Weyl Cosmology

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Selected Topics in Mathematical Physics — 2021

A little bit of mathematics
Differential geometry

Metric tensor $g_{\mu\nu}$ $\Rightarrow$ interval

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \qquad (g_{\mu\nu}g^{\nu\lambda} = \delta^\lambda_\mu)$$

Connections $\Gamma^\lambda_{\mu\nu}$ $\Rightarrow$ covariant derivative

$$\nabla_\mu l^\nu = l^\nu_{,\mu} + \Gamma^\nu_{\sigma\mu} l^\sigma, \ \ldots$$

Curvature tensor $R^\mu{}_{\nu\lambda\sigma}$

$$R^\mu{}_{\nu\lambda\sigma} = \frac{\partial\Gamma^\mu_{\nu\sigma}}{\partial x^\lambda} - \frac{\partial\Gamma^\mu_{\nu\lambda}}{\partial x^\sigma} + \Gamma^\mu_{\varkappa\lambda}\Gamma^\varkappa_{\nu\sigma} - \Gamma^\mu_{\varkappa\sigma}\Gamma^\varkappa_{\nu\lambda}$$

Ricci tensor $R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}$

Curvature scalar $R = g^{\mu\lambda}R_{\mu\lambda}$

$$\boxed{g_{\mu\nu}(x), \quad \Gamma^\lambda_{\mu\nu}(x)}$$

A little bit of mathematics

Differential geometry II

Three tensors $g_{\mu\nu}(x)$, $S^{\lambda}_{\mu\nu}$, $Q_{\lambda\mu\nu}$

Torsion $S^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}$

Nonmetricity $Q_{\lambda\mu\nu} = \nabla_{\lambda} g_{\mu\nu}$

Connections $\Gamma^{\lambda}_{\mu\nu} = C^{\lambda}_{\mu\nu} + K^{\lambda}_{\mu\nu} + L^{\lambda}_{\mu\nu}$

$$C^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\varkappa}(g_{\varkappa\mu,\nu} + g_{\varkappa\nu,\mu} - g_{\mu\nu,\varkappa})$$

$$Q_{\lambda\mu\nu} = \nabla_{\lambda} g_{\mu\nu} \quad \Rightarrow \quad Q_{\lambda\mu\nu} = Q_{\lambda\nu\mu}$$

$$K^{\lambda}{}_{\mu\nu} = \frac{1}{2}(S^{\lambda}{}_{\mu\nu} - S_{\mu}{}^{\lambda}{}_{\nu} - S_{\nu}{}^{\lambda}{}_{\mu})$$

$$L^{\lambda}_{\mu\nu} = \frac{1}{2}(Q^{\lambda}{}_{\mu\nu} - Q_{\mu}{}^{\lambda}{}_{\nu} - Q_{\nu}{}^{\lambda}{}_{\mu})$$

A little bit of mathematics
Differential geometry III

Riemann geometry

$$S^{\lambda}_{\mu\nu} = 0, \quad Q_{\lambda\mu\nu} = 0$$

$$S^{\lambda}_{\mu\nu} = 0, \quad Q_{\lambda\mu\nu} = 0, \quad \Rightarrow \quad \Gamma^{\lambda}_{\mu\nu} = C^{\lambda}_{\mu\nu}$$

Weyl geometry

$$S^{\lambda}_{\mu\nu} = 0 \quad \Rightarrow \quad \Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$$

$$Q_{\lambda\mu\nu} = A_{\lambda}(x) g_{\mu\nu}(x)$$

$$\Gamma^{\lambda}_{\mu\nu} = C^{\lambda}_{\mu\nu} + W^{\lambda}_{\mu\nu}$$

$$W^{\lambda}_{\mu\nu} = -\frac{1}{2}(A_{\mu}\delta^{\lambda}_{\nu} + A_{\nu}\delta^{\lambda}_{\mu} - A^{\lambda} g_{\mu\nu})$$

$A_{\lambda}(x)$ — "Weyl vector"

Local conformal transformation

$$ds^2 = \Omega^2(x) d\hat{s}^2 = \Omega^2 \hat{g}_{\mu\nu}(x) dx^\mu dx^\nu$$

$$g_{\mu\nu} = \Omega^2 \hat{g}_{\mu\nu}, \quad g^{\mu\nu} = \frac{1}{\Omega^2} \hat{g}^{\mu\nu}$$

$$\sqrt{-g} = \Omega^4(x) \sqrt{-\hat{g}}$$

$$C^\lambda_{\mu\nu} = \hat{C}^\lambda_{\mu\nu} + \left(\frac{\Omega_{,\nu}}{\Omega} \delta^\lambda_\mu + \frac{\Omega_{,\mu}}{\Omega} \delta^\lambda_\nu - g^{\lambda\varkappa} \frac{\Omega_{,\varkappa}}{\Omega} \hat{g}_{\mu\nu} \right)$$

