Transformation of the action under the Legendre transform by an arbitrary parameter and membrane dust quantization

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Initial action

We consider an arbitrary action without higher derivatives

$$S[\varphi^a(X^M)] = \int L(X^M, \varphi^a, \partial_M \varphi^a)\, d^D X. \tag{1}$$

Here $\varphi^a(X^M) = \big(\varphi^1(X^1, \dots, X^D), \dots, \varphi^n(X^1, \dots, X^D)\big)$, etc.

$$\frac{\delta S}{\delta \varphi^a} = \frac{\partial L}{\partial \varphi^a} - \partial_M \frac{\partial L}{\partial(\partial_M \varphi^a)} \tag{2}$$

Let Lagrangian is represented in the following form Пусть лагранжиан представлен в виде

$$L(X^M, \varphi^a, \partial_M \varphi^a) = L_f(X^M, \varphi^a, \partial_M \varphi^a, f^i(X^M, \varphi^a, \partial_M \varphi^a)), \tag{3}$$

where f^i are arbibrary smooth C_∞ functios. The representation (3) is not unique even if functions L and f^i are fixed. We utilize this ambiguity considering the Nambu-Goto string action and the action for co-sting duct.

Introducing of Lagrange multipliers

Now F^i are independent fields, $F^i = f^i(X^M, \varphi^a, \partial_M \varphi^a)$ are constraints, G_i are Lagrange multipliers.

$$S_F[\varphi^a(X^M), F^i(X^M), G_i(X^M)] = \tag{4}$$
$$= \int \left\{ L_f(X^M, \varphi^a, \partial_M \varphi^a, F^i) + G_i[f^i(X^M, \varphi^a, \partial_M \varphi^a) - F^i] \right\} d^D X.$$

New field equations are equivlent to (2) if one exclude F^i and G_i

$$\frac{\delta S_F}{\delta G_i} = f^i(X^M, \varphi^a, \partial_M \varphi^a) - F^i = 0, \tag{5}$$
$$\frac{\delta S_F}{\delta F^i} = \frac{\partial L_f}{\partial F^i} - G_i = 0, \tag{6}$$
$$\frac{\delta S_F}{\delta \varphi^a} = \frac{\partial L_f}{\partial \varphi^a} + G_i \frac{\partial f^i}{\partial \varphi^a} - \partial_M \frac{\partial L_f}{\partial(\partial_M \varphi^a)} - \partial_M \left(G_i \frac{\partial f^i}{\partial(\partial_M \varphi^a)} \right) = 0. \tag{7}$$

The equations (5) and (6) are of first order. The equations (7) are of first or second order. If (sufficient condition) L_f and f^i are linear in $\partial_M \varphi^a$, then (7) are of first order too.

Exclusion of auxiliary fields F^i (case 1)

According to (6)

$$G_i = \frac{\partial L_F(X^M, \varphi^a, \partial_M \varphi^a, F^i)}{\partial F^i}. \tag{8}$$

If $\det \frac{\partial G_i}{\partial F_j} = \det \frac{\partial^2 L_F}{\partial F^i \partial F^j} \neq 0$, then

$$F^i = F^i(X^M, \varphi^a, \partial_M \varphi^a, G_j). \tag{9}$$

Excluding F^i from the action (4) by (9), one get a new action

$$S_G[\varphi^a(X^M), G_i(X^M)] = \int \Big\{ L_f(X^M, \varphi^a, \partial_M \varphi^a, F^i(X^M, \varphi^a, \partial_M \varphi^a, G_j)) + \\ + G_i[f^i(X^M, \varphi^a, \partial_M \varphi^a) - F^i(X^M, \varphi^a, \partial_M \varphi^a, G_j)] \Big\} d^D X. \tag{10}$$

