Noise Resistant Quantum Search algorithms

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Stony Brook

Selected Topics in Mathematical Physics,
75-th Anniversary of I. V. Volovich.

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Beginning

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Thank you for inviting me. I am happy to celebrate the anniversary of professor Volovich. I know Igor for many years. I learned a lot form him. His book on open quantum systems is central for modern quantum technology. I use the book for teaching. I will congratulate Igor in Russian and then switch back to English. Today I will talk about quantum search algorithm. Famous Grover search works faster than classical search. In our days small quantum computers are available on the internet. If I run Grover algorithm on an IBM quantum computer it will be plugged by noise. The result will be as bad as random pick. So we reformulated the quantum search to make it resistive to noise.

Outline

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1. Introduction of Grover's algorithm
2. Improved quantum search
3. Implementation on IBM quantum computers
4. Conclusion/Outlook

Search problem

- Each item of the database is a basis vector in the Hilbert space.
- One-way function (oracle) f:

$$f : \{0,1\}^n \to \{0,1\}$$

  Quantum query checks all the witnesses simalteniously.
- Evaluate the one-way function $f(x)$ is polynomial fast. Evaluation of the inverse $f^{-1}(1) = t$ is exponentially slow (like the telephone book).

- Search problem is to find $t \in \{0,1\}^n$ which gives $f(t) = 1$, given by the one-way function.

Grover's algorithm

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By H we denote the Hadamard gate.

- Initial state is the average of all the items in the database: $|s_n\rangle = H^{\otimes n}|0\rangle^{\otimes n}$.
- Query to the oracle (phase kickback): $U_t = \mathbb{1}_{2^n} - 2|t\rangle\langle t|$ The standard construction of QIS. One target item $|t\rangle$.
- (Global) Diffusion operator uses the n-qubit Toffoli gate:

$$D_n = 2|s_n\rangle\langle s_n| - \mathbb{1}_{2^n}$$

  Reflection in the average. This will be modified. We shall separate the database into several blocks of equal size and replace $|s_n\rangle$ by averