EXTENDED Z_3-GRADED LORENTZ SYMMETRY AND QUARK CHROMODYNAMICS

Richard Kerner
Laboratoire de Physique Théorique de la Matière Condensée
Sorbonne-Université,, Paris, France

Selected Topics in Mathematical Physics
Workshop in honor of Igor Volovich
Steklov Institute,
Moscow, RUSSIA
September 27-30, 2021

- **We propose a modification of standard QCD description of the colour triplet of quarks by introducing a 12-component colour generalization of Dirac spinor, with built-in Z_3 grading playing an important algebraic role in quark confinement.**

- We propose a modification of standard QCD description of the colour triplet of quarks by introducing a 12-component colour generalization of Dirac spinor, with built-in Z_3 grading playing an important algebraic role in quark confinement.
- In "colour Dirac equations" the $SU(3)$ colour symmetry is entangled with the Z_3-graded generalization of Lorentz symmetry, containing three 6-parameter sectors related by Z_3-graded maps.

- We propose a modification of standard QCD description of the colour triplet of quarks by introducing a 12-component colour generalization of Dirac spinor, with built-in Z_3 grading playing an important algebraic role in quark confinement.
- In "colour Dirac equations" the $SU(3)$ colour symmetry is entangled with the Z_3-graded generalization of Lorentz symmetry, containing three 6-parameter sectors related by Z_3-graded maps.
- The generalized Lorentz covariance requires simultaneous presence of 12 colour Dirac multiplets, which lead to the description of all internal symmetries of quarks: besides $SU(3) \times SU(2) \times U(1)$, the flavour symmetries and three quark families.

From Pauli to Dirac

- **After the discovery of spin of the electron (the Stern-Gerlach experiment), Pauli understood that a Schroedinger-type equation involving only one complex-valued wave function is not enough to take into account this new degree of freedom.**

From Pauli to Dirac

- **After the discovery of spin of the electron (the Stern-Gerlach experiment), Pauli understood that a Schroedinger-type equation involving only one complex-valued wave function is not enough to take into account this new degree of freedom.**
- **He proposed then to describe the dichotomic spin variable by introducing a two-component function forming a column on which Hermitean matrices can act as linear operators.**

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} \tag{1}$$

The simplest linear relation between the operators of energy, mass and momentum acting on a column vector (called *a Pauli spinor*) would read then:

$$\begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} mc^2 & 0 \\ 0 & mc^2 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} + c \, \boldsymbol{\sigma} \cdot \mathbf{p} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}, \qquad (2)$$

where

$$\boldsymbol{\sigma} \cdot \mathbf{p} = \sigma_1 \, p^1 + \sigma_2 \, p^2 + \sigma_3 \, p^3 = \begin{pmatrix} p^3 & p^1 - i \, p^2 \\ p^1 + i \, p^2 & -p^3 \end{pmatrix}.$$

- **According to the "correspondence principle", in Schroedinger's version of quantum mechanics the energy E and the momentum $\mathbf{p}$ are replaced by Hermitean operators $-i\hbar\partial_t$ and $-i\hbar\boldsymbol{\nabla}$**

- According to the "correspondence principle", in Schroedinger's version of quantum mechanics the energy E and the momentum $\mathbf{p}$ are replaced by Hermitean operators $-i\hbar\partial_t$ and $-i\hbar\boldsymbol{\nabla}$
- The Schroedinger-like two-component equation (2) can be written in a more concise form

$$E\mathbb{1}_2\,\psi = mc^2\mathbb{1}_2\,\psi + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi, \tag{3}$$

where $\psi = \begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix}$, $\mathbb{1}_2$ is the 2×2 unit matrix (obviously Hermitean), and the three Pauli matrices composing the 3-dimensional 2×2-matrix valued vector $\boldsymbol{\sigma} = [\sigma_1,\ \sigma_2,\ \sigma_3]$ is also Hermitean, composed of three *Pauli matrices*.

- **The basis of complex traceless 2×2 Hermitean matrices contains just three elements since then known as *Pauli matrices*:**

- **The basis of complex traceless 2×2 Hermitean matrices contains just three elements since then known as *Pauli matrices*:**
- $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
  $$\boldsymbol{\sigma} = [\sigma_1, \sigma_2, \sigma_3].$$

- **The three Pauli matrices multiplied by $\frac{i}{2}$ span the three dimensional Lie algebra: let $\tau_k = \frac{i}{2}\sigma_k$, then**

$$[\tau_1, \tau_2] = \tau_3, \;\; [\tau_2, \tau_3] = \tau_1, \;\; [\tau_3, \tau_1] = \tau_2.$$

- **The three Pauli matrices multiplied by $\frac{i}{2}$ span the three dimensional Lie algebra: let $\tau_k = \frac{i}{2}\sigma_k$, then**

$$[\tau_1, \tau_2] = \tau_3, \quad [\tau_2, \tau_3] = \tau_1, \quad [\tau_3, \tau_1] = \tau_2.$$

- **On the other hand, the three Pauli matrices form the Clifford algebra related to the Euclidean 3-dimensional metric:**

$$\sigma_i \sigma_k + \sigma_k \sigma_i = 2\delta_{ik} \mathbb{1}_2$$

ensuring that

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \mid \mathbf{p} \mid^2 \mathbb{1}_2.$$

However, the equation **(2)**:

$$E\,\psi = mc^2\,\psi + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi. \qquad (4)$$

is not invariant under Lorentz transformations. Indeed, by iterating, i.e. taking the square of this operator, we arrive at the following relation between the operators of energy and momentum and the mass of the particle:

$$E^2 = m^2c^4 + 2\,mc^3\mid\mathbf{p}\mid^2\boldsymbol{\sigma}\cdot\mathbf{p} + c^2\mathbf{p}^2, \qquad (5)$$

instead of the relativistic relation

$$E^2 - c^2\,\mathbf{p}^2 = m^2c^4. \qquad (6)$$

- The double product in the expression for the energy squared can be removed if one introduces a second Pauli spinor satisfying a smilar equation, and intertwining the two spinors.

- **The double product in the expression for the energy squared can be removed if one introduces a second Pauli spinor satisfying a smilar equation, and intertwining the two spinors.**
- **So let us denote the first Pauli spinor by ψ_+ and the second one by ψ_-, and let them satisfy the following coupled system of equations:**

$$E\,\psi_+ = mc^2\,\psi_+ + \boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,$$

$$E\,\psi_- = -mc^2\,\psi_- + \boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+, \tag{7}$$

(by the way, here $-1 = e^{i\pi}$, a complex number!)

- The double product in the expression for the energy squared can be removed if one introduces a second Pauli spinor satisfying a smilar equation, and intertwining the two spinors.
- So let us denote the first Pauli spinor by ψ_+ and the second one by ψ_-, and let them satisfy the following coupled system of equations:

$$E\,\psi_+ = mc^2\,\psi_+ + \boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,$$

$$E\,\psi_- = -mc^2\,\psi_- + \boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+, \tag{7}$$

(by the way, here $-1 = e^{i\pi}$, a complex number!)

- which coincides with the relativistic equation for the electron found by Dirac a few years later.

The negative mass along with the positive one in the spectrum of the Dirac equation introduces in fact a complex extension of mass.

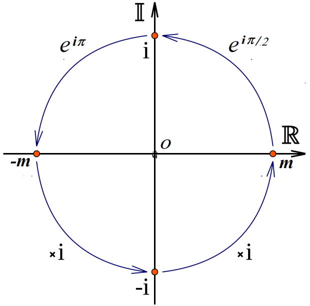

Figure: Rotations in the complex plane. The negative mass is in fact a complex number $m\, e^{i\pi}$.

Therefore the standard Dirac equation for the electron (or any **spin $\frac{1}{2}$** particle with non-zero mass m) may be interpreted as a pair of coupled equations involving two Pauli spinors,

$$\psi_+ = \begin{pmatrix} \psi_+^1 \\ \psi_+^2 \end{pmatrix} \quad \text{and} \quad \psi_- = \begin{pmatrix} \psi_-^1 \\ \psi_-^2 \end{pmatrix},$$

$$E\psi_+ = mc^2\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\;\psi_-,$$

$$E\psi_- = -mc^2\psi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\;\psi_+,$$

where as usual

$$E = -i\hbar\,\partial_t, \quad \mathbf{p} = -i\hbar\,\mathbf{grad}$$

- **The relativistic invariance is now manifest: due to the negative mass term in the second equation, the iteration leads to the separation of variables, and all the components satisfy the desired relation**

$$[E^2 - c^2\,\mathbf{p}^2]\psi_+ = m^2 c^4\ \psi_+, \quad [E^2 - c^2\,\mathbf{p}^2]\,\psi_- = m^2 c^4\ \psi_-.$$

- The relativistic invariance is now manifest: due to the negative mass term in the second equation, the iteration leads to the separation of variables, and all the components satisfy the desired relation

$$[E^2 - c^2\,\mathbf{p}^2]\psi_+ = m^2c^4\;\psi_+, \quad [E^2 - c^2\,\mathbf{p}^2]\,\psi_- = m^2c^4\;\psi_-.$$

- In a more appropriate basis the Dirac equation becomes manifestly relativistic: $[\gamma^\mu p_\mu - mc]\;\psi = 0$, with $p_0 = \frac{E}{c}$,

$$\gamma^0 = \sigma_3 \otimes \mathbb{1}_2 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^k = (i\sigma_2)\otimes\sigma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}.$$

It can be written in a compact way as follows:

$$\gamma^\mu \, p_\mu \, \psi = mc \, \psi \quad \text{with} \quad \psi = (\psi_+, \psi_-)^T \, , \tag{8}$$

where $p_\mu{=}-i\hbar\partial_\mu$, $\psi_\pm$ are two complex 2-component Pauli spinors, and as Dirac matrices γ^μ one can choose

$$\gamma^0{=}\sigma_3 \otimes 1\!\!1_2, \qquad \gamma^k{=}(i\sigma_2) \otimes \sigma^k, \tag{9}$$

where $\sigma_0 = 1\!\!1_2$, and σ^k $(k{=}1,2,3)$ are Pauli matrices. The Dirac matrices realize the 4-dimensional Clifford algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu{=}2\,\eta^{\mu\nu} 1\!\!1_4, \quad \eta^{\mu\nu}{=}\mathrm{diag}(+,-,-,-). \tag{10}$$

- **Under the Lorentz transformation**

$$x^{\mu} \rightarrow x^{\mu'} = \Lambda^{\mu'}_{\ \nu}\, x^{\nu} \tag{11}$$

- **Under the Lorentz transformation**

$$x^{\mu} \rightarrow x^{\mu'} = \Lambda^{\mu'}_{\ \nu}\, x^{\nu} \tag{11}$$

- **the spinor field** $\psi = \psi^{A}$ $(A = 1, 2, 3, 4)$ **transforms as follows:**

$$\psi'(x^{\rho'}) = \psi'(\Lambda^{\rho'}_{\ \mu} x^{\mu}) = S\psi(x^{\mu})\ . \tag{12}$$

In order to ensure the standard Lorentz covariance, the condition relating the vectorial and spinorial realizations of the Lorentz group $O(3,1) \simeq SL(2,\mathbf{C})$ is:

$$S\gamma^{\mu'}S^{-1} = \Lambda^{\mu'}{}_{\nu}(S)\gamma^{\nu} \ . \tag{13}$$

The spinorial representation S is given by the formula

$$S = exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right), \tag{14}$$

- where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and the corresponding infinitesimal vectorial representation is given by the formula

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \text{ where} \tag{15}$$

- where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, and the corresponding infinitesimal vectorial representation is given by the formula

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \text{ where} \tag{15}$$

-

$$\omega_{\mu\nu} = \eta_{\mu\lambda}\,\omega^\lambda{}_\nu = -\omega_{\nu\mu}. \tag{16}$$

with three independent Lorentz boosts $(\omega_{0k} = -\omega_{k0})$ and three independent spatial rotations $(\omega_{ij} = -\omega_{ji})$.

