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Stationary solutions of the Vlasov-Poisson kinetic equations on torus

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Introduction

The Vlasov-Poisson system of equations regarding to density distribution functions of charged particles and electric potential describes the kinetics of high-temperature plasma in a fusion reactor. If a considerable part of particles reaches the boundary, this can lead either to destruction of the reactor, or to cooling the plasma due to its contact with the reactor wall. Therefore it is necessary to provide plasma confinement at some distance from the vacuum container wall. In most models of fusion reactors an external magnetic field is used as a control ensuring plasma confinement. From the point of view of differential equations this means that one has to prove the existence of solutions of the Vlasov-Poisson system with external magnetic field for which the supports of density distribution functions do not intersect with the boundary.

In this lecture we consider the boundary value problem for the Vlasov-Poisson system for two-component high-temperature plasma with external magnetic field.

Introduction

$$-\Delta\varphi(x,t) = 4\pi \sum_{\beta=\pm} q_\beta \int_{\mathbb{R}^3} f^\beta(x,v,t)\,dv$$

$$(x \in Q, 0 < t < T), \quad (1)$$

$$\partial_t f^\beta + \left(v, \nabla_x f^\beta\right) + \frac{q_\beta}{m_\beta}\left(-\nabla_x\varphi + \frac{v \times B}{c}, \nabla_v f^\beta\right) = 0$$

$$(x \in Q, v \in \mathbb{R}^3, 0 < t < T, \beta = \pm 1), \quad (2)$$

$$\varphi(x,t) = 0, \quad (x \in \partial Q, 0 \le t < T). \quad (3)$$

Introduction

System (1)–(3) describes the evolution of distribution functions of the density for the charged particles in a rarefied plasma. Here
– $f^{\beta}(x, v, t)$ is an unknown distribution function (for positively charged ions if $\beta = +1$ and for electrons if $\beta = -1$) at the point x with velosity v and at the moment t,
– $\varphi(x, t)$ is an unknown potential of electric field,
– ∇_x and ∇_v are the gradients with respect to x and v,
– m_{+1} and m_{-1} are the masses of ion and electron, respectively,
– q_- is the charge of electron,
– q_+ is the charge of ion,
– c is the velocity of light,
– $B = B(x)$ is the strength of external magnetic field,
– $(\cdot, \cdot)$ is the scalar product in $\mathbb{R}^3$,
– $v \times B$ is the vector product in $\mathbb{R}^3$,
– $Q \subset \mathbb{R}^3$ is a domain with the boundary $\partial Q \in C^\infty$.

Fusion reactors

Mathematical model of a fusion reactor is described by the boundary value problems (1)–(3) for Vlasov equations in domains with boundary regarding to density distribution functions for particles of both signs and electric potential.

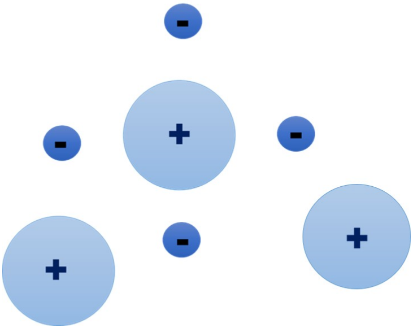

Fig 1.

Fusion reactors

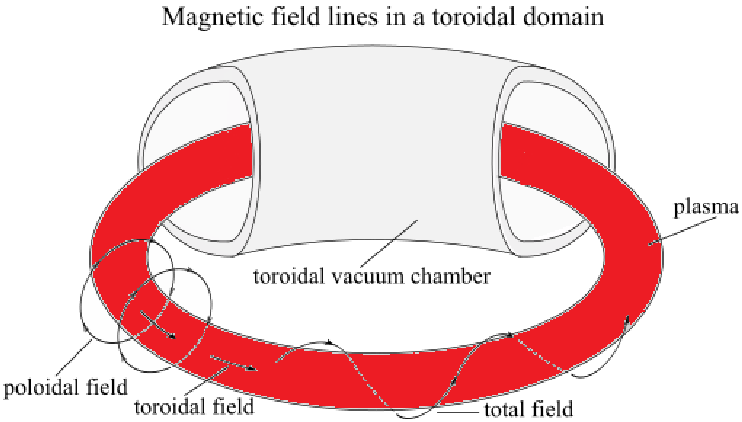

Fig 2. Tokamak scheme

Fusion reactors

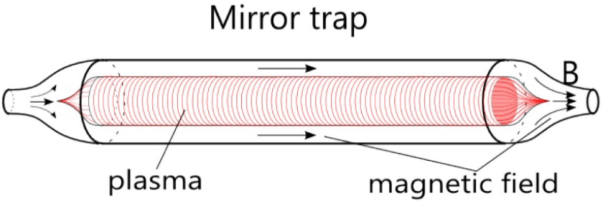

Fig 3.

Thermonuclear reactor

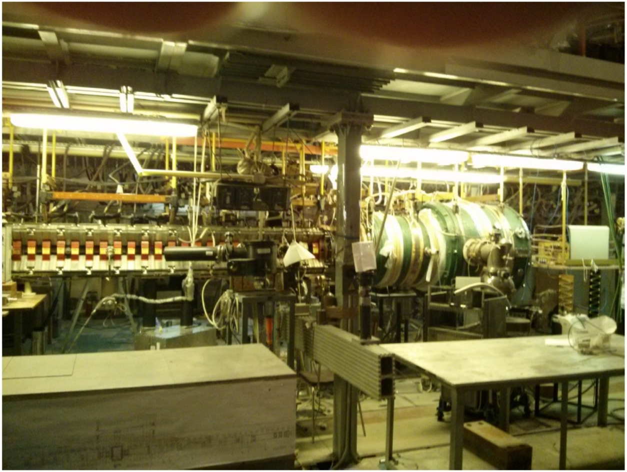

Fig 4.

Fusion reactors

In order to obtain a high-temperature plasma in reactor, it is necessary to hold plasma strictly inside domain during some time. Usually creation of control system providing existence of plasma in reactor is based on usability of external magnetic field.
In the terms of differential equations this means that we must guarantee existence of solutions to the Vlasov–Poisson equations containing external magnetic field with compact supports inside domain. If sufficiently many particles reach a boundary, then either the wall of reactor will be destroyed, or plasma will get cold due to its cooling on the wall of reactor.

Stationary Symmetric Models for Tokamak

In this section we formulate the main result of this lecture devoted to existence of stationary solution of Vlasov–Poisson system with external magnetic field in the case of Tokamak.
The group $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ acts isometrically on $\mathbb{R}^3$ via rotations around the x_3-axis. For $\theta \in S^1$ and $x \in \mathbb{R}^3$, this action is denoted by

$$\theta * x = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Let now $Q \subset \mathbb{R}^3$ be a smooth bounded domain which is invariant under the S^1-action and which does not contain a point of the form $(0, 0, x_3)$ in its closure.
For $x_0 \in \mathbb{R}^3$, $\delta > 0$, we define the toroidal neighborhoods

$$\mathcal{O}_\delta(x_0) := \left\{ x \in \mathbb{R}^3 : \mathrm{dist}\left(x, S^1 * x_0\right) < \delta \right\} := S^1 * B_\delta(x_0). \quad (4)$$

Stationary Symmetric Models for Tokamak

Definition 1. A function $f : \overline{Q} \times \mathbb{R}^3 \to \mathbb{R}$, $\varphi : \overline{Q} \to \mathbb{R}$ resp., is said to be S^1-invariant if $f(\theta * x, \theta * v) = f(x, v)$, $\varphi(\theta * x) = \varphi(x)$ resp., for all $\theta \in S^1$, $x \in \overline{Q}$, $v \in \mathbb{R}^3$.

Definition 2. A vector field $B : \overline{Q} \to \mathbb{R}^3$ is S^1-equivariant if $B(\theta * x) = \theta * B(x)$ for all $\theta \in S^1$, $x \in \overline{Q}$.

For arbitrary fixed $x \in \overline{Q}$, let $P_x : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the plane spanned by the vectors $(x_1, x_2, 0)$ and $(0, 0, 1)$. We use this projection to decompose a magnetic field $B : \overline{Q} \to \mathbb{R}^3$ into its poloidal part $P_x B(x)$ and its toroidal part $(I - P_x)B(x)$, where $I : \mathbb{R}^3 \to \mathbb{R}^3$ is the identical operator.

Stationary Symmetric Models for Tokamak

For $x_0 \in Q$, we set $r_0 := \sqrt{x_{0,1}^2 + x_{0,2}^2} > 0$ and $z_0 := x_{0,3}$. For the confinement of the spatial supports of f^+ and f^-, we use the magnetic field $B^{x_0} : \overline{Q} \to \mathbb{R}^3$ given by

$$B^{x_0}(x) := \frac{1}{x_1^2 + x_2^2} \begin{pmatrix} x_1(x_3 - z_0) \\ x_2(x_3 - z_0) \\ -\sqrt{x_1^2 + x_2^2}\left(\sqrt{x_1^2 + x_2^2} - r_0\right) \end{pmatrix}. \qquad (5)$$

Note that $P_x B^{x_0}(x) = B^{x_0}(x)$, i.e. B^{x_0} is a poloidal field. Clearly $\mathrm{div} B^{x_0}(x) = 0$ $(x \in Q)$.

