Spherically symmetric ‘t Hooft-Polyakov monopoles

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Notation

$\mathbb{R}^3$ - Euclidean space with Cartesian coordinates $x=(x^\mu), \quad \mu=1,2,3$

$A_\mu{}^i(x)$ - local connection form (Yang-Mills fields) for $\mathbb{SU}(2)$ group $\quad i=1,2,3$

$\varphi^i(x)$ - triplet of scalar fields in the adjoint representation

$\varepsilon_{ijk},\ \varepsilon_{123}=1$ - the totally antisymmetric tensor

$$\boxed{F_{\mu\nu}{}^i=\varepsilon_{\mu\nu\rho}\nabla^\rho\varphi^i}$$

- the Bogomol’nyi equations (9 equations for 12 unknowns $A_\mu{}^i$ and φ^i)

where:

$$F_{\mu\nu}{}^i:=\partial_\mu A_\nu{}^i-\partial_\nu A_\mu{}^i+A_\mu{}^j A_\nu{}^k\varepsilon_{jk}{}^i$$

- local curvature form (the Yang-Mills field strength)

$$\nabla_\mu\varphi^i:=\partial_\mu\varphi^i+A_\mu{}^j\varphi^k\varepsilon_{jk}{}^i$$

- the covariant derivative

The result: a general spherically symmetric solution of the Bogomol’nyi equations is found.

Why this is interesting? 't Hooft Nucl.Phys.B (1974); Polyakov JETP Letters (1974)

The $\mathbb{SU}(2)$ gauge model:

$$L = -\frac{1}{4} F^{\alpha\beta i} F_{\alpha\beta i} + \frac{1}{2} \nabla^{\alpha} \varphi^{i} \nabla_{\alpha} \varphi_{i} - \frac{1}{4} \lambda (\varphi^{2} - a^{2})$$

where $\lambda > 0, \quad a > 0$ are coupling constants

x^{α} are Cartesian coordinates in Minkowskian space $\mathbb{R}^{1,3}$

$$\alpha, \beta = 0,1,2,3 \qquad \varphi^{2} := \varphi^{i} \varphi_{i}$$

Static solutions minimize the energy

$$E = \frac{1}{4} F^{\mu\nu i} F_{\mu\nu i} + \frac{1}{2} \nabla^{\mu} \varphi^{i} \nabla_{\mu} \varphi_{i} + \frac{1}{4} \lambda (\varphi^{2} - a^{2})$$

- the energy density in the time gauge $A_{0}{}^{i} = 0$

We are looking for solutions with $\varphi(\infty) \neq 0$

$Q \in \mathbb{Z}$ is the index of the map (topological charge): $\varphi(\infty) : \mathbb{S}^{2} \to \mathbb{S}^{2}$

$$\mathfrak{E} = 4\pi Q a + \frac{1}{4} \int dx \underbrace{\left(F^{\mu\nu i} - \varepsilon^{\mu\nu\rho} \nabla_{\rho} \varphi^{i}\right)}_{\parallel \atop 0} \left(F_{\mu\nu i} - \varepsilon_{\mu\nu\lambda} \nabla^{\lambda} \varphi_{i}\right)$$

- the total energy

- the Bogomol'nyi equations

Theorem. Any solution of the Bogomol'nyi equations satisfies also the equations of motion for $\lambda = 0$.

Spherically symmetric ansatz

$$A_\mu{}^i \mapsto S^{-1}{}_\mu{}^\nu A_\nu{}^j S_j{}^i, \qquad S \in \mathbb{SO}(3)$$

$$A_\mu{}^i = \varepsilon_\mu{}^{ij} \frac{x_j}{r^2}\big(K(r)-1\big) + \delta_\mu^i \frac{M(r)}{r} + \frac{x_\mu x^i}{r^3} L(r)$$

$$\varphi^i = \frac{x^i}{r^2} H(r)$$

$K(r),\ L(r),\ M(r),\ H(r)$ - unknown functions on radius

$$\lim_{r\to\infty} A_\mu{}^i = 0, \qquad \lim_{r\to\infty} \varphi^i = \frac{x^i}{r} a$$

- boundary conditions ➡ $Q = 1$

The ‘t Hooft-Polyakov monopole corresponds to $M = L = 0$

The Bogomol’nyi equations:

$$\begin{aligned} rK' + M(L+M) &= KH, \\ -rK' + K^2 - 1 - LM &= rH' - H - KH, \\ rM' - K(L+M) &= MH. \end{aligned}$$

A general solution of this system of equations for $H = 0$ was found in

Katanaev M., Volkov O. Mod.Phys.Lett. (2020)

Up to now only two exact solutions of the Bogomol’nyi equations were known:

1) Prasad M., Sommerfield C. Phys.Rev.Lett.(1975);
   Bogomolny E. Sov.J.Nucl.Phys.(1976)

2) Singleton D. Phys.Rev.D (1995)

Theorem. A general spherically symmetric solution of the Bogomol'nyi equations is

$$M(\xi) = \pm\sqrt{1 - K^2 + H_\xi - H},$$

$$L(\xi) = \mp\frac{K_\xi - K^2 + 1 + H_\xi - (K+1)H}{\sqrt{1 - K^2 + H_\xi - H}}, \qquad (*)$$

where $H(\xi)$ is a solution of the Riccati equation

$$H_\xi + H - H^2 = C\mathrm{e}^{2\xi}$$

with arbitrary constant $C \in \mathbb{R}$ and $K(r)$ is an arbitrary function satisfying inequality

$$1 - K^2 + H_\xi - H \geq 0.$$

The upper and lower signs in Eqs. $(*)$ must be chosen simultaneously.

$\boxed{r \mapsto \xi := \ln r}$ - new variable

Proposition (Special case). If $K^2 = 1 + rH' - H = 0$, then a general spherically symmetric solution of the Bogomol'nyi equations is

$$M = 0, \qquad H = C_1 r,$$

The function $L(r)$ being arbitrary.

Substitution $H(r) := \dfrac{r}{Z(z)} + 1 \quad \Rightarrow \quad Z' + CZ^2 = -1$ - special Riccati equation

The solution going through the point $Z(0) = Z_0$:

$$Z(r) = \begin{cases} \dfrac{Z_0\sqrt{-C} - \tanh\left(\sqrt{-C}r\right)}{\sqrt{-C} + CZ_0\tanh\left(\sqrt{-C}r\right)}, & C < 0, \\ Z_0 - r, & C = 0, \\ \dfrac{Z_0\sqrt{C} - \tan\left(\sqrt{C}r\right)}{\sqrt{C} + CZ_0\tan\left(\sqrt{C}r\right)}, & C > 0. \end{cases}$$

The constant of integration $C \neq 0$ can be absorbed by rescaling

$$Z \mapsto \sqrt{|C|}Z, \quad r \mapsto \sqrt{|C|}r$$

$$Z(r) = \begin{cases} \dfrac{Z_0 - \tanh r}{1 - Z_0\tanh r}, & C < 0, \\ \dfrac{Z_0 - \tan r}{1 + Z_0\tan r}, & C > 0. \end{cases}$$

Special case $\quad C<0, \quad Z_0=0, \quad K^2=1+rH'-H$

The solution is $\quad M=L=0, \quad H=1-\dfrac{r}{\tanh r}, \quad K=\pm\dfrac{r}{\sinh r}$

- the Bogomol'nyi-Prasad-Sommerfield solution

Special case $\quad C=0, \quad K^2=1+rH'-H$

The solution is $\quad M=L=0, \quad H=\dfrac{Z_0}{Z_0-r}, \quad K=\pm\dfrac{r}{Z_0-r}$

- the Singleton solution

Conclusion

All spherically symmetric ‘t Hooft-Polyakov monopole solutions minimizing the energy in the $Q=1$ sector for $\lambda=0$ are found analytically.