Extra symmetries

The three important symmetries of the Dirac equation are the following:

1) The spin inversion and space inversion,

$$\boldsymbol{\sigma} \rightarrow -\boldsymbol{\sigma}, \quad \mathbf{p} \rightarrow -\mathbf{p};$$

2) The charge conjugation,

$$m \rightarrow -m, \quad \psi^1 \rightarrow \psi^2, \ \psi^2 \rightarrow \psi^1$$

3) Global gauge invariance:

$$\psi \rightarrow e^{i\lambda}\psi, \quad \psi^\dagger \rightarrow \psi^\dagger e^{-i\lambda}.$$

In case of *local* gauge transformation

$$\psi(x) \rightarrow \tilde{\psi}(x) = e^{i\lambda(x)}\, \psi(x) \tag{17}$$

to keep Dirac's equation invariant we have to introduce the *gauge field* $A_\mu(x)$ coupled to the Dirac spinor field according to the *minimal interaction* as follows:

$$\gamma^\mu\, (p_\mu - eA_\mu(x))\psi = mc\psi.$$

where e is the elementary electric charge. We have

$$p_\mu\, \tilde{\psi}(x) = -i\hbar\partial_\mu(e^{i\lambda(x)}\psi(x)) = \partial_\mu\lambda(x)\tilde{\psi}(x) + e^{i\lambda(x)}p_\mu\psi(x).$$

The Dirac equation will keep its form unchanged if the gauge field $A_\mu(x)$ is simultaneously transformed as follows:

$$A_\mu \rightarrow \tilde{A}_\mu = A_\mu + \frac{\hbar}{e}\,\partial_\mu \lambda. \tag{18}$$

Then we have:

$$(p_\mu - e\tilde{A}_\mu)\tilde{\psi} = e^{i\lambda(x)}(p_\mu - eA_\mu)\,\psi. \tag{19}$$

- In currently widely accepted Quantum Chromo-Dynamics (QCD) the extra color variable and the new symmetry it represents are taken into account by introducing three Dirac spinors, $\psi^A, \quad A = 1, 2, 3$, and the free Lagrangian is invariant under the action of the fundamental representation of the $SU(3)$ group:

$$\psi^{B'} = U^{B'}_{A} \, \psi^A.$$

- **In currently widely accepted Quantum Chromo-Dynamics (QCD) the extra color variable and the new symmetry it represents are taken into account by introducing three Dirac spinors, $\psi^A, \quad A = 1, 2, 3$, and the free Lagrangian is invariant under the action of the fundamental representation of the $SU(3)$ group:**

$$\psi^{B'} = U^{B'}_{A} \, \psi^A.$$

- **The action of the Lorentz group (identically on each of the Dirac spinors forming the color triplet) commutes with the action of the $SU(3)$ group.**

Explicitly, the fundamental representation of the $SU(3)$ group acts on the following triplet of Dirac spinors:

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} \tag{20}$$

The Lorentz group acts simultaneously on each of the "coloured" Dirac spinors via its standard 4-D spinorial representation

- We shall extend the $Z_2 \times Z_2$ symmetry by Z_3 **group**, so that the system will mix not only the two **spin** $\frac{1}{2}$ states and particles with anti-particles, but the three colours as well.

- We shall extend the $Z_2 \times Z_2$ symmetry by Z_3 **group**, so that the system will mix not only the two **spin** $\frac{1}{2}$ states and particles with anti-particles, but the three colours as well.
- The standard Dirac equation **(8)** expressed in terms of two entangled Pauli spinors $\psi_{\pm}$ in (7) will be extended so as to incorporate six entangled Pauli spinors, to which three colours and three anti-colours are attributed.

- **The Z_3 symmetry can be combined with the Z_2 symmetry; 3 and 2 being prime numbers, the Cartesian product of the two is isomorphic with another cyclic group,**

$$Z_3 \times Z_2 = Z_6$$

.

- The Z_3 symmetry can be combined with the Z_2 symmetry; 3 and 2 being prime numbers, the Cartesian product of the two is isomorphic with another cyclic group,

$$Z_3 \times Z_2 = Z_6$$

.

- The generalized Dirac equation is invariant under the discrete group $Z_3 \times Z_2 \times Z_2 \simeq Z_6 \times Z_2$ (which is not isomorphic with Z_{12} because 6, being divisible by 2 and by 3, is not a prime number).

The cyclic group Z_6 is represented in the complex plane by its generator $q = e^{\frac{2\pi i}{6}} = e^{\frac{\pi i}{3}}$, and its powers from 1 to 6. In terms of the Z_3 group generated by j and Z_2 group generated by -1, we have $q = -j^2, \;\; q^2 = j, \; q^3 = -1, \; q^4 = j^2, \; q^5 = -j, \; q^6 = 1$, as shown in the figure (2) below.

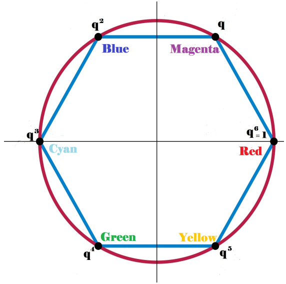

Figure: The six complex numbers q^k can be put into correspondence with three colours and three anti-colours.

- **In analogy with colours labeling quark fields, if the "white" combination is represented by 0, then we have *two* linear colourless sums of three powers of q, namely**

$$1 + q^2 + q^4 = 0 \text{ and } q + q^3 + q^5 = 0,$$

- **In analogy with colours labeling quark fields, if the "white" combination is represented by 0, then we have *two* linear colourless sums of three powers of q, namely**

$$1 + q^2 + q^4 = 0 \text{ and } q + q^3 + q^5 = 0,$$

- **and *three* white combinations of colour with its anti-colour,**

$$q + q^4 = 0, \; q^2 + q^5 = 0, \; q^3 + q^6 = 0,$$

just like a fermion and its antiparticle, or three bosons (like e.g. mesons π^0, π^+ and π^-).

A Z_3-graded analog of Pauli's exclusion principle and the Z_3-graded Dirac's equation were introduced in our papers in 2017, 2018, 2019.

R. Kerner, *Ternary generalization of Pauli's principle and the Z_6-graded algebras*, *Physics of Atomic Nuclei*, **80** (3), pp. 529-531 (2017). also: arXiv:1111.0518, arXiv:0901.3961

R. Kerner, *Ternary $Z_2 \times Z_3$ graded algebras and ternary Dirac equation*, *Physics of Atomic Nuclei* **81** (6), pp. 871-889 (2018), also: arXiv:1801.01403

R. Kerner, *The Quantum nature of Lorentz invariance*, *Universe*, **5** (1), p.1, (2019). https://doi.org/10.3390/universe5010001 (2019).

R. Kerner and J. Lukierski, *Z_3-graded colour Dirac equation for quarks, confinementt and generalized Lorentz symmetries*, *Phys. Letters B*, Vol. 792, pp. 233-237 (2019), also: arXiv:1901.10936 [hep-th]

Alternative proposal: colors first.

The generalized Dirac equation incorporating colour degrees of freedom in a Z_3-symmetric way was proposed in publications cited above; after introducing three pairs of independent Pauli spinors

$$\varphi_+ = \begin{pmatrix} \varphi^1_+ \\ \varphi^2_+ \end{pmatrix}, \;\; \varphi_- = \begin{pmatrix} \varphi^1_- \\ \varphi^2_- \end{pmatrix}, \;\; \chi_+ = \begin{pmatrix} \chi^1_+ \\ \chi^2_+ \end{pmatrix},$$

$$\chi_- = \begin{pmatrix} \chi^1_- \\ \chi^2_- \end{pmatrix}, \;\; \psi_+ = \begin{pmatrix} \psi^1_+ \\ \psi^2_+ \end{pmatrix}, \;\; \psi_- = \begin{pmatrix} \psi^1_- \\ \psi^2_- \end{pmatrix}. \tag{21}$$

with Pauli sigma-matrices acting on them in a natural way.

- These three Pauli spinors φ_+, χ_+ and ψ_+ are conventionally named "red", "blue" and "green", while their antiparticle counterparts φ_-, χ_- and ψ_- are called, respectively, "cyan", "yellow" and "magenta".

