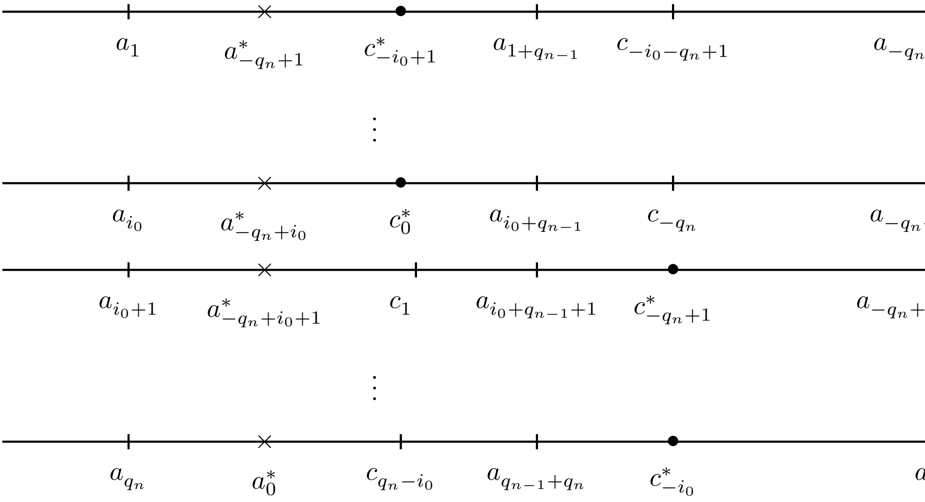


Fig. 3.1.2. $c^*_0 \in f^{i_0}((a_{-q_n}, a_{q_{n-1}}])$

3.1. Case I and Case III

From theorem 1.4.3 we know that, any Herman's map h topologically conjugated by linear rotation f_ρ. We apply these Lemmas to a Herman's maps h with irrational rotation number ρ_h.

Let n be odd. Using the break points of h^{q_n}, we define the following subintervals:

$$A_n := \bigcup_{s=1}^{i_0} [c^*_{-i_0+s},\ a^*_{-q_n+s}], \quad B_n := \bigcup_{s=i_0+1}^{q_n} [c^*_{-i_0-q_n+s},\ a^*_{-q_n+s}].$$

In Case III the subintervals are given by

$$[a^*_{-q_n+s}, c^*_{-i_0+s}],\ 1 \leq s \leq i_0,$$

respectively

$$[a^*_{-q_n+s}, c^*_{-i_0-q_n+s}],\ i_0+1 \leq s \leq q_n,$$

which we combine to the subsets

$$A_n := \bigcup_{s=1}^{i_0} [a^*_{-q_n+s}, c^*_{-i_0+s}], \quad B_n := \bigcup_{s=i_0+1}^{q_n} [a^*_{-q_n+s}, c^*_{-i_0-q_n+s}].$$

For n even, the orientation of the above intervals has to be reversed. Therefore in Case I we have the following system of disjoint intervals

$$[a^*_{-q_n+s}, c^*_{-i_0+s}],\ \ 1 \le s \le i_0,$$

respectively,

$$[a^*_{-q_n+s}, c^*_{-i_0-q_n+s}],\ \ i_0+1 \le s \le q_n.$$

In Case III one finds

$$[c^*_{-i_0+s}, a^*_{-q_n+s}],\ \ 1 \le s \le i_0,$$

respectively

$$[c^*_{-i_0-q_n+s}, a^*_{-q_n+s}],\ i_0+1 \le s \le q_n.$$

In Case I and n even, respectively in Case II and n odd, the subsets A_n and B_n can be defined as before. The above constructions show, that the boundaries of every interval in the subsets A_n and B_n is an interval whose boundaries consist of break points from $PB_n(a^*_0)$ respectively, $PB_n(c^*_0)$.

We put $\sigma = \sigma_h(a_0)$ and formulate the main result of the section 3.1.