We write the action (10) using the Legendre transform with respect to F^i

$$\mathcal{H}(X^M, \varphi^a, \partial_M \varphi^a, G_i) = G_i F^i(X, \varphi, \partial\varphi, G) - L_F(X, \varphi, \partial\varphi, F(X, \varphi, \partial\varphi, G)).$$

$$\frac{\partial \mathcal{H}}{\partial G_i} = F^i(X, \varphi, \partial\varphi, G) + G_j \frac{\partial F_j}{\partial G_i} - \underbrace{\frac{\partial L_F}{\partial F^j}}_{G_j} \frac{\partial F_j}{\partial G_i} = F^i(X, \varphi, \partial\varphi, G).$$

Legendre transform is the method to rewrite an action

$$G_i = \frac{\partial L_F(X^M, \varphi^a, \partial_M \varphi^a, F^i)}{\partial F^i}.$$

$$\mathcal{H}(X^M, \varphi^a, \partial_M \varphi^a, G_i) = G_i F^i(X, \varphi, \partial\varphi, G) - L_F(X, \varphi, \partial\varphi, F(X, \varphi, \partial\varphi, G)). \tag{11}$$

Using (11) one can rewrite the action (10) as

$$S_G[\varphi^a(X^M), G_i(X^M)] = \int \left[G_i f^i(X^M, \varphi^a, \partial_M \varphi^a) - \mathcal{H}(X^M, \varphi^a, \partial_M \varphi^a, G_i)\right] d^D X. \tag{12}$$

$$\frac{\delta S_G}{\delta G_i} = f^i(X^M, \varphi^a, \partial_M \varphi^a) - \frac{\partial \mathcal{H}}{\partial G_i} = f^i(X^M, \varphi^a, \partial_M \varphi^a) - F^i(X^M, \varphi^a, \partial_M \varphi^a, G_j). \tag{13}$$

$$\frac{\delta S_G}{\delta \varphi^a} = G_i \frac{\partial f^i}{\partial \varphi^a} - \frac{\partial \mathcal{H}}{\partial \varphi^a} - \partial_M \left[G_i \frac{\partial f^i}{\partial(\partial_M \varphi^a)} - \frac{\partial \mathcal{H}}{\partial(\partial_M \varphi^a)}\right]. \tag{14}$$

New field equations are equivlent to (2) if one exclude G_i.
Legendre transformations, if the order of the dynamics equations decreases from 2 to 1, can be considered a generalization of the transition to the Hamiltonian formalism.

Additional reparametrization

One can locally reparametrize the fields G_i using an arbitrary non-degenerate variable transformation

$$G_i(h_k), \qquad \det \frac{\partial G_i}{\partial h_k} \neq 0.$$

Now the action (12) has the form

$$S_h[\varphi^a(X), h_k(X)] = \int \left[G_i(h)\, f^i(X, \varphi, \partial\varphi) - \mathcal{H}(X, \varphi, \partial\varphi, G(h)) \right] \, d^D X \tag{15}$$

Exclusion of auxiliary fields F^i (case 2)

Let L_F linear in F^i, then the equation does not involve F^i.

$$\frac{\delta S_F}{\delta F^i} = \frac{\partial L_f}{\partial F^i} - G_i = 0,$$

Let the equation be reduced to the form

$$f^i(X^M, \varphi^a, \partial_M \varphi^a) = \tilde{F}^i(X^M, \varphi^a, \partial_M \varphi^a, G_j), \tag{16}$$

using (5) we get

$$F^i = \tilde{F}^i(X^M, \varphi^a, \partial_M \varphi^a, G_j). \tag{17}$$

We use (17) to exclude fields F^i from the action (4) despite of $\frac{\partial^2 L_F}{\partial F^i \partial F^j} = 0$.

$$S_{\tilde{G}}[\varphi^a(X^M), G_i(X^M)] = \int \Big\{ L_F(X^M, \varphi^a, \partial_M \varphi^a, \tilde{F}^i(X^M, \varphi^a, \partial_M \varphi^a, G_j)) + \\ + G_i[f^i(X^M, \varphi^a, \partial_M \varphi^a) - \tilde{F}^i(X^M, \varphi^a, \partial_M \varphi^a, G_j)] \Big\} \, d^D X. \tag{18}$$