- **These three Pauli spinors φ_+, χ_+ and ψ_+ are conventionally named "red", "blue" and "green", while their antiparticle counterparts φ_-, χ_- and ψ_- are called, respectively, "cyan", "yellow" and "magenta".**
- **The cyclic group Z_3 is represented on the complex plane by multiplicative group of three complex numbers, generated by powers of $j = e^{\frac{2\pi i}{3}}$, namely:**

$$j = e^{\frac{2\pi i}{3}}, \ \ j^2 = e^{\frac{4\pi i}{3}}, \ \ j^3 = 1, \ \ 1 + j + j^2 = 0. \tag{22}$$

The resulting system of equation is as follows:

$$\begin{aligned}
E\,\varphi_+ &= mc^2\,\varphi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_-,\\
E\,\chi_- &= -j\,mc^2\,\chi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+,\\
E\,\psi_+ &= j^2\,mc^2\,\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\varphi_-,\\
E\,\varphi_- &= -mc^2\,\varphi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_+\\
E\,\chi_+ &= j\,mc^2\,\chi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,\\
E\,\psi_- &= -j^2\,mc^2\,\psi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,.\varphi_+
\end{aligned} \tag{23}$$

The color content is better seen in the following alternative basis:

$$\begin{aligned}
E\,\varphi_+ &= mc^2\,\varphi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_-,\\
E\,\varphi_- &= -mc^2\,\varphi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\chi_+\\
E\,\chi_+ &= j\;mc^2\,\chi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_-,\\
E\,\chi_- &= -j\;mc^2\,\chi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi_+\\
E\,\psi_+ &= j^2\;mc^2\,\psi_+ + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,\varphi_-,\\
E\,\psi_- &= -j^2\;mc^2\,\psi_- + c\,\boldsymbol{\sigma}\cdot\mathbf{p}\,.\varphi_+
\end{aligned} \tag{24}$$

- **The particle-antiparticle Z_2-symmetry appears as** $m \to -m$ **and simultaneously** $(\varphi_+, \chi_+, \psi_+) \to (\varphi_-, \chi_-, \psi_-)$ **and vice versa; the Z_3-colour symmetry is realized by multiplication of mass** m **by** j **each time the colour changes, i.e. more explicitly,** Z_3 **symmetry is realized as follows:**

- **The particle-antiparticle Z_2-symmetry appears as $m \to -m$ and simultaneously $(\varphi_+, \chi_+, \psi_+) \to (\varphi_-, \chi_-, \psi_-)$ and vice versa; the Z_3-colour symmetry is realized by multiplication of mass m by j each time the colour changes, i.e. more explicitly, Z_3 symmetry is realized as follows:**
-

$$m \to jm, \quad \varphi_\pm \to \chi_\pm \to \psi_\pm \to \varphi_\pm, \tag{25}$$

$$m \to j^2 m, \quad \varphi_\pm \to \psi_\pm \to \chi_\pm \to \varphi_\pm, \tag{26}$$

- **The energy operator is obviously diagonal, and its action on the spinor-valued column-vector can be represented as a 6×6 operator valued unit matrix. The mass operator is diagonal, too, but its elements represent all powers of the sixth root of unity $q = e^{\frac{2\pi i}{6}}$, which are**

$$q = -j^2, \; q^2 = j, \; q^3 = -1, \; q^2 = j^2, \; q^5 = -j \text{ and } q^6 = 1.$$

- The energy operator is obviously diagonal, and its action on the spinor-valued column-vector can be represented as a 6×6 operator valued unit matrix. The mass operator is diagonal, too, but its elements represent all powers of the **sixth root of unity** $q = e^{\frac{2\pi i}{6}}$, which are

$$q = -j^2,\ q^2 = j,\ q^3 = -1,\ q^2 = j^2,\ q^5 = -j \text{ and } q^6 = 1.$$

- The system (23) was formulated in a basis in which the "coloured" Pauli spinors alternate with their antiparticles; however, if we want to put forward the colour content, it is better to choose an alternative basis in the space of spinors arranged as follows:

$$\left(\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-\right)^T . \tag{27}$$

Then the mass and momentum operators take on the following form:

$$M = \begin{pmatrix} m & 0 & 0 & 0 & 0 & 0 \\ 0 & -m & 0 & 0 & 0 & 0 \\ 0 & 0 & jm & 0 & 0 & 0 \\ 0 & 0 & 0 & -jm & 0 & 0 \\ 0 & 0 & 0 & 0 & j^2 m & 0 \\ 0 & 0 & 0 & 0 & 0 & -j^2 m \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 & 0 & \boldsymbol{\sigma} \cdot \mathbf{p} & 0 & 0 \\ 0 & 0 & \boldsymbol{\sigma} \cdot \mathbf{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boldsymbol{\sigma} \cdot \mathbf{p} \\ 0 & 0 & 0 & 0 & \boldsymbol{\sigma} \cdot \mathbf{p} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \mathbf{p} & 0 & 0 & 0 & 0 \\ \boldsymbol{\sigma} \cdot \mathbf{p} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

- **The dimension of the two matrices M and P displayed above is 12×12: all the entries in the first one are proportional to the 2×2 identity matrix, so that in the definition one should read $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ instead of m, $\begin{pmatrix} jm & 0 \\ 0 & jm \end{pmatrix}$ instead of $j\, m$, etc.**

- The dimension of the two matrices M and P displayed above is 12×12: all the entries in the first one are proportional to the 2×2 identity matrix, so that in the definition one should read $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}$ instead of m, $\begin{pmatrix} jm & 0 \\ 0 & jm \end{pmatrix}$ instead of $j\,m$, etc.
- The entries in the second matrix P contain 2×2 **Pauli's sigma-matrices**, so that P is also a 12×12 **matrix**. The energy operator E is proportional to the 12×12 **identity matrix**.

- Only even powers of σ-matrices are proportional to $\mathbb{1}_2$, and only the powers of circulant 3×3 circulant matrix that are multiplicities of 3 are proportional to $\mathbb{1}_3$.
  The diagonalization of the system is achieved only at the sixth iteration. The final result is extremely simple: all the components satisfy the same sixth-order equation,

- Only even powers of σ-matrices are proportional to $\mathbb{1}_2$, and only the powers of circulant 3×3 circulant matrix that are multiplicities of 3 are proportional to $\mathbb{1}_3$.
  The diagonalization of the system is achieved only at the sixth iteration. The final result is extremely simple: all the components satisfy the same sixth-order equation,
- $$E^6 \, \varphi_+ = m^6 c^{12} \, \varphi_+ + c^6 \mid \mathbf{p} \mid^6 \; \varphi_+,$$
  $$E^6 \, \varphi_- = m^6 c^{12} \, \varphi_- + c^6 \mid \mathbf{p} \mid^6 \; \varphi_-. \tag{28}$$

  and similarly for all other components.

Using a more rigorous approach the three operators can be expressed in terms of tensor products of matrices of lower dimensions. Let us introduce two following 3×3 matrices:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix} \quad \text{and} \quad Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{29}$$

whose products and powers generate the $U(3)$ Lie group algebra, or the $SU(3)$ algebra if we remove the unit matrix.

The standard 3×3 matrix basis of ternary Clifford algebra (first considered in XIX-th century by Cayley and Sylvester, who called its elements "*nonions*") looks as follows:

$$Q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \; Q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \; Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (30)$$

$$Q_1^\dagger = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \; Q_2^\dagger = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}, \; Q_3^\dagger = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad (31)$$

where j is the third primitive root of unity,

$$j = e^{\frac{2\pi i}{3}}, \;\; j^2 = e^{\frac{4\pi i}{3}}, \;\; 1 + j + j^2 = 0. \qquad (32)$$

and $\mathcal{M}^\dagger$ denotes the Hermitean conjugate of matrix $\mathcal{M}$. We see that all the matrices (30, 31) are non-Hermitean.

To complete the basis of 3×3 traceless matrices, we must add to (30) and (31) the following two linearly independent *diagonal* matrices:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \quad B^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}. \tag{33}$$

We shall also use alternative notation I_A, $A = 1, 2, ...8$, with

$$I_1 = Q_1, \; I_2 = Q_2, I_3 = Q_3, \; I_4 = Q_1^\dagger, \; I_5 = Q_2^\dagger, \; I_6 = Q_6^\dagger, \; I_7 = B, \; I_8 = B^\dagger \tag{34}$$

and can also add $I_0 = \mathbb{1}_3$. The Hermitean conjugation $I_A^\dagger \; (A = 1, 2, ..., 8)$:

$$I_A^\dagger = (Q_1^\dagger, \; Q_2^\dagger, \; Q_3^\dagger, \; Q_1, \; Q_2, \; Q_3, \; B^\dagger, \; B) = I_{A^\dagger} \tag{35}$$

provides the following permutation of indices $A \rightarrow A^\dagger$:

$$A = (1, 2, 3, 4, 5, 6, 7, 8) \rightarrow A^\dagger = (4, 5, 6, 1, 2, 3, 8, 7). \tag{36}$$

We can introduce as well the standard complex conjugation $\mathcal{M} \rightarrow \bar{\mathcal{M}}$, which leads to the relations

$$\bar{I}_A = (\bar{Q}_1 = Q_2, \bar{Q}_2 = Q_1, \bar{Q}_3 = Q_3, \ \ \bar{Q}_1^\dagger = Q_2^\dagger, \ \bar{Q}_2^\dagger = Q_1^\dagger \ \ \bar{B} = B^\dagger) = I_{\bar{A}}, \tag{37}$$

which corresponds to another permutation of indices A,

$$A = (1, 2, 3, 4, 5, 6, 7, 8) \rightarrow \bar{A} = (2, 1, 3, 5, 4, 6, 8, 7). \tag{38}$$

The 3×3 matrices Q_3 and $Q_3^\dagger$ are real, while $Q_2 = \bar{Q}_1$ are mutually complex conjugated, as well as their Hermitean counterparts $Q_2^\dagger = \bar{Q}_1^\dagger$.

The six matrices Q_k and $Q_j^\dagger$, $i, j = 1, 2, 3$ are endowed with natural $\mathbb{Z}_3$-grading

$$\text{grade}(Q_i) = 1, \quad \text{grade}(Q_k^\dagger) = 2, \tag{39}$$

Out of three independent $\mathbb{Z}_3$-grade 0 ternary (i.e. three-linear) combinations, only one leads to a non-vanishing result. One can simply check that both j and j^2 *ternary skew commutators* do vanish

$$\{Q_1, Q_2, Q_3\}_j = Q_1 Q_2 Q_3 + j Q_2 Q_3 Q_1 + j^2 Q_3 Q_1 Q_2 = 0, \tag{40}$$

$$\{Q_1, Q_2, Q_3\}_{j^2} = Q_1 Q_2 Q_3 + j^2 Q_2 Q_3 Q_1 + j Q_3 Q_1 Q_2 = 0, \tag{41}$$

as well as the odd permutation, e.g.
$Q_2 Q_1 Q_3 + j Q_1 Q_3 Q_2 + j^2 Q_3 Q_2 Q_1 = 0.$

In contrast, the totally symmetric combination does not vanish but is proportional to the 3×3 identity matrix $I_0 = \mathbb{1}_3$:

$$Q_a Q_b Q_c + Q_b Q_c Q_a + Q_c Q_a Q_b = 3\, \eta_{abc}\, \mathbb{1}_3, \quad a, b, ... = 1, 2, 3. \quad (42)$$

with η_{abc} given by the following non-zero components

$$\eta_{111} = \eta_{222} = \eta_{333} = 1, \;\; \eta_{123} = \eta_{231} = \eta_{312} = j^2, \\ \eta_{213} = \eta_{321} = \eta_{132} = j \quad (43)$$

and all other components vanishing. The above relation can be used as definition of *ternary Clifford algebra* **.**

An analogous set of relations is formed by Hermitean conjugates $Q^\dagger_{\dot a} := \bar{Q}^T_a$ of matrices Q_a, which we shall endow with dotted indices $\dot a, \dot b, ... = 1, 2, 3$. They satisfy the relation

$$Q_a^2 = Q^\dagger_{\dot a} \tag{44}$$

as well as the identities conjugate to the ones in (42)

$$Q^\dagger_{\dot a} Q^\dagger_{\dot b} Q^\dagger_{\dot c} + Q^\dagger_{\dot b} Q^\dagger_{\dot c} Q^\dagger_{\dot a} + Q^\dagger_{\dot c} Q^\dagger_{\dot a} Q^\dagger_{\dot b} = 3\, \eta_{\dot a \dot b \dot c}\, \mathbb{1}_3, \text{ with } \eta_{\dot a \dot b \dot c} = \bar{\eta}_{cba}. \tag{45}$$

The 12×12 matrices M and P can be represented as the following tensor products:

$$M = m\, B \otimes \sigma_3 \otimes \mathbb{1}_2, \qquad P = Q_3 \otimes \sigma_1 \otimes (\boldsymbol{\sigma} \cdot \mathbf{p}) \tag{46}$$

with as usual,

$$\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- Let us rewrite the matrix operator generating the system **(24)** when it acts on the column vector containing twelve components of three "colour" fields, in the basis **(27)** $[\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-]$:

- Let us rewrite the matrix operator generating the system **(24)** when it acts on the column vector containing twelve components of three "colour" fields, in the basis **(27)** $[\varphi_+, \varphi_-, \chi_+, \chi_-, \psi_+, \psi_-]$:
-
$$E \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2 = mc^2 \; B \otimes \sigma_3 \otimes \mathbb{1}_2 + Q_3 \otimes \sigma_1 \otimes c \, \boldsymbol{\sigma} \cdot \mathbf{p}$$

with energy and momentum operators on the left hand side, and the mass operator on the right hand side:

$$E \mathbb{1}_2 \otimes \mathbb{1}_3 \otimes \mathbb{1}_2 - Q_3 \otimes \sigma_1 \otimes c \, \boldsymbol{\sigma} \cdot \mathbf{p} = mc^2 \; B \otimes \sigma_3 \otimes \mathbb{1}_2 \quad (47)$$

- **Like with the standard Dirac equation, let us transform this equation so that the mass operator becomes proportional the the unit matrix. To do so, we multiply the equation (47) from the left by the matrix** $B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2$.

- Like with the standard Dirac equation, let us transform this equation so that the mass operator becomes proportional the the unit matrix. To do so, we multiply the equation **(47)** from the left by the matrix $B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2$.
- Now we get the following equation which enables us to interpret the energy and the momentum as the components of a Minkowskian four-vector $c\, p^\mu = [E,\ c\mathbf{p}]$:

$$E\ B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2 - Q_2 \otimes (i\sigma_2) \otimes c\,\boldsymbol{\sigma} \cdot \mathbf{p} = mc^2 \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \mathbb{1}_2, \quad (48)$$

where we used the fact that under matrix multiplication, $\sigma_3\sigma^3 = \mathbb{1}_2$, $B^\dagger B = \mathbb{1}_3$ and $B^\dagger Q_3 = Q_2$.

- **The sixth power of this operator gives the same result as before,**

$$\left[E\ B^\dagger \otimes \sigma_3 \otimes 1\!\!1_2 - Q_2 \otimes (i\sigma_2) \otimes c\, \boldsymbol{\sigma} \cdot \mathbf{p}\right]^6 = \left[E^6 - c^6 \mathbf{p}^6\right] 1\!\!1_{12}$$
$$= m^6 c^{12} 1\!\!1_{12} \tag{49}$$

- The sixth power of this operator gives the same result as before,

$$\left[E\ B^{\dagger} \otimes \sigma_3 \otimes \mathbb{1}_2 - Q_2 \otimes (i\sigma_2) \otimes c\,\boldsymbol{\sigma} \cdot \mathbf{p}\right]^6 = \left[E^6 - c^6 \mathbf{p}^6\right] \mathbb{1}_{12} = m^6 c^{12} \mathbb{1}_{12} \qquad (49)$$

- It is also worth to note that taking the determinant on both sides of the eq. **(48)** yields the twelfth-order equation:

$$\left(E^6 - c^6\ \mid \mathbf{p} \mid^6\right)^2 = m^{12} c^{24}. \qquad (50)$$

- The equation **(48)** can be written in a concise manner using the Minkowskian indices and the usual pseudo-scalar product of two four-vectors as follows:

$$\Gamma^{\mu} p_{\mu} \, \Psi = mc \mathbb{1}_{12} \, \Psi, \quad \text{with} \;\; p^0 = \frac{E}{c}, \;\; p^k = [\, p^x, p^y, p^z \,]. \tag{51}$$

- The equation **(48)** can be written in a concise manner using the Minkowskian indices and the usual pseudo-scalar product of two four-vectors as follows:

$$\Gamma^{\mu} p_{\mu} \, \Psi = mc \mathbb{1}_{12} \, \Psi, \quad \text{with} \;\; p^0 = \frac{E}{c}, \;\; p^k = [\, p^x, p^y, p^z \,]. \tag{51}$$

- with 12×12 matrices Γ^{μ}, $(\mu = 0, 1, 2, 3)$ defined as follows:

$$\Gamma^0 = B^{\dagger} \otimes \sigma_3 \otimes \mathbb{1}_2, \qquad \Gamma^k = Q_2 \otimes (i\sigma_2) \otimes \sigma^k \tag{52}$$

There is still certain arbitrariness in the choice of 3×3 matrix factors $B^\dagger$ and Q_2 in the colour Dirac operator.

This is due to the choice of $j = e^{\frac{2\pi i}{3}}$ as the generator of the representation of the finite Z_3-symmetry group.

If j^2 is chosen instead, in (48) the matrix $B^\dagger$ will be replaced by B, Q_2 by Q_1, which is its complex conjugate; the remaining terms keep the same form.

The equation (47) can be written in a concise manner by introducing the 12×12 matrix colour Dirac operator $\Gamma^\mu p_\mu$ using Minkowskian indices and metric $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$:

$$\Gamma^\mu p_\mu \, \Psi = mc \, \mathbb{1}_{12} \, \Psi, \quad \text{with} \;\; p^0 = \frac{E}{c}, \;\; p^k = [\, p^x, p^y, p^z \,]. \qquad (53)$$

with 12×12 matrices Γ^μ $(\mu = 0, 1, 2, 3)$ defined as follows:

$$\Gamma^0 = B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2, \qquad \Gamma^k = Q_2 \otimes (i\sigma_2) \otimes \sigma^k \qquad (54)$$

The 12-component colour Dirac equation is invariant under an arbitrary similarity transformation, i.e. if we set

$$\Psi' = \mathcal{R}\,\Psi, \quad (\Gamma^\mu)' = \mathcal{R}\,\Gamma^\mu\,\mathcal{R}^{-1} \quad \text{then} \quad (\Gamma^\mu)' p_\mu\,\Psi' = mc\,\Psi', \tag{55}$$

we get obviously

$$\left[(\Gamma^\mu)' p_\mu\right]^6 = (p_0^6 - \mid \mathbf{p} \mid^6) \mathbb{1}_{12} \tag{56}$$

Following the formulae (54) for the colour Dirac Γ^μ-matrices we see that they are neither real ($\bar{\Gamma}^\mu \neq \Gamma^\mu$) nor Hermitean ($(\Gamma^\mu)^\dagger \neq \Gamma^\mu$).

From the colour Dirac equation (48) one gets the equations for complex-conjugated $\bar{\Psi}$ and Hermitean-conjugated $\Psi^\dagger$:

$$\bar{\Gamma}^\mu p_\mu \, \bar{\Psi} = mc \, \bar{\Psi}, \qquad p_\mu \Psi^\dagger (\Gamma^\mu)^\dagger = mc \Psi^\dagger, \tag{57}$$

where $\bar{\Psi}$ is a column , $\Psi^\dagger$ is a row , $\bar{\sigma}_k = -\sigma_2 \sigma_k \sigma_2$, $\sigma_k = \sigma^k$, $\sigma_0 = \sigma^0 = \mathbb{1}_2$, **and**

$$\bar{\Gamma}^0 = B \otimes \sigma_3 \otimes \mathbb{1}_2, \quad \bar{\Gamma}^k = Q_1 \otimes (i\sigma_2) \otimes \bar{\sigma}^k,$$

$$(\Gamma^0)^\dagger = B \otimes \sigma_3 \otimes \mathbb{1}_2, \quad (\Gamma^k)^\dagger = Q_1 \otimes \sigma_3 \otimes \sigma^k, \tag{58}$$

The second equation of (57) can be written in terms of matrices Γ^μ if we introduce the Hermitean-adjoint colour Dirac spinor $\Psi^H = \Psi^\dagger C$, where the 12×12-matrix C satisfies the relation

$$(\Gamma^\mu)^\dagger C = C \Gamma^\mu. \tag{59}$$

It can be also shown that neither $\bar{\Gamma}^\mu$ nor $(\Gamma^\mu)^\dagger$ can be obtained via similarity transformation.

To obtain a general solution of the colour Dirac equation one should use its Fourier transformed version. In the momentum space it becomes:

$$\left(\Gamma^{\mu} p_{\mu} - m \mathbb{1}_{12}\right) \hat{\Psi}(p) = 0. \tag{60}$$

The sixth power of the matrix $\Gamma^{\mu} p_{\mu}$ is diagonal and proportional to m^6, so that we have

$$(\Gamma^{\mu} p_{\mu})^6 - m^6 \mathbb{1}_{12} = \left(p_0^6 - \mid \mathbf{p} \mid^6 - m^6\right) \mathbb{1}_{12} = 0. \tag{61}$$

Now we should find the inverse of the matrix $(\Gamma^\mu p_\mu - m \mathbb{1}_{12})$. Let us note that the sixth-order expression on the left-hand side in (61) can be factorized as follows:

$$(\Gamma^\mu p_\mu)^6 - m^6 = \left((\Gamma^\mu p_\mu)^2 - m^2 \right) \left((\Gamma^\mu p_\mu)^2 - j\, m^2 \right) \left((\Gamma^\mu p_\mu)^2 - j^2\, m^2 \right). \tag{62}$$

The first factor can be expressed as the product of two linear operators, one of which defines the colour Dirac equation (51), (60):

$$(\Gamma^{\mu} p_{\mu})^2 - m^2 = (\Gamma^{\mu} p_{\mu} - m)\ (\Gamma^{\mu} p_{\mu} + m) \tag{63}$$

Therefore the inverse of the Fourier transform of the linear operator defining the colour Dirac equation (60) is given by the following matrix:

$$[\Gamma^{\mu} p_{\mu} - m]^{-1} = \frac{(\Gamma^{\mu} p_{\mu} + m)\ \left((\Gamma^{\mu} p_{\mu})^2 - j\ m^2\right)\ \left((\Gamma^{\mu} p_{\mu})^2 - j^2\ m^2\right)}{\left(p_0^6 - \mid \mathbf{p} \mid^6 - m^6\right)}. \tag{64}$$

- The inverse of the six-order polynomial can be decomposed into a sum of three expressions with second-order denominators, multiplied by the common factor of the fourth order. Let us denote by Ω the sixth root of $(\mid \mathbf{p} \mid^6 + m^6)$,

$$\Omega = \sqrt[6]{\mid \mathbf{p} \mid^6 + m^6}, \tag{65}$$

- **The inverse of the six-order polynomial can be decomposed into a sum of three expressions with second-order denominators, multiplied by the common factor of the fourth order. Let us denote by Ω the sixth root of** $(\mid \mathbf{p} \mid^6 + m^6)$,

$$\Omega = \sqrt[6]{\mid \mathbf{p} \mid^6 + m^6}, \tag{65}$$

- **along with five other root values obtained via multiplication by consecutive powers of the sixth root of unity,** $q = e^{\frac{2\pi i}{6}}$ **. Recalling the definition of** j **and that** $q^2 = j$**, we have the identity**

$$(p_0^6 - \Omega^6) = (p_0^2 - \Omega^2)((p_0^2 - j\Omega^2)((p_0^2 - j^2\Omega^2) \tag{66}$$

which leads to the decomposition formula

$$\frac{1}{\left(p_0^6 - \mid \mathbf{p} \mid^6 - m^6\right)} = \frac{1}{3\,\Omega^4}\left[\frac{1}{p_0^2 - \Omega^2} + \frac{j}{p_0^2 - j\Omega^2} + \frac{j^2}{p_0^2 - j^2\Omega^2}\right] \tag{67}$$

or equivalently,

$$\frac{1}{\left(p_0^6 - \mid \mathbf{p} \mid^6 - m^6\right)} = \frac{1}{3\,\Omega^4}\left[\frac{1}{p_0^2 - \Omega^2} + \frac{1}{j^2 p_0^2 - \Omega^2} + \frac{1}{j p_0^2 - \Omega^2}\right] \tag{68}$$

After such a substitution in (64), six Z_6-graded simple poles do appear, Figure (3) illustrating the location of these six poles in the complex energy plane.

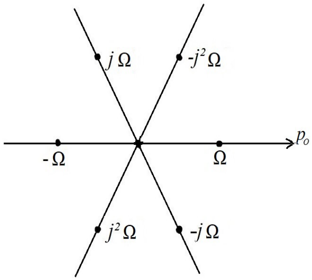

Figure: The six simple poles in the Fourier-transform of the propagator (67), with two real ones $\pm\Omega$ and two conjugate Lee-Wick poles $\pm j\Omega$, $\pm j^2\Omega$.

As long as there is a non-zero mass term, we do not encounter the infrared divergence problem at $\mid \mathbf{p} \mid \rightarrow 0$. Each of the three inverses of a second-order polynomial can be in turn expressed as a sum of simple first-order poles, e.g.

$$\frac{1}{p_0^2 - j\Omega^2} = \frac{j}{2\,\Omega}\left[\frac{1}{p_0 - j^2\Omega} - \frac{1}{p_0 + j^2\Omega}\right] = \frac{j^2}{2\,\Omega}\left[\frac{1}{jp_0 - \Omega} - \frac{1}{jp_0 + \Omega}\right], \tag{69}$$

and similarly for other terms in (67).

- **In order to introduce the propagators in the coordinate space, one has to perform the contour integrals in complex energy plane. The inverse Fourier transformation from the 4-momentum into the space-time dependent functions implies the extension of the p_0 component (the energy) into the complex domain.**

- In order to introduce the propagators in the coordinate space, one has to perform the contour integrals in complex energy plane. The inverse Fourier transformation from the 4-momentum into the space-time dependent functions implies the extension of the p_0 component (the energy) into the complex domain.
- The first term in the decomposition **(67)** of the coulour Dirac propagator presents two simple poles on the real line, while the second and the third terms display two simple poles each, located on complex straight lines $Imp_0 = jRep_0$ and $Imp_0 = j^2 Rep_0$.

- One can add that in the propagators given by formula **(67)** the non-standard residua $\pm j$ and $\pm j^2$ should be justified by suitable form of the Z_3-graded commutators describing quantum oscillator algebra of colour quark field excitations.

- **One can add that in the propagators given by formula (67) the non-standard residua $\pm j$ and $\pm j^2$ should be justified by suitable form of the Z_3-graded commutators describing quantum oscillator algebra of colour quark field excitations.**
- **The colour Dirac equation (57) breaks the Lorentz symmetry $\mathcal{O}(1,3)$ reducing it to $\mathcal{O}_3$, because the 3×3-matrices describing "colour" are *different* for the Γ^0 and Γ^k components.**

- One can add that in the propagators given by formula (67) the non-standard residua $\pm j$ and $\pm j^2$ should be justified by suitable form of the Z_3-graded commutators describing quantum oscillator algebra of colour quark field excitations.
- The colour Dirac equation (57) breaks the Lorentz symmetry $\mathcal{O}(1,3)$ reducing it to $\mathcal{O}_3$, because the 3×3-matrices describing "colour" are *different* for the Γ^0 and Γ^k components.
- However we shall show in the following Section that one can introduce a Z_3-graded generalization of the Lorentz transformations, acting in covariant way on three "replicas" of the energy-momentum four-vector introduced above.

The mass shell condition

$$E^6 - c^6 \mid \mathbf{p} \mid^6 = m^6 c^{12} \tag{70}$$

can be decomposed into the usual relativistic Klein-Gordon invariant multiplied by a strictly positive factor:

$$C_6 = p_0^6 - \Omega^6 = (p_0^2 - \mid \mathbf{p} \mid^2)(p_0^4 + p_0^2 \mid \mathbf{p} \mid^2 + \mid \mathbf{p} \mid^4) = m^6 c^6, \tag{71}$$

The sixth-order polynomial C_6 can be further decomposed into the product of the following three second-order polynomials,

$$C_6 = \overset{(0)}{C_2} \, \overset{(1)}{C_2} \, \overset{(2)}{C_2} \,, \tag{72}$$

$$\text{with} \quad \overset{(0)}{C_2} = p_0^2 - \mathbf{p}^2, \quad \overset{(1)}{C_2} = j \, p_0^2 - \mathbf{p}^2, \quad \overset{(2)}{C_2} = j^2 \, p_0^2 - \mathbf{p}^2. \tag{73}$$

Let us denote by superscripts (0), (1) and (2) the four-momenta with quadratic invariants given by $\overset{(0)}{C_2}$, $\overset{(1)}{C_2}$ and $\overset{(2)}{C_2}$. We get explicitly

$$(p_0)^2 - (\overset{(0)}{\mathbf{p}})^2 = \overset{(0)}{C_2} \; ,$$

$$(\overset{(1)}{p_0})^2 - (\overset{(1)}{\mathbf{p}})^2 = \overset{(1)}{C_2} \; ,$$

$$(\overset{(2)}{p_0})^2 - (\overset{(2)}{\mathbf{p}})^2 = \overset{(2)}{C_2} \; , \tag{74}$$

From any real four-vector $\overset{(0)}{p}_{0\mu}$ one can define its two **"replicas"**

$0\overset{(1)}{\mathbf{p}}_{\mu}$ and $\overset{(2)}{\mathbf{p}}_{\mu}$ with p_0 in the complex plane, obtained by rotations by j and by j^2 as follows:

Let us introduce three 4×4 **matrices** acting on Minkowskian four-vectors:

$$\overset{(0)}{A} = \mathrm{diag}\,(1,1,1,1) = \mathbb{1}_4, \quad \overset{(1)}{A} = \mathrm{diag}\,(j^2,1,1,1), \quad \overset{(2)}{A} = \mathrm{diag}\,(j,1,1,1), \tag{75}$$

providing a (reducible) matrix representation of the cyclic Z_3 group,

$$\overset{(r)}{A}\overset{(s)}{A} = \overset{(r+s)}{A}. \tag{76}$$

The superscripts $(r+s)$ are added modulo 3, e.g. $1+2 \to 0,\ 2+2 \to 1$, etc.

Acting on a given four-vector $p_\mu = (p_0, \mathbf{p})$ by one of the matrices $\overset{(r)}{A}$ we produce its three Z_3-graded "replicas" belonging correspondingly to sectors $\overset{(r)}{C_2}$:

$$\overset{(r)}{p} = \overset{(r)}{A}\, p : \quad \overset{(0)}{p_\mu} \to \begin{pmatrix} p_0 \\ \mathbf{p} \end{pmatrix}, \quad \overset{(1)}{p_\mu} \to \begin{pmatrix} j^2 p_0 \\ \mathbf{p} \end{pmatrix}, \quad \overset{(2)}{p_\mu} \to \begin{pmatrix} j p_0 \\ \mathbf{p} \end{pmatrix}. \tag{77}$$

In what follows, we shall use a short-hand notation:

$$p'_\mu = L_\mu{}^\nu p_\nu \to p' = Lp, \qquad \overset{(r)}{p'_\mu} = \overset{(r)}{A}_\mu{}^\nu \overset{(0)}{p_\nu} \to \overset{(r)}{p'} = \overset{(r)}{A}\,\overset{(0)}{p} \tag{78}$$

It should be stressed here that the spacetime remains Minkowskian, with one real time and three real spatial coordinates; however, the components of $\overset{(1)}{p}_\mu$ and $\overset{(2)}{p}_\mu$ can take on particular Z_3-graded complex values.

Three "replicas" (77) are the images of the same four-vector which can be obtained by Z_3-valued rotations in the complex energy plane.

Example of three replicas

The same object can be seen in three different manners:

Figure: A triple mirror giving three different images of one and the same object.

Let us denote by $\overset{(0)}{L}_{00}$ the classical Lorentz transformations which map the real Minkowskian momenta $\overset{(0)}{p}_{\nu}$ into $\overset{(0)}{p}{}'_{\nu}$

$$(\overset{(0)}{L}_{00})_{\mu}{}^{\nu}\overset{(0)}{p}_{\nu} = \overset{(0)}{p}{}'_{\mu} \rightarrow \overset{(0)}{L}_{00}\overset{(0)}{p} = \overset{(0)}{p}{}' , \qquad (79)$$

where lower indices (00) mean that we transform $\overset{(0)}{C}_2$ into itself, and the superscript (0) says that we deal with the classical Lorentz transformations.

The zero-grade Lorentz transformations can be extended to the mappings of four-vectors $\overset{(r)}{p}$ belonging to sector $\overset{(r)}{C_2}$ onto four-vectors $\overset{(s)}{p'}$ belonging to sector $\overset{(s)}{C_2}$, with $r, s = 0, 1, 2$. **Let us apply a Lorentz boost transforming a four-vector from the sector s, $\overset{(s)}{p}$ into a vector from another sector r, $\overset{(r)}{p'}$. With notations notations using the definition of A-matrices , we have:**

$$\overset{(0)}{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \overset{(1)}{A} = \begin{pmatrix} j & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \overset{(2)}{A} = \begin{pmatrix} j^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

the following formula

$$\overset{(r)}{p'} = \overset{(r)}{A}\ \overset{(0)}{p} = \overset{(r)}{A}\ \overset{(0)}{L_{00}}\ \overset{(0)}{p} = \overset{(r)}{A}\ \overset{(0)}{L_{00}}\ \overset{(s)}{A^{-1}}\ \overset{(s)}{A}\ \overset{(0)}{p} = \overset{(r-s)}{L_{rs}}\ \overset{(s)}{p} \tag{80}$$

$$\text{where} \quad \overset{(r-s)}{L_{rs}} = \overset{(r)}{A}\ \overset{(0)}{L_{00}}\ \overset{(s)}{A^{-1}}, \tag{81}$$

describes the Lorentz transformation from sector s onto sector r, and where the superscript $(r - s)$ accordingly to the Z_3-grading is taken modulo 3.

- **To provide the formulae for Z_3-graded boosts in explicit form we choose the four-vectore $p_\mu = (p_0,\ \mathbf{p})$ restricted to the plane $(0, 1)$, with the three-vector $\mathbf{p}$ aligned along the first spatial axis.**

- To provide the formulae for Z_3-graded boosts in explicit form we choose the four-vectore $p_\mu = (p_0,\ \mathbf{p})$ restricted to the plane $(0,1)$, with the three-vector $\mathbf{p}$ aligned along the first spatial axis.
- In such a frame the Lorentz rotations reduce only to the boost in $(0,1)$ plane, given by the following transformation:

$$\begin{pmatrix} p'_0 \\ p'_1 \end{pmatrix} = \begin{pmatrix} \mathrm{ch}u & \mathrm{sh}u \\ \mathrm{sh}u & \mathrm{ch}u \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}, \qquad (82)$$

Subsequently, we get the following triplet of homogeneous transformations:

$\overset{(0)}{L}_{00}$, $\overset{(0)}{L}_{11}$ and $\overset{(0)}{L}_{22}$:

$$\overset{(0)}{L}_{00}(u) = \begin{pmatrix} \mathrm{ch}u & \mathrm{sh}u \\ \mathrm{sh}u & \mathrm{ch}u \end{pmatrix}, \quad \overset{(0)}{L}_{11}(u) = \begin{pmatrix} \mathrm{ch}u & j^2\mathrm{sh}u \\ j\mathrm{sh}u & \mathrm{ch}u \end{pmatrix}, \quad \overset{(0)}{L}_{22}(u) = \begin{pmatrix} \mathrm{ch}u & j\mathrm{sh}u \\ j^2\mathrm{sh}u & \mathrm{ch}u \end{pmatrix} \tag{83}$$

preserving respectively the bilinear forms $\overset{(r)}{C}_2$.

- The matrices **(83)** are self-adjoint:

$$\overset{(0)\dagger}{L}_{00} = \overset{(0)}{L}_{00}\,, \quad \overset{(0)\dagger}{L}_{11} = \overset{(0)}{L}_{11}\,, \quad \overset{(0)\dagger}{L}_{22} = \overset{(0)}{L}_{22} \tag{84}$$

- The matrices **(83)** are self-adjoint:

$$\overset{(0)\dagger}{L}_{00} = \overset{(0)}{L}_{00}\,, \quad \overset{(0)\dagger}{L}_{11} = \overset{(0)}{L}_{11}\,, \quad \overset{(0)\dagger}{L}_{22} = \overset{(0)}{L}_{22} \tag{84}$$

- The generalized Lorentz boosts **(83)** conserve the group property: the product of two Lorentz boosts acting in the r-th sector is a boost of the same type. Indeed, we see from **(83)** that the product of two boosts acting in the r-th sector **(**$r = 0, 1, 2$**)** looks as follows (no summation over r!):

$$\overset{(0)}{L}_{rr}(u) \cdot \overset{(0)}{L}_{rr}(v) = \overset{(0)}{L}_{rr}(u + v). \tag{85}$$

If we look at three fourdimensional Lorentz boost transformations on planes $(0,i)$, $i = 1,2,3$, the respective set of three independent "classical" Lorentz boosts belonging to $\overset{(0)}{L_{00}}$ requires the introduction of three 4×4 matrices with three independent parameters u, v, w:

$$\begin{pmatrix} \mathrm{ch}u & \mathrm{sh}u & 0 & 0 \\ \mathrm{sh}u & \mathrm{ch}u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathrm{ch}v & 0 & \mathrm{sh}v & 0 \\ 0 & 1 & 0 & 0 \\ \mathrm{sh}v & 0 & \mathrm{ch}v & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathrm{ch}w & 0 & 0 & \mathrm{sh}w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{sh}w & 0 & 0 & \mathrm{ch}w \end{pmatrix} \tag{86}$$

Next, let us consider the general set of matrices transforming the s-th sector into the r-th one,

$$\overset{(r)}{p'}_{\mu} = (\overset{(r-s)}{L_{rs}})_{\mu}{}^{\nu} \overset{(s)}{p_{\nu}} , \quad r, s = 0, 1, 2, \quad r \neq s. \tag{87}$$

There are two types of such matrices: raising and lowering the Z_3-grade by 1. For the sake of simplicity, let us firstly consider the two-dimensional case (i.e. $\mu, \nu = 0, 1$ in (87).

The 2×2-matrices raising the Z_3 index (r) of the generalized four-momenta $(\overset{(r)}{p}_\mu \rightarrow \overset{(r+1)}{p}_\mu$ are:

$$\overset{(1)}{L}_{10} = \begin{pmatrix} j^2 \mathrm{ch} u & j^2 \mathrm{sh} u \\ \mathrm{sh} u & \mathrm{ch} u \end{pmatrix}, \; \overset{(1)}{L}_{21} = \begin{pmatrix} j^2 \mathrm{ch} u & j \mathrm{sh} u \\ j \mathrm{sh} u & \mathrm{ch} u \end{pmatrix}, \; \overset{(1)}{L}_{02} = \begin{pmatrix} j^2 \mathrm{ch} u & \mathrm{sh} u \\ j^2 \mathrm{sh} u & \mathrm{ch} u \end{pmatrix} \tag{88}$$

The determinants of the matrices (88) are equal to j^2.

The matrices lowering the Z_3 index by one (or increasing it by 2, what is equivalent from the point of view of the Z_3-grading) are:

$$\overset{(2)}{L}_{01} = \begin{pmatrix} j\,\mathrm{ch}u & \mathrm{sh}u \\ j\,\mathrm{sh}u & \mathrm{ch}u \end{pmatrix}, \quad \overset{(2)}{L}_{12} = \begin{pmatrix} j\,\mathrm{ch}u & j^2\mathrm{sh}u \\ j^2\mathrm{sh}u & \mathrm{ch}u \end{pmatrix}, \quad \overset{(2)}{L}_{20} = \begin{pmatrix} j\,\mathrm{ch}u & j\,\mathrm{sh}u \\ \mathrm{sh}u & \mathrm{ch}u \end{pmatrix} \tag{89}$$

The determinants of the matrices (89) are equal to j.

The above two sets of three matrices each are mutually **Hermitean-adjoint**:

$$\overset{(1)\dagger}{L}_{01} = \overset{(2)}{L}_{10}\,, \quad \overset{(1)\dagger}{L}_{12} = \overset{(2)}{L}_{21}\,, \quad \overset{(1)\dagger}{L}_{20} = \overset{(2)}{L}_{02} \qquad (90)$$

We recall that the superscript over each matrix $\overset{(t)}{L}_{rs}$ is equal to the difference of its lower indices, i.e. $(t) = (r - s)$.

The matrices $\overset{(1)}{L}_{rs}$ and $\overset{(2)}{L}_{rs}$ $(r, s = 0, 1, 2)$ raising or lowering respectively the Z_3-grade of the four-momentum vectors $\overset{(r)}{p}_{\mu}$ do not form a Lie group.

However, together with matrices $\overset{(0)}{L}_{rs}$ they can be used as building blocks in bigger 12×12 matrices forming a Z_3-graded generalization of the Lorentz group.

This construction is possible due to the chain rule obeyed by these matrices, which due to the definition **(81)** display the group property. We have:

$$\overset{(r-s)}{L_{rs}}\ (p_0, p_1;\ u)\ \overset{(s-t)}{L_{st}}\ (p_0, p_1;\ v) = \overset{(r-t)}{L_{rt}}\ (p_0, p_1;\ (u+v)). \quad (91)$$

In order to pass to arbitrary four-momentum vectors $\overset{(r)}{p_\mu}$, $\mu = 0, 1, 2, 3$ one should embed the 2×2 matrices **(88 - 89)** into 4×4 matrices in a way analogous to passing from the 2×2 boost matrices $\overset{(0)}{L_{00}}$ to the triplet of boosts in planes $(0, i), i = 1, 2, 3$ described by the 4×4 matrices (86).

If we write a Z_3-extended four-momentum vector $(\overset{(0)}{p^\mu}, \overset{(1)}{p^\mu}, \overset{(2)}{p^\mu})^T$ as a column with 12 entries, we can introduce three boost sectors $\overset{(r)}{\Lambda}$, $(r = 0, 1, 2)$ of the generalized Z_3-graded Lorentz group as 12×12 matrices as follows:

$$\overset{(0)}{\Lambda} : \begin{pmatrix} \overset{(0)}{L}_{00} & 0 & 0 \\ 0 & \overset{(0)}{L}_{11} & 0 \\ 0 & 0 & \overset{(0)}{L}_{22} \end{pmatrix} \overset{(1)}{\Lambda} : \begin{pmatrix} 0 & 0 & \overset{(1)}{L}_{02} \\ \overset{(1)}{L}_{10} & 0 & 0 \\ 0 & \overset{(1)}{L}_{21} & 0 \end{pmatrix} \overset{(2)}{\Lambda} : \begin{pmatrix} 0 & \overset{(2)}{L}_{01} & 0 \\ 0 & 0 & \overset{(2)}{L}_{12} \\ \overset{(2)}{L}_{20} & 0 & 0 \end{pmatrix}. \tag{92}$$

In each of the 12×12 matrices $\overset{(r)}{\Lambda}$, $r = 0, 1, 2$ the triplets of 4×4 matrices $\overset{(r-s)}{L_{rs}}$ are obtained from the standard classical Lorentz boosts by using the definition (81), i.e. each $\overset{(r)}{\Lambda}$-matrix depends exclusively on three parameters defining three independent classical Lorentz boosts.
One can show that our matrices display the following Z_3-graded multiplication rules:

$$\overset{(0)}{\Lambda} \cdot \overset{(r)}{\Lambda} \subset \overset{(r)}{\boldsymbol{\Lambda}}, \quad \overset{(1)}{\Lambda} \cdot \overset{(r)}{\Lambda} \subset \overset{(r+1)}{\boldsymbol{\Lambda}}, \quad \overset{(2)}{\Lambda} \cdot \overset{(r)}{\Lambda} \subset \overset{(r+2)}{\boldsymbol{\Lambda}}, \tag{93}$$

where $\overset{(r)}{\boldsymbol{\Lambda}}$ $(r = 0, 1, 2)$ denote the Z_3-graded sectors of the full set of 12×12 matrix Lorentz group which includes also the Z_3-graded $O(3)$ spatial rotations.

The multiplication rules (eq. 93) with the Z_3-graded structure can be described in a compact way using the bold-face symbols $\overset{(r)}{\mathbf{\Lambda}}$ as follows:

$$\overset{(r)}{\mathbf{\Lambda}} \cdot \overset{(s)}{\mathbf{\Lambda}} \subset \overset{(r+s)|_3}{\mathbf{\Lambda}}, \quad \text{with} \;\; r, s, .. = 0, 1, 2, \;\; (r+s) \text{ taken modulo } 3. \tag{94}$$

The construction of Z_3-graded $O(3)$ rotations completing the Z_3-graded boosts $\overset{(r)}{\mathbf{\Lambda}}$ is a follows.

Let us denote by R_i the usual space rotation around the i-th axis, represented as a 3×3 matrix. When incorporated into the four-vector representation of the Lorentz group, it becomes a sub-matrix of a 4×4 Lorentzian matrix according to the formula $\overset{(0)}{R}_i = \begin{pmatrix} 1 & 0 \\ 0 & R_i \end{pmatrix}$. The Z_3-graded space rotations supplementing the Z_3-graded boosts **(92)** are constructed as the following 12×12 matrices:

$$\overset{(0)}{\mathcal{R}}_i = \mathbb{1}_3 \otimes \overset{(0)}{R}_i, \quad \overset{(1)}{\mathcal{R}}_i = Q_3^\dagger \otimes \overset{(0)}{R}_i, \quad \overset{(2)}{\mathcal{R}}_i = Q_3 \otimes \overset{(0)}{R}_i, \tag{95}$$

where the choice of the colour generators $Q_3^\dagger$ and Q_3 is consistent with the initial definition of the colour Dirac equations.

The Z_3-**graded** infinitesimal generators of the Lorentz boosts can be obtained by considering the matrices $\overset{(r)}{\Lambda}$ with infinitesimal boost parameters what amounts to the replacements of the entries $\mathrm{sh}u$ by **1**, and of all other entries, $\mathrm{ch}u$ and 1 alike, by 0, i.e. taking the differential. The resulting 12×12 **matrices** are the Lie algebra generators of the generalized Lorentz boosts, which we shall denote as $\overset{(r)}{K_i}$, $r = 0, 1, 2$. By taking their commutators we obtain the Z_3-**graded generators** of space rotations $(r + s)$ modulo 3):

$$[\overset{(r)}{K_i}, \overset{(s)}{K_j}] = -\epsilon_{ijk} \overset{(r+s)}{J_k} \qquad (96)$$

In this way we obtained the full set of generators of the Z_3-graded Lorentz algebra which satisfy the following commutation relations:

$$\left[J_i^{(r)}, J_k^{(s)}\right] = \epsilon_{ikl} J_l^{(r+s)}, \quad \left[J_i^{(r)}, K_k^{(s)}\right] = \epsilon_{ikl} K_l^{(r+s)}, \quad \left[K_i^{(r)}, K_k^{(s)}\right] = -\epsilon_{ikl} J_l^{(r+s)}. \tag{97}$$

which were firstly introduced and studied in: R. Kerner and J. Lukierski, Physics Letters B (2019)

Let us consider 12×12 component matrices $\overset{(r)}{\Lambda}$ as 3×3 matrices with their matrix entries represented by 4×4 blocks $\overset{(t)}{L}_{rs}$ (see (87))

The matrices $\overset{(0)}{\Lambda}$ are Hermitean by virtue of formula (84), while
$(\overset{(1)}{\Lambda})^\dagger = \overset{(2)}{\Lambda}$ or equivalently, $(\overset{(2)}{\Lambda})^\dagger = \overset{(1)}{\Lambda}$ (see formula 90).

The group structure of 12×12 matrices $\mathbf{\Lambda} = (\overset{(0)}{\Lambda}, \overset{(1)}{\Lambda}, \overset{(2)}{\Lambda})$ is preserved under the similarity transformations,

$$\Lambda \rightarrow \tilde{\Lambda} = \mathcal{U} \Lambda \mathcal{U}^{-1}, \tag{98}$$

but the above Hermitean properties of Λ-matrices are conserved only if the transformation matrices are unitary. The proof is immediate: let us denote by $\mathcal{U} = U \otimes \mathbb{1}_4$ a 12×12 matrix where U is a 3×3 complex valued matrix by the unit 4×4 matrix $\mathbb{1}_4$ and denote $\mathcal{U}^{\dagger} = U^{\dagger} \otimes \mathbb{1}_4$.

Consider $\overset{(0)}{\Lambda} \rightarrow \mathcal{U} \overset{(0)}{\Lambda} \mathcal{U}^{-1}$ and impose the **Hermiticity conditions** on the transformed matrices $\mathcal{U} \overset{(0)}{\Lambda} \mathcal{U}^{-1}$. The matrix $\overset{(0)}{\Lambda}$ being Hermitean, we get

$$\left(\mathcal{U} \overset{(0)}{\Lambda} \mathcal{U}^{-1} \right)^{\dagger} = (\mathcal{U}^{-1})^{\dagger} \overset{(0)}{\Lambda}{}^{\dagger} \mathcal{U}^{\dagger} = \mathcal{U} \overset{(0)}{\Lambda} \mathcal{U}^{-1}. \qquad (99)$$

The matrix $\mathcal{U} \overset{(0)}{\Lambda} \mathcal{U}^{-1}$ is Hermitean as well if the similarity matrices $\mathcal{U}$ are *unitary*, i.e. if $\mathcal{U}^{\dagger} = \mathcal{U}^{-1}$, according to the formula $\mathcal{U} = U \otimes \mathbb{1}_4$ it follows that $U^{\dagger} = U^{-1}$. If the similarity matrices are unitary, the Hermitean conjugation relations between the matrices $\overset{(1)}{\Lambda}$ and $\overset{(2)}{\Lambda}$ are also preserved.

In this way we introduced the symmetry $SU(3)$ acting on the vector representation of the Z_3-graded Lorentz group. The 3×3 matrices U appearing in the 12×12 matrices $\mathcal{U}$ during the unitary similarity transformations leave the 4×4 Lorentzian blocks unaffected, in agreement with the well known "no-go theorems" by Coleman and Mandula and O'Raifeartaigh.

- **In order to obtain the entire Z_3-graded Lorentz group we should add as well the Z_3-graded extension of space rotations, also represented as 12×12 matrices, given by 3×3 matrices with 4×4-dimensional entries, as the Z_3-graded boosts.**

- **In order to obtain the entire Z_3-graded Lorentz group we should add as well the Z_3-graded extension of space rotations, also represented as 12×12 matrices, given by 3×3 matrices with 4×4-dimensional entries, as the Z_3-graded boosts.**
- **As in the case of Lorentz boosts, besides the rotations that leave the transformed 3-momentum in the same sector, one gets also 12×12 matrices with non diagonal 4×4 entries (95), which map one of the Z_3-graded sector onto another one.**

- In order to obtain the entire Z_3-graded Lorentz group we should add as well the Z_3-graded extension of space rotations, also represented as 12×12 matrices, given by 3×3 matrices with 4×4-dimensional entries, as the Z_3-graded boosts.
- As in the case of Lorentz boosts, besides the rotations that leave the transformed 3-momentum in the same sector, one gets also 12×12 matrices with non diagonal 4×4 entries **(95)**, which map one of the Z_3-graded sector onto another one.
- We conclude that the full set of Z_3-graded $O(3)$ subgroup elements can be represented by 12×12 matrices and incorporated in the Z_3-graded Lorentz group.

The Z_3-graded Lorentz group should also act on the coloured Dirac spinors through certain linear action, which should be realized as a generalized spinorial representatopn..

Such construction was introduced in our most recent publication:

R. Kerner and J. Lukierski, *Internal quark symmetries and colour* $SU(3)$ *entangled with* Z_3*-graded orentz algebra*, *Nuclear Physics B*, Vol. 972, (November 2021), 115529

which we shall briefly present in what follows.

What we want to find is a set of 12×12 **matrices** $J_i^{(r)}$, and $K_j^{(r)}$, $i, j.. = 0, 1, 2$ satisfying the same **Z_3-graded commutation relations** as their vectorial counterparts:

$$\left[J_i^{(r)}, J_k^{(s)}\right] = \epsilon_{ikl} J_l^{(r+s)}, \quad \left[J_i^{(r)}, K_k^{(s)}\right] = \epsilon_{ikl} K_l^{(r+s)}, \quad \left[K_i^{(r)}, K_k^{(s)}\right] = -\epsilon_{ikl} J_l^{(r+s)}. \tag{100}$$

The spinor representation of the zeroth sector $L^{(0)}$ of the **Z_3-graded Lorentz algebra** is obtained in a simplest possible manner, by tensorising the spinorial generators of the usual representation on Dirac spinors by the unit 3×3 **matrix**:

$$J_l^{(0)} = -\frac{i}{2} \mathbb{1}_3 \otimes \mathbb{1}_2 \otimes \sigma_l, \quad K_i^{(0)} = -\frac{1}{2} \mathbb{1}_3 \otimes \sigma_1 \otimes \sigma_i. \tag{101}$$

satisfying classical Lorentz algebra commutation relations:

$$\left[J_i^{(0)}, J_k^{(0)}\right] = \epsilon_{ikl} J_l^{(0)}, \quad \left[J_i^{(0)}, K_k^{(0)}\right] = \epsilon_{ikl} K_l^{(0)}, \\ \left[K_i^{(0)}, K_k^{(0)}\right] = -\epsilon_{ikl} J_l^{(0)}. \tag{102}$$

The two extra Lorentz sectors, $L^{(1)}$ and $L^{(2)}$, are constructed as the following 12×12 **matrices**:

$$J_l^{(1)} = -\frac{i}{2}\, Q_3 \otimes \mathbb{1}_2 \otimes \sigma_l, \quad K_i^{(1)} = -\frac{1}{2}\, Q_3 \otimes \sigma_1 \otimes \sigma_i. \qquad (103)$$

$$J_i^{(2)} = -\frac{i}{2}\, Q_3^\dagger \otimes \mathbb{1}_2 \otimes \sigma_i, \quad K_m^{(2)} = -\frac{1}{2}\, Q_3^\dagger \otimes \sigma_1 \otimes \sigma_m. \qquad (104)$$

Let us recall once more the notation $I_A,\ A = 1, 2, ...8$**, with**

$$I_1 = Q_1,\ I_2 = Q_2, I_3 = Q_3,\ I_4 = Q_1^\dagger,\ I_5 = Q_2^\dagger,\ I_6 = Q_6^\dagger,\ I_7 = B,\ I_8 = B^\dagger \tag{105}$$

We can also add $I_0 = \mathbb{1}_3$**. The Hermitean conjugation** $I_A^\dagger\ (A = 1, 2, ..., 8)$:

$$I_A^\dagger = (Q_1^\dagger,\ Q_2^\dagger,\ Q_3^\dagger,\ Q_1,\ Q_2,\ Q_3,\ B^\dagger,\ B) = I_{A^\dagger} \tag{106}$$

which provides the following permutation of indices $A \rightarrow A^\dagger$:

$$A = (1, 2, 3, 4, 5, 6, 7, 8) \rightarrow A^\dagger = (4, 5, 6, 1, 2, 3, 8, 7). \tag{107}$$

The formulae (102), (103) and (104) describe the spinorial realization of the Lie algebra $\mathcal{L}$ which is implied by the initial choice of matrices Γ^μ. Let us introduce a unified notation englobing all possible choices of Γ^μ-matrices $(A \neq B)$

$$\Gamma^0_{(A;\alpha)} = l_A \otimes \sigma_\alpha \otimes \sigma^0, \quad \Gamma^i_{(B;\beta)} = l_B \otimes (i\sigma_\beta) \otimes \sigma^i, \qquad (108)$$

where $l_0 = \mathbb{1}_3$, l_A **with** $A = 1, 2, ..., 8$ colormagenta **are given in** (34), **and** $\alpha, \beta = 2, 3$ **but** $\{\sigma_\alpha, \ \sigma_\beta\}_+ = 0$ **i.e. we always have either** $\alpha = 2, \ \beta = 3$ **or** $\alpha = 3, \beta = 2$.

The choice $\alpha = 1$ **is not present in the formula (108) because it is reserved for the description of symmetry generators $\mathcal{L}$ ((103), (104)). Further, eight colour** 3×3 **matrices** l_A **(** $A = 1, 2, ...8$ **) span the ternary basis of the** $SU(3)$ **algebra.**

- The characteristic feature of "colour" Γ-matrices is that the 3×3 matrices I_A appearing as the first tensorial factors in (108) are *different* for temporal and spatial components of the matrix-valued 4-vector Γ^{μ}. We see that the choice of the colour factor in (108) depends on two sets of values of the four-vector index: $\mu = 0$ or $\mu = i, \;\; i = 1, 2, 3$. This property can be interpreted as the entanglement of colour and Lorentz symmetry degrees of freedom.

- **The characteristic feature of "colour" Γ-matrices is that the 3×3 matrices l_A appearing as the first tensorial factors in (108) are *different* for temporal and spatial components of the matrix-valued 4-vector Γ^μ. We see that the choice of the colour factor in (108) depends on two sets of values of the four-vector index: $\mu = 0$ or $\mu = i, \;\; i = 1, 2, 3$. This property can be interpreted as the entanglement of colour and Lorentz symmetry degrees of freedom.**
- **In the notation (109) basic Γ-matrices appearing in the first version of the colour Dirac equation can be denoted as**

$$\Gamma^0_{(8,3)} = B^\dagger \otimes \sigma_3 \otimes \mathbb{1}_2, \quad \Gamma^i_{(2;2)} = Q_2 \otimes (i\sigma_2) \otimes \sigma^i. \qquad (109)$$

- **In order to get a closed formula for the adjoint action $\mathcal{S}^{(0)}\Gamma^{\mu}[\mathcal{S}^{(0)}]^{-1}$ of classical spinorial Lorentz group, where $a^i, b^k, \;\; (i,k=1,2,3)$ are the six real $SL(2,\mathbf{C})$ Lie group parameters**

$$S^{(0)} = \exp\left(a^i K_i^{(0)} + b^k J_k^{(0)}\right) \qquad (110)$$

- In order to get a closed formula for the adjoint action $\mathcal{S}^{(0)}\Gamma^{\mu}[\mathcal{S}^{(0)}]^{-1}$ of classical spinorial Lorentz group, where $a^i, b^k, \ (i,k=1,2,3)$ are the six real $SL(2,\mathbf{C})$ Lie group parameters

$$S^{(0)} = \exp\,\left(a^i K_i^{(0)} + b^k J_k^{(0)}\right) \tag{110}$$

- we should introduce the following pairs of Γ^{μ}-matrices

$$\Gamma^{\mu} = (\Gamma^i_{(A;2)},\ \Gamma^0_{(B;3)}) \text{ and } \tilde{\Gamma}^{\mu} = (\Gamma^i_{(B;2)},\ \Gamma^0_{(A;3)}), \tag{111}$$

where we have chosen $\alpha = 3$ and $\beta = 2$.

Although for *any* choice of the first factor I_A in $\Gamma^{\mu}_{(A;\alpha)}$'s we have

$$\left[J^{(0)}_i , \Gamma^j_{(A;\alpha)} \right] = \epsilon_{ijk} \Gamma^k_{(A;\alpha)}, \quad \left[J^{(0)}_i , \Gamma^0_{(A;\alpha)} \right] = 0, \tag{112}$$

the boosts $K^{(0)}_i$ act covariantly only on doublets $\left(\Gamma^{\mu}, \tilde{\Gamma}^{\mu} \right)$, with $(A \neq B)$, because only for such a choice we can get the closure of commutation relations:

$$[K^{(0)}_i, \Gamma^j_{(A;2)}] = \delta^j_i \, \Gamma^0_{(A;3)}, \quad [K^{(0)}_i, \Gamma^0_{(B;3)}] = \Gamma^i_{(B;2)},$$

$$[K^{(0)}_i, \Gamma^j_{(B;2)}] = \delta^j_i \, \Gamma^0_{(B;3)}, \; [K^{(0)}_i, \Gamma^0_{(A;3)}] = \Gamma^i_{(A;2)}. \tag{113}$$

It follows from **(112), (113)** that the standard Lorentz covariance requires the pair of coloured Dirac equations described by the *doublet* $(\Gamma^\mu, \tilde{\Gamma}^\mu)$ of coloured Dirac matrices, which we shall call "*Lorentz doublets*". In particular, the Γ^μ matrices should be supplemented by the following Lorentz doublet partner:

$$\tilde{\Gamma}^0 = \Gamma^0_{(2;3)} = Q_2 \otimes \sigma_3 \otimes \mathbb{1}_2, \quad \tilde{\Gamma}^i = \Gamma^i_{(8;2)} = B^\dagger \otimes (i\sigma_2) \otimes \sigma^i. \quad (114)$$

- The Lorentz doublets of Γ^μ-matrices required by the standard Lorentz covariance can be used for the description of weak isospin (flavour) doublets of the $SU(2) \times U(1)$ electroweak symmetry.

- The Lorentz doublets of Γ^μ-matrices required by the standard Lorentz covariance can be used for the description of weak isospin (flavour) doublets of the $SU(2) \times U(1)$ electroweak symmetry.
- In such a way one can show that the internal symmetries $SU(3) \times SU(2) \times U(1)$ of Standard Model are linked with the presence of standard Lorentz covariance which generates three 24-component Lorentz doublets of colour Dirac spinors.

- By calculating the multicommutators of $\left(J_i^{(1)}, K_l^{(1)}\right) \in L^{(1)}$ with the set $\Gamma^{\mu}_{(a)}$, $(a = 1, 2...6)$, we will show that the following sextet of Γ-matrices which break the Lorentz covariance is closed under the action of $L^{(1)}$:

- By calculating the multicommutators of $\left(J_i^{(1)}, K_l^{(1)}\right) \in L^{(1)}$ with the set $\Gamma^{\mu}_{(a)}$, $(a = 1, 2...6)$, we will show that the following sextet of Γ-matrices which break the Lorentz covariance is closed under the action of $L^{(1)}$:
-

$$\begin{aligned}
\Gamma^{\mu}_{(1)} &= \left(\Gamma^0_{(8;3)}, \ \Gamma^i_{(2;2)}\right); \quad & \Gamma^{\mu}_{(4)} &= \left(\Gamma^0_{(8;2)}, \ \Gamma^i_{(2;3)}\right); \\
\Gamma^{\mu}_{(2)} &= \left(\Gamma^0_{(2;2)}, \ \Gamma^i_{(4;3)}\right); \quad & \Gamma^{\mu}_{(5)} &= \left(\Gamma^0_{(2;3)}, \ \Gamma^i_{(4;2)}\right); \\
\Gamma^{\mu}_{(3)} &= \left(\Gamma^0_{(4;3)}, \ \Gamma^i_{(8;2)}\right); \quad & \Gamma^{\mu}_{(6)} &= \left(\Gamma^0_{(4;2)}, \ \Gamma^i_{(8;3)}\right).
\end{aligned} \tag{115}$$

It is easy to see that from the six components of the sextet **(115)** one can construct as well the set of six Γ^{μ}-matrices $\Gamma^{\mu}_{(A;\alpha)}$, $A = 2, 4, 8$ and $\alpha = 2, 3$, which can be described as well as three Lorentz doublets **(111)** , with $(A, B) = (2, 8), (2, 4)$ **and** $(4, 8)$**..** More explicitly,

$$\left(\Gamma^0_{(A;\alpha)} = I_A \otimes \sigma_\alpha \otimes \mathbb{1}_2, \; \Gamma^i_{(B;\beta)} = I_B \otimes (i\sigma_\beta) \otimes \sigma^i \right), \qquad (116)$$

Standard Quark content reproduced

The bottom line is:

imposing the Z_3-graded Lorentz invariance on the initial $12-component$ **generalized Dirac spinor describing a coloured quark state and on the corresponding coloured Dirac equation generates a set of six equivalent representations of this equation.**

The set of six coloured spinors which splits naturally into three "Lorentz doublets" describes the set of three families (generations) with two flavours each.

According to this model, leptons can be considered as "colourless quarks".

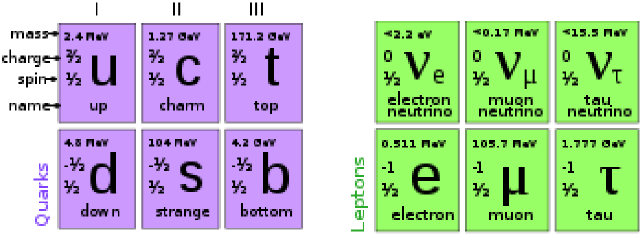

Figure: Three quark generations with two flavours each, and three types of leptons with their neutrinos.