Theorem 3.1.1.[12]

Let h be a PL circle homeomorphism with irrational rotation number ρ_h and two break points $a_0^* = 0$ and$c_0^* := c_0$, whose total jump ratio $\sigma_h = 1$, and which lie on different orbits. Assume c_0^* fulfills the assumptions of Case I respectively Case II for some i_0 with $0 \leq i_0 < q_{n-1}$. Then in Case I

$$(Dh^{q_n}(x))^{(-1)^n} = \begin{cases} \sigma^{\mu_h(A_n \cup B_n)-1}, & \text{if } x \in A_n \cup B_n \\ \sigma^{\mu_h(A_n \cup B_n)}, & \text{if } x \in S^1 \setminus (A_n \cup B_n); \end{cases}$$

respectively in Case III,

$$(Dh^{q_n}(x))^{(-1)^{n+1}} = \begin{cases} \sigma^{\mu_h(A_n \cup B_n)-1}, & \text{if } x \in A_n \cup B_n \\ \sigma^{\mu_h(A_n \cup B_n)}, & \text{if } x \in S^1 \setminus (A_n \cup B_n). \end{cases}$$

[12] A. Dzhalilov, A. Jalilov, D.Mayer, A remark on Denjoy's inequality for Pl circle homeomorphisms with two break points, J. Math. Anal. Appl. (2017) http://dx.doi.org/10.1016/j.jmaa.2017.09.003

In section 3.2 we consider cases II and IV.
Case II. Let $c_0^* \notin O(a_0^*)$ and $c_0^* \in f^{j_0}((a_0, a_{-q_n}])$, for some $0 \leq j_0 < q_n$;

Case IV. $a_{i_0}^* := f^{i_0}(a_0) = c_0^*$, $0 < i_0 < q_n$.

The following lemma shows the locations of break points of f^{q_n} for the case II.

Lemma 3.2.1.

Assume $c_0^* \in f^{i_0}((a_0^*, a_{-q_n}])$ for some$0 \leq i_0 < q_n$. Then the break points of f^{q_n} belong to the following elements of the dynamical partition $\mathcal{P}_n(c_{-i_0}^*)$ of the break point $c_{-i_0}^*$:

- $c_{-i_0}^*, a_0^* \in I_0^{(n)}(c_{-i_0}^*)$;
- $c_{-i_0+s}^* = f^s(c_{-i_0}^*), a_{-q_n+s}^* = f^s(a_{-q_n}) \in f^s([c_{-i_0}^*, a_{-q_n}]) \subset I_s^{(n-1)}(c_{-i_0}^*)$, $1 \leq s \leq i_0$;
- $c_{-q_n-i_0+s}^* = f^s(c_{-q_n-i_0}^*), a_{-q_n+s}^* = f^s(a_{-q_n}) \in f^s([c_{-i_0}, c_{-q_n}]) \subset I_s^{(n-1)}(c_{-i_0}^*)$,$i_0 + 1 \leq s \leq q_n - 1$.

Then the following theorem holds for the Case II.

Theorem 3.2.1[13]

Let h be a Herman's map with two break points $a_0^* = a_0$ and $c_0^* = c_0$ with $\sigma_h = 1$, which lie on different orbits. Assume c_0^* fulfills the assumption of Lemma 3.2.1. for some i_0 with $0 \le i_0 < q_n$. Then for all $n \ge 1$

$$(Dh^{q_n}(x))^{(-1)^n} = \begin{cases} \sigma^{\mu_h(A_n)-\mu_h(B_n)-1}, & \text{if } x \in A_n, \\ \sigma^{\mu_h(A_n)-\mu_h(B_n)+1}, & \text{if } x \in B_n, \\ \sigma^{\mu_h(A_n)-\mu_h(B_n)}, & \text{if } x \notin A_n \cup B_n. \end{cases}$$

[13] A. Dzhalilov, A. Jalilov, D.Mayer, A remark on Denjoy's inequality for Pl circle homeomorphisms with two break points, J. Math. Anal. Appl. (2017) http://dx.doi.org/10.1016/j.jmaa.2017.09.003

Finally, we consider a P-homeomorphism f with irrational rotation number ρ_f and two break points $a_0^* := a_0$, $c_0^* := f^{i_0}(a_0)$, $i_0 > 0$, on the same orbit.

Lemma 3.2.2.

Assume f is a P-homeomorphism with irrational rotation number ρ_f and two break points $a_0^* := a_0$, $a_{i_0}^* := f^{i_0}(a_0), i_0 > 0$, on the same orbit. Choose $n > n_{i_0}$.