Legendre transform is the method to rewrite an action

$$\tilde{\mathcal{H}}(X^M, \varphi^a, \partial_M \varphi^a, G_i) = G_i \tilde{F}^i(X, \varphi, \partial\varphi, G) - L_F(X, \varphi, \partial\varphi, \tilde{F}(X, \varphi, \partial\varphi, G)).$$

$$\frac{\partial \mathcal{H}}{\partial G_i} = \tilde{F}^i(X, \varphi, \partial\varphi, G) + G_j \frac{\partial \tilde{F}_j}{\partial G_i} - \underbrace{\frac{\partial L_F}{\partial F^j}}_{G_j} \frac{\partial \tilde{F}_j}{\partial G_i} = \tilde{F}^i(X, \varphi, \partial\varphi, G).$$

The action (18) transforms to the form

$$S_{\tilde{G}}[\varphi^a(X^M), G_i(X^M)] = \int \left[G_i f^i(X^M, \varphi^a, \partial_M \varphi^a) - \tilde{\mathcal{H}}(X^M, \varphi^a, \partial_M \varphi^a, G_i) \right] d^D X. \tag{19}$$

$$\frac{\delta S_{\tilde{G}}}{\delta G_i} = f^i(X^M, \varphi^a, \partial_M \varphi^a) - \frac{\partial \tilde{\mathcal{H}}}{\partial G_i} = f^i(X^M, \varphi^a, \partial_M \varphi^a) - \tilde{F}^i(X^M, \varphi^a, \partial_M \varphi^a, G_j).$$

$$\frac{\delta S_{\tilde{G}}}{\delta \varphi^a} = G_i \frac{\partial f^i}{\partial \varphi^a} - \frac{\partial \tilde{\mathcal{H}}}{\partial \varphi^a} - \partial_M \left[G_i \frac{\partial f^i}{\partial(\partial_M \varphi^a)} - \frac{\partial \tilde{\mathcal{H}}}{\partial(\partial_M \varphi^a)} \right].$$

New field equations are equivlent to (2) if one exclude G_i.

Example

$$S[\varphi(x)] = -\frac{1}{2}\int (\partial_M \varphi)(\partial^M \varphi)\, d^D x.$$

$$\frac{\delta S}{\delta \varphi} = \partial_M \partial^M \varphi$$

Let $F^M = \partial^M \varphi$. Consider the case 1 and case 2.

$$L_{F1} = -\frac{1}{2} F^M F_M, \qquad L_{F2} = -\frac{1}{2} F^M \partial_M \varphi.$$

Example (case 1)

$$S_{F1}[\varphi(x), F^M(x), G_M(x)] = \int \left[-\frac{1}{2} F^M F_M + G_M(\partial^M \varphi - F^M) \right] d^D x.$$

$$\frac{\delta S_{F1}}{\delta G_M} = \partial^M \varphi - F^M, \quad \frac{\delta S_{F1}}{\delta F^M} = -F_M - G_M, \quad \frac{\delta S_{F1}}{\delta \varphi} = -\partial_M G^M.$$

$$\begin{aligned} S_{G1}[\varphi(x), G_M(x)] &= \int \left[-\frac{1}{2} G^M G_M + G_M(\partial^M \varphi + G^M) \right] d^D x = \\ &= \int \Big[\underbrace{\frac{1}{2} G^M G_M}_{-\mathcal{H}} + G_M \partial^M \varphi \Big] d^D x. \end{aligned} \tag{20}$$

$$\frac{\delta S_{G1}}{\delta G_M} = G^M + \partial^M \varphi, \qquad \frac{\delta S_{G1}}{\delta \varphi} = -\partial_M G^M.$$