If $A_\mu =$ gauge field

$$A_\mu = \hat{A}_\mu + 2 \frac{\Omega_{,\mu}}{\Omega} \quad \Rightarrow$$

$$\Gamma^\lambda_{\mu\nu} = \hat{\Gamma}^\lambda_{\mu\nu} \quad \Rightarrow$$

$$R^\mu{}_{\nu\lambda\sigma} = \hat{R}^\mu{}_{\nu\lambda\sigma}, \quad R_{\mu\nu} = \hat{R}_{\mu\nu}$$

Strength tensor

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu}$$

Weyl gravity

$$\mathcal{L}_{\mathrm{W}} = \alpha_1 R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R^2 + \alpha_4 F_{\mu\nu} F^{\mu\nu}$$

$$S_{\mathrm{W}} = \int \mathcal{L}_{\mathrm{W}} \sqrt{-g}\, d^4x, \quad \frac{\delta S_{\mathrm{W}}}{\delta \Omega} = 0$$

Dynamical variables: $g_{\mu\nu}(x)$, $A_\mu x$

Matter fields $S_{\mathrm{tot}} = S_{\mathrm{W}} + S_{\mathrm{m}}$, $S_{\mathrm{m}} = \int \mathcal{L}_{\mathrm{m}} \sqrt{-g}\, d^4x$
S_{m} is not necessary conformal invariant
δS_{m} does!

$$\delta S_{\mathrm{m}} \stackrel{\mathrm{def}}{=} -\frac{1}{2} \int T^{\mu\nu} (\delta g_{\mu\nu}) \sqrt{-g}\, d^4x - \int G^\mu (\delta A_\mu) \sqrt{-g}\, d^4x$$

$$+ \int \frac{\delta \mathcal{L}_{\mathrm{m}}}{\delta \Psi} (\delta \Psi) \sqrt{-g}\, d^4x = 0$$

Ψ — collective dynamical variable for matter fields, $\delta \mathcal{L}_{\mathrm{m}} / \delta \Psi = 0$
$T^{\mu\nu}$ — energy-momentum tensor, G^μ — “Weyl current”

Weyl gravity II

$$\delta g_{\mu\nu} = \frac{2}{\Omega} g_{\mu\nu}(\delta\Omega) + \Omega^2(\delta\hat{g}_{\mu\nu})$$

$$\delta A_\mu = (\delta A_\mu) + 2(\delta(\log\Omega)_{,\mu}$$

$$\delta\hat{g}_{\mu\nu}: \quad \frac{\delta S_{\rm m}}{\delta\hat{g}_{\mu\nu}} \stackrel{\rm def}{=} -\frac{1}{2}\hat{T}_{\mu\nu} \quad \Rightarrow \quad \hat{T}_{\mu\nu} = \Omega^2 T_{\mu\nu}$$

$$(\hat{T}^\nu_\mu = \Omega^4 T^\nu_\mu, \ \hat{T}^{\mu\nu} = \Omega^6 T_{\mu\nu})$$

$$\delta\hat{A}_\mu: \quad \frac{\delta S_{\rm m}}{\delta\hat{A}_\mu} \stackrel{\rm def}{=} -\hat{G}^\mu \quad \Rightarrow \quad \hat{G}^\mu = \Omega^4 G^\mu$$

Self-consistency condition

$$\delta\Omega: \quad 2G^\mu_{;\mu} = Trace[T^{\mu\nu}]$$

";" — covariant derivative with $C^\lambda_{\mu\nu}$

Cosmology

Cosmology $=$ homogeneity and isotropy
Robertson-Walker metric

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right), \quad k = 0, \pm 1$$

$$A_\mu = (A_0(t), 0, 0, 0) \quad \Rightarrow$$

$$\boxed{F_{\mu\nu} \equiv 0}$$

$$T^\mu_\nu = (\,T^0_0(t), T^1_1(t) = T^2_2 = T^3_3)$$

Special gauge

$$A_0(t) = 0$$

Field equations

$$\delta A_\mu : \quad -6\gamma\dot{R} = G^0, \quad \gamma = \frac{1}{3}(\alpha_1+\alpha_2+3\alpha_3)$$