1) If $\sigma_f = 1$, then one finds for the break points $a^*_{-q_n+s+1}$, a^*_{s+1} of f^{q_n} that $a^*_{-q_n+s+1},\ a^*_{s+1} \in f^s([a_1^*, a^*_{-q_n+1}]) \subset I_{s+1}^{(n-1)}(a_0^*) \in \mathcal{P}_n(a_0^*),\ \ 0 \le s \le i_0 - 1$;

2) if $\sigma_f \neq 1$, we have

- $a_0^* \subset I_0^{(n-1)}(a_0^*)$;
- $a^*_{-q_n+1+s},\ a^*_{1+s} \in f^s([a_{i_0+1}, a^*_{-q_n+i_0+1}]) \subset I^{(n-1)}_{i_0+1+s}(a_0^*), \quad 0 \le s \le q_n - i_0 - 2$;
- $a^*_{s+1} \in f^s([a_1^*, a^*_{-q_n+1}]) \subset I^{(n-1)}_{1+s}(a_0^*), \quad i_0 \le s \le q_n - i_0 - 1.$

Denote by $U_n(a_s^*),\ 1 \le s \le i_0$, the closed intervals with endpoints a_s^* and $a^*_{-q_n+s}$. Obviously these subintervals are disjoint. Lemma 3.2.2 implies, that $U_n(a_s^*) \subset I_s^{(n-1)}(a_0^*),\ 1 \le s \le i_0$. Next we define for every $n \ge 1$

$$U_n = \bigcup_{s=1}^{i_0} U_n(a_s^*).$$

Then one has.

Theorem 3.2.2.[14]

Let h be a Herman's map with two break points $a_0^* = 0$ and $a_{i_0}^* = h^{i_0}(a_0)$, $i_0 > 0$, with $\sigma_h = 1$, which lie on the same orbit. Put $n_{i_0} := min\{n : q_n \ge i_0\}$. For $n > n_{i_0}$ one finds

$$(Dh^{q_n}(x))^{(-1)^{n+1}} = \begin{cases} \sigma^{\mu_h(U_n)}, & \text{if } \ x \in U_n \\ \sigma^{\mu_h(U_n)-1}, & \text{if } x \in S^1 \setminus U_n, \end{cases}$$

[14] A. Dzhalilov, A. Jalilov, D.Mayer, A remark on Denjoy's inequality for Pl circle homeomorphisms with two break points, J. Math. Anal. Appl. (2017) http://dx.doi.org/10.1016/j.jmaa.2017.09.003

3.3. Herman maps and sturmian sequences

We consider Herman's maps $h(x) := h_{\beta,\lambda,\theta}(x) = H_{\beta,\lambda}(x) + \theta \mod 1, \;\; x \in S^1$.
for some $\lambda > 1$, $\beta > 0$ and the parameter $0 \leq \theta \leq 1$.
Let us denote

$$\rho_\beta := \frac{\beta}{\beta+1} \text{ and } D_n := \frac{1}{\log \sigma} \log Dh^n(x)$$

for all $n \geq 1$ and $D_0 = x$, $x \in S^1$.
Next, we define the following intervals:

$$I_0(\beta, \lambda, \theta) := [0, \rho_{\beta,\lambda,\theta}), \;\; I_1(\beta, \lambda, \theta) := [\rho_{\beta,\lambda,\theta}, 1)$$

corresponding to the map h and

$$J_0(\beta) := [0, \rho_\beta), \;\; J_1(\beta) := [\rho_\beta, 1)$$

corresponding to the linear rotation $f_{\rho_\beta}(x) = x + \rho_\beta \mod 1$, for some irrational ρ_β.
We introduce the following two coding functions ν_1 and ν_2 :

$$\nu_1(x) := \begin{cases} 0, & \text{if } x \in I_0, \\ 1, & \text{if } x \in I_1 \end{cases} \text{ and } \;\; \nu_2(x) := \begin{cases} 0, & \text{if } x \in J_0, \\ 1, & \text{if } x \in J_1. \end{cases}$$

Fix arbitrary $x \in S^1$. Now, using these coding functions we define two infinite binary sequences corresponding to the orbit $\{h^n(x),\ n \geq 0\}$ and sequence $\{D_n,\ n \geq 0\}$, correspondingly:

$$\omega_1(x) := (\nu_1(x), \nu_1(h(x)), ..., \nu_1(h^n(x)), ...) \qquad (3.3.1)$$

and

$$\omega_2(x) := (\nu_2(x), \nu_2(D_1), ..., \nu_2(D_n), ...) \qquad (3.3.2)$$

Now we formulate the main result[15] of this section.

Theorem 3.3.2.

Let h be an Herman's map with two break points a_0 and c_0 with $\sigma_h = 1$, which lie on different orbits. Suppose the rotation number $\rho_{\beta,\lambda,\theta}$ and β are irrational numbers. Then the sequences $\omega_1(x)$, $\omega_2(x)$ in (3.3.1) and (3.3.2) are Sturmian sequences.

[15] Jalilov A.A. Herman's Maps and Sturmian Sequences, Bulletin of Institute of mathematics, V2, 2020.

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