Example (case 2)

$$S_{F2}[\varphi(x), F^M(x), G_M(x)] = \int \left[-\frac{1}{2} F^M \partial_M \varphi + G_M(\partial^M \varphi - F^M) \right] d^D x.$$

$$\frac{\delta S_{F2}}{\delta G_M} = \partial^M \varphi - F^M, \quad \frac{\delta S_{F2}}{\delta F^M} = -\frac{1}{2}\partial_M \varphi - G_M, \quad \frac{\delta S_{F2}}{\delta \varphi} = -\frac{1}{2}\partial_M F^M - \partial_M G^M.$$

Exclude F^M using $F^M = -2G^M$

$$\begin{aligned} S_{G2}[\varphi(x), G_M(x)] &= \int \Big[G^M \partial_M \varphi + \underbrace{G_M(\partial^M \varphi + 2G^M)}_{-\tilde{\mathcal{H}}} \Big] d^D x = \\ &= \int \left[2G_M G^M + 2G^M \partial_M \varphi \right] d^D x. \qquad (21) \end{aligned}$$

$$\frac{\delta S_{G2}}{\delta G_M} = 4G^M + 2\partial^M \varphi, \qquad \frac{\delta S_{G2}}{\delta \varphi} = -2\partial_M G^M.$$

The actions (20) and (21) coinside up to field reparametrizaton $G_M \to 2G_M$.

Dirac action for membrane $\to$ Polykov-type action ($\mathsf{d} \neq 2$)

$$S_x[x^M(\xi)] = -\int_{\mathbf{V}} d^{\mathsf{d}}\xi \sqrt{|\gamma|}, \tag{22}$$

$$\gamma = \det \gamma_{ij}, \qquad \gamma_{ij}(x^M, \partial_i x^M) = g_{MN}(x(\xi)) \frac{\partial x^M}{\partial \xi^i} \frac{\partial x^N}{\partial \xi^j}. \tag{23}$$

$$H^{ij} = \frac{\partial L}{\partial \gamma_{ij}} = \frac{1}{2}\sqrt{|\gamma|}\gamma^{ij}, \quad \gamma^{ij}\gamma_{jk} = \delta^i_k, \quad \det H^{ij} = \frac{\mathrm{sgn}\gamma}{2^{\mathsf{d}}} |\gamma|^{\frac{\mathsf{d}-2}{2}}.$$

$$\mathcal{H}(x^M, \partial_i x^M, H^{ij}) = \tfrac{\mathsf{d}-2}{2} \sqrt[\mathsf{d}-2]{\mathrm{sgn}\gamma\, 2^{\mathsf{d}} \det H^{ij}}. \tag{24}$$

$$S_H[x, H] = -\int_{\mathbf{V}} \left[H^{ij}\gamma_{ij}(x, \partial x) - \tfrac{\mathsf{d}-2}{2} \sqrt[\mathsf{d}-2]{\mathrm{sgn}\gamma\, 2^{\mathsf{d}} \det H^{ij}}\right] d^{\mathsf{d}}\xi \tag{25}$$

$$H^{ij} = \frac{1}{2}\sqrt{|\det h_{ij}|}\, h^{ij}, \quad h^{ij}h_{jk} = \delta^i_k, \quad \mathcal{H}(x^M, \partial_i x^M, h_{ij}) = \tfrac{\mathsf{d}-2}{2}\sqrt{|h|}.$$

H^{ij} is **not tensor**, h_{ij} is **tensor**.

$$S_h[x^M(\xi), h_{ij}(\xi)] = -\int_{\mathbf{V}} \left[\tfrac{1}{2} h^{ij} g_{MN} \frac{\partial x^M}{\partial \xi^i}\frac{\partial x^N}{\partial \xi^j} - \tfrac{\mathsf{d}-2}{2}\right]\sqrt{|h|}\, d^{\mathsf{d}}\xi \tag{26}$$