Definition 3. A triple (f^+, f^-, φ) of time independent functions $f^\pm \in \mathcal{C}^1(\overline{Q} \times \mathbb{R}^3)$, $f^\pm \geq 0$, $\varphi \in \mathcal{C}^2(\overline{Q})$ with $\int_{\mathbb{R}^3} f^\pm(\cdot, v)\, dv \in \mathcal{C}^0(\overline{Q})$ such that these functions satisfy the equations (2) (with $\partial_t f^\beta = 0$), $\beta = \pm 1$, and the boundary value problem (1), (3), is called a stationary solution of Vlasov–Poisson system (1), (2) with boundary condition (3).

The total charge of the βth component of a stationary solution (f^+, f^-, φ) is defined by $\mathcal{Q}^\beta := q_\beta \|f^\beta\|_{L^1(Q\times\mathbb{R}^3)}$.

Theorem 1. Let $x_0 \in Q$ and $B = B_b : \overline{Q} \to \mathbb{R}^3$ be a S^1-equivariant magnetic field with poloidal part $P_x B_b(x) = bB^{x_0}(x)$, where $b > 0$ is a parameter.

(i) Let $b > 0$ be fixed. Then, for any numbers $0 < \delta_\pm < \mathrm{dist}(x_0, \partial Q)$, $\varepsilon_\pm > 0$, and $c > 0$, there exists a S^1-invariant stationary solution (f^+, f^-, φ) of the Vlasov-Poisson system (1), (2) with boundary condition (3) such that $\mathrm{supp}\, f^\pm \subset \mathcal{O}_{\delta_\pm}(x_0) \times B_{\varepsilon_\pm}(0)$ and $\mathcal{Q}^+ = c\,|\mathcal{Q}^-| > 0$.

(ii) Let (f^+, f^-, φ) be a solution from (i) associated with the parameter values $b, \delta_\pm, \varepsilon_\pm$, and c. Then for any $\lambda \in (0, \infty)$ the Vlasov-Poisson system (1), (2), (3) considered with magnetic field $B = B_{\lambda b}$ has a stationary solution $(f_\lambda^+, f_\lambda^-, \varphi_\lambda)$ with $\mathrm{supp}\, f_\lambda^\pm \subset \mathcal{O}_{\delta_\pm}(x_0) \times B_{\lambda\varepsilon_\pm}(0)$ and total charges $\mathcal{Q}_\lambda^\pm = \lambda^2 \mathcal{Q}^\pm$.

Corollary 1. For any $\delta_\pm \in (0, \mathrm{dist}(x_0, \partial Q))$, $c_+ > 0$, and $c_- < 0$, there exists $b > 0$ such that the problem (1), (2), (3) with the magnetic field B_b has a stationary solution (f^+, f^-, φ) with $\mathrm{supp}\, f^\pm \subset\subset \mathcal{O}_{\delta_\pm}(x_0) \times \mathbb{R}^3$ and $\mathcal{Q}^\pm = c_\pm$.

From Theorem 1 it follows that the toroidal part of the magnetic field does not play any role for the existence of confined stationary solutions. Moreover, due to the drift of charged particles a purely toroidal field can not be used to guarantee the existence of stationary solutions with spatial supports strictly inside Q. In some sense toroidal magnetic field in tokamak can be considered as analogue of homogeneous magnetic field directed along the axis of mirror trap. However in contrast to the case of tokamak, homogeneous magnetic field in mirror trap provides existence of stationary solutions with spatial supports of $f^{\pm}$ strictly contained in Q, see [12]. From physical point of view "the poloidal field does most of the work in tokamak confinement. The toroidal field enhances stability, as well as improving thermal insulation", see [5, Section 1.5].

Part (i) of Theorem 1 shows that the strength of the external magnetic field, which corresponds to the value of the parameter $b > 0$, is not important if one is only interested in the existence of some stationary solutions with supports in a prescribed region. If one want to confine a given amount of plasma, measured in terms of the total charges $\mathcal{Q}^{\pm}$, a sufficiently strong magnetic field becomes crucial.

Note also that in particular if $\mathcal{Q}^{+} \neq |\mathcal{Q}^{-}|$, the electric potential φ is non-trivial. Moreover, the results extend to the extreme cases $\mathcal{Q}^{+} = 0$, $\mathcal{Q}^{-} < 0$ and $\mathcal{Q}^{+} > 0$, $\mathcal{Q}^{-} = 0$, i.e. we also can find confined stationary solutions for the two different one-component systems modelling a plasma consisting only of ions or electrons.

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Thank You