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2}\right)$$

$$\delta g_{\mu\nu} : \quad -12\gamma\left\{\frac{\dot{a}}{a}\dot{R} + R\left(\frac{R}{12} + \frac{\dot{a}^2+k}{a^2}\right)\right\} = T^0_0$$

$$-4\gamma\left\{\ddot{R} + 2\frac{\dot{a}}{a}\dot{R} - R\left(\frac{R}{12} + \frac{\dot{a}^2+k}{a^2}\right)\right\} = T^1_1$$

Self-consistency condition:

$$2\frac{(G^0a^3)^{\cdot}}{a^3} = T^0_0 + 3\,T^1_1$$

Vacuum solutions

$$G^0 = 0 \quad \Rightarrow \quad \dot{R} = 0 \quad \Rightarrow \quad R = R_0 \quad \Rightarrow$$

$$a^2 + k = -\frac{R_0}{12}a^2 + \frac{Q_0}{a^2} \quad \Rightarrow$$

$$R_0 Q_0 = 0$$

1. $Q_0 = 0$
   $R_0 < 0 \quad \Rightarrow \quad$ de Sitter
   $R_0 > 0 \quad \Rightarrow \quad$ AdS
   $R_0 = 0 \quad \Rightarrow \quad$ Milne universe
2. $Q_0 \neq 0 \quad R_0 = 0$
   $4Q_0(t - t_0)^2 = a^2, \quad k = 0 \quad (Q_0 > 0)$
   $a^2 + (t - t_0)^2 = Q_0, \quad k = \pm 1 \quad (Q_0 > 0)$
   $(t - t_0)^2 - a^2, \quad k = -1$

New possibilities

Single particle u^μ

Riemann geometry

$$S_{\rm part} = -m\int ds$$

Weyl geometry — new invariant $B = A_\mu u^\mu$

$$S_{\rm part} = \int f_1(B)ds + \int f_2(B)d\tau = \int \{ f_1(B)\sqrt{g_{\mu\nu}u^\mu u^\nu} + f_2(B)\}d\tau$$

Equations of motion

$$\begin{aligned} f_1(B)u_{\lambda;\mu}u^\mu &= \left((f_1^{'}(B)+f_2^{''}(B))A_\lambda - f_1^{'}(B)u_\lambda\right)B_{,\mu}u^\mu \\ &+ (f_1^{'}(B)+f_2^{'}(B))F_{\lambda\mu}u^\mu \end{aligned}$$

Since $F_{\lambda\mu}u^\lambda u^\mu \equiv 0$ and $u_{\lambda;\sigma}u^\lambda \equiv 0$

Either $(f_1^{'}+f_2^{''})B = f_1^{'}$ or $B_{,\mu}u^\mu = 0$

Independent observer

Conformal invariance

Main problem — gauge fixing

Independent observer

Independent $=$ not a dynamical variable

Simplest choice $\Rightarrow$

$$S_{\mathrm{obs}} = \int A_\mu \xi^\mu \sqrt{-g}\, d^4x,$$

ξ^μ — proportional to the four-velocity

$$T^{\mu\nu}_{\mathrm{obs}} = -(A_\lambda \xi^\lambda) g^{\mu\nu}$$

Cosmology: it singles out the gauge $A_\mu = 0$

Vacuum cosmological solutions

$A_\mu = 0$, $G^\mu = 0$, $T^{\mu\nu} = 0$

Field equations

$$-6\gamma \dot{R} = \xi^0$$

$$-12\gamma\left\{\frac{\dot{a}}{a}\dot{R} + R\left(\frac{R}{12} + \frac{\dot{a}^2 + k}{a^2}\right)\right\} = 0$$

$$-4\gamma\left\{\ddot{R} + 2\frac{\dot{a}}{a}\dot{R} - R\left(\frac{R}{12} + \frac{\dot{a}^2 + k}{a^2}\right)\right\} = 0$$

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + k}{a^2}\right)$$

Self-consistency condition:

$$2\frac{(G^0 a^3)^{\dot{}}}{a^3} = 0 \quad \Rightarrow \quad \dot{R} = \frac{b}{a^3}, \qquad b = const$$

Vacuum cosmological solutions II

$A_\mu = 0,\ G^\mu = 0,\ T^{\mu\nu} = 0$

$$\boxed{\ddot{a}^2 a^2 = (\dot{a}^2 + k)^2 - \frac{1}{3}\dot{a} b}$$

$\dot{a} > 0,\ a \gg a_0 \quad \Rightarrow \quad \dot{a} \propto a \quad \Rightarrow \quad$ exponential growth!

The End