Nambu-Goto action $\to$ Polykov action ($\mathrm{d} = 2$ and $\forall \mathrm{d} \in \mathbb{N}$)

$$S_x[x^M(\xi)] = -\int\limits_{\mathbf{V}} d^{\mathrm{d}}\xi \sqrt{|\gamma|},$$

$$L(\Gamma_{ij}, \gamma_{ij}) = |\Gamma|^{\varepsilon} \cdot |\gamma|^{\frac{1}{2}-\varepsilon}, \qquad \Gamma = \det \Gamma_{ij}(x, \partial x), \quad \gamma = \det \gamma_{ij}(x, \partial x)$$

$$\Gamma_{ij}(x^M, \partial_i x^M) = \gamma_{ij}(x^M, \partial_i x^M) = g_{MN} \frac{\partial x^M}{\partial \xi^i} \frac{\partial x^N}{\partial \xi^j},$$

$$S_H[x, H] = -\int\limits_{\mathbf{V}} \left\{ H^{ij}\gamma_{ij} - \left[(\tfrac{1}{2} - \varepsilon)\mathrm{d} - 1\right] |\Gamma|^{\varepsilon} \left[\frac{\mathrm{sgn}\gamma \ \det \frac{H^{ij}}{|\Gamma|^{\varepsilon}}}{(\frac{1}{2} - \varepsilon)^{\mathrm{d}}} \right]^{\frac{1}{\mathrm{d} - (\frac{1}{2} - \varepsilon)^{-1}}} \right\} d^{\mathrm{d}}\xi.$$

$$\frac{H^{ij}}{|\Gamma|^{\varepsilon}} = (\tfrac{1}{2} - \varepsilon) \cdot |\det h_{ij}|^{\frac{1}{2}-\varepsilon} h^{ij}$$

$$S_h[x, h] = -\int\limits_{\mathbf{V}} \frac{|\Gamma|^{\varepsilon}}{|h|^{\varepsilon}} \left\{ (\tfrac{1}{2} - \varepsilon) h^{ij} g_{MN} \frac{\partial x^M}{\partial \xi^i} \frac{\partial x^N}{\partial \xi^j} - \left[\tfrac{\mathrm{d}-2}{2} - \varepsilon \mathrm{d} \right] \right\} \sqrt{|h|}\, d^{\mathrm{d}}\xi.$$

$$\mathrm{d} = 2, \ \varepsilon \to 0: \quad S_{\text{N-G}}[x^M(\xi), h_{ij}(\xi)] = -\int\limits_{\mathbf{V}} \tfrac{1}{2} h^{ij} g_{MN} \frac{\partial x^M}{\partial \xi^i} \frac{\partial x^N}{\partial \xi^j} \sqrt{|h|}\, d^{\mathrm{d}}\xi.$$

Polykov action $\forall \mathrm{d} \in \mathbb{N}$

$$S_h[x^M(\xi), h_{ij}(\xi)] = -\int\limits_{\mathbf{V}} \left[\tfrac{1}{2} h^{ij} g_{MN} \frac{\partial x^M}{\partial \xi^i} \frac{\partial x^N}{\partial \xi^j} - \tfrac{\mathrm{d}-2}{2}\right] \sqrt{|h|}\, d^{\mathrm{d}}\xi$$

$$\frac{1}{\sqrt{|h|}} \frac{\delta S_h}{\delta x^M} = \frac{1}{\sqrt{|h|}} \frac{\partial}{\partial \xi^i} \left(\sqrt{|h|}\, h^{ij}\, g_{MN}\, \frac{\partial x^N}{\partial \xi^j} \right).$$

$$\frac{1}{\sqrt{|h|}} \frac{\delta S_h}{\delta h_{ij}} = \frac{1}{2} \underbrace{g_{MN} \frac{\partial x^M}{\partial \xi^m} \frac{\partial x^N}{\partial \xi^n}}_{\gamma_{mn}} \left(h^{mi} h^{nj} - \frac{1}{2} h^{ij} h^{mn} \right) + \frac{\mathrm{d}-2}{4} h^{ij}.$$

$$\mathrm{d} \neq 2: \quad \frac{\delta S_h}{\delta h_{ij}} = 0 \;\Rightarrow\; h_{ij} = \gamma_{ij} = g_{MN} \frac{\partial x^M}{\partial \xi^i} \frac{\partial x^N}{\partial \xi^j}.$$

$$\mathrm{d} = 2: \quad \frac{\delta S_h}{\delta h_{ij}} = 0 \;\Rightarrow\; h_{ij} = \gamma_{ij} \cdot \mathrm{scalar}(\xi)$$

Membrane dust

$$S[\varphi^\alpha(X)] = -\int \sqrt{|\det \mathcal{G}^{\alpha\beta}|}\,\sqrt{|g|}\,d^D X.$$

$$\alpha,\beta = 1,\dots,\bar{\mathrm{d}}, \qquad \mathcal{G}^{\alpha\beta}(\varphi,\partial\varphi) = g^{MN}\frac{\partial\varphi^\alpha}{\partial X^M}\frac{\partial\varphi^\beta}{\partial X^N}.$$

$$\text{world\_surface} = \{p \mid \varphi^\alpha(p) = \text{const},\ \alpha = 1,\dots,\bar{\mathrm{d}}\}.$$

For $\bar{\mathrm{d}} \neq 2$:

$$S_\rho[\varphi^\alpha,\rho_{\alpha\beta}] = -\int \left\{\rho_{\alpha\beta}g^{MN}\frac{\partial\varphi^\alpha}{\partial X^M}\frac{\partial\varphi^\beta}{\partial X^N} - \tfrac{\bar{\mathrm{d}}-2}{2}\sqrt[\bar{\mathrm{d}}-2]{\operatorname{sgn}\mathcal{G}\,2^{\bar{\mathrm{d}}}\det\rho_{\alpha\beta}}\right\}\sqrt{|g|}\,d^D X.$$

$\forall\bar{\mathrm{d}} \in \mathbb{N}$:

$$S_k[\varphi^\alpha,k^{\alpha\beta}] = -\int \left\{\frac{1}{2}k_{\alpha\beta}g^{MN}\frac{\partial\varphi^\alpha}{\partial X^M}\frac{\partial\varphi^\beta}{\partial X^N} - \frac{\bar{\mathrm{d}}-2}{2}\right\}\sqrt{|\det k^{\alpha\beta}|}\,\sqrt{|g|}\,d^D X.$$

$$\rho_{\alpha\beta} = k_{\alpha\beta}\sqrt{|\det k^{\alpha\beta}|}.$$

$\rho_{\alpha\beta}$ is **tensor**, $k_{\alpha\beta}$ is **not tensor**.

Domain wall dust ($\bar{\mathrm{d}} = 1$)

$$S_\varrho[\varphi,\varrho] = -\int \frac{1}{2}\left\{\varrho\, g^{MN}\,\frac{\partial\varphi}{\partial X^M}\,\frac{\partial\varphi}{\partial X^N} + \frac{1}{\varrho}\right\}\sqrt{|g|}\,d^D X.$$

Field equations are Hamilton-Jacobi type tachyonic equation

$$\frac{1}{\sqrt{|g|}}\frac{\delta S_\varrho}{\delta\varrho} = -\frac{1}{2}\left\{g^{MN}\,\frac{\partial\varphi}{\partial X^M}\,\frac{\partial\varphi}{\partial X^N} - \frac{1}{\varrho^2}\right\} = -\frac{1}{2}\left\{\|d\varphi\|^2 - \frac{1}{\varrho^2}\right\} = 0$$

and tachyonic continuity equation

$$\frac{1}{\sqrt{|g|}}\frac{\delta S_\varrho}{\delta\varphi} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial X^M}\left(\sqrt{|g|}\,\varrho\, g^{MN}\frac{\partial\varphi}{\partial X^N}\right) = \hat{\boldsymbol{\delta}}(\varrho d\varphi) = 0.$$

"Tachyonic" means $d\varphi$ is spacelike, but perturbations are causal.

Membrane dust $\forall \bar{\mathrm{d}} \in \mathbb{N}$

$$S_k[\varphi^\alpha, k^{\alpha\beta}] = -\int \left\{ \frac{1}{2} k_{\alpha\beta} g^{MN} \frac{\partial \varphi^\alpha}{\partial X^M} \frac{\partial \varphi^\beta}{\partial X^N} - \frac{\bar{\mathrm{d}} - 2}{2} \right\} \sqrt{|\det k^{\alpha\beta}|} \sqrt{|g|}\, d^D X.$$

$$\alpha, \beta = 1, \ldots, \bar{\mathrm{d}}, \quad k^{\alpha\beta} \text{ is not tensor}, \quad \rho_{\alpha\beta} = k_{\alpha\beta} \sqrt{|\det k^{\alpha\beta}|} \text{ is tensor.}$$

$$\frac{1}{\sqrt{|g|}} \frac{\delta S_k}{\delta \varphi^\alpha} = \frac{1}{\sqrt{|g|}} \partial_M \left(\sqrt{|g|}\, \rho_{\alpha\beta}\, \partial^M \varphi^\beta \right)$$

$$\frac{1}{\sqrt{|g|}} \frac{\delta S_k}{\delta k^{\alpha\beta}} = \frac{\sqrt{|\det k^{\alpha\beta}|}}{2} g^{MN} \frac{\partial \varphi^\mu}{\partial x^M} \frac{\partial \varphi^\nu}{\partial x^N} \left[k_{\alpha\mu} k_{\beta\nu} - \frac{1}{2} k_{\alpha\beta} k_{\mu\nu} \right] + \frac{\bar{\mathrm{d}} - 2}{4} k_{\alpha\beta}.$$

$$\bar{\mathrm{d}} \neq 2: \quad \frac{\delta S_k}{\delta k^{\alpha\beta}} = 0 \ \Rightarrow \rho^{\alpha\beta} = \mathcal{G}^{\alpha\beta} = g^{MN} \frac{\partial \varphi^\mu}{\partial x^M} \frac{\partial \varphi^\nu}{\partial x^N}$$

$$\bar{\mathrm{d}} = 2: \quad \frac{\delta S_k}{\delta k^{\alpha\beta}} = 0 \ \Rightarrow k^{\alpha\beta} = \mathcal{G}^{\alpha\beta} \cdot \mathrm{scalar}(x)$$

To quantization of membrane dust

Background:

$$g_{MN} = \mathrm{diag}(-,+,\ldots,+), \qquad \varphi_0^\alpha = x^\alpha, \qquad \rho_0^{\alpha\beta} = \mathcal{G}_0^{\alpha\beta} = \delta^{\alpha\beta}.$$

Coordinates numbering:

$$M,N = \underbrace{0,\ldots,D-1}_{\text{all coordinates}},\ \ i,j = \underbrace{0,\ldots,D-1-\bar{\mathrm{d}}}_{\text{tangential}},\ \ \alpha,\beta = \underbrace{D-\bar{\mathrm{d}},\ldots,D-1}_{\text{transversal}}.$$

Perturbation:

$$\varphi^\alpha = \varphi_0^\alpha + \psi^\alpha, \qquad \Box\psi^\alpha = 0.$$

$$\nabla_\alpha\psi^\beta = 0, \quad (\nabla\psi^\alpha, \nabla\psi^\beta) = 0 \ \Rightarrow\ \rho^{\alpha\beta} = \mathcal{G}^{\alpha\beta} = \delta^{\alpha\beta}.$$

Exact solution (no linearization!) is massless perturbation in tangential direction

$$\psi^\alpha(x^M) = \mathrm{Re}\,\left[C^\alpha\cdot\exp(x^i k_i)\right], \qquad k^i k_i = 0.$$

The solution corresponds to results for small perturabation in linear order.

Thank you for your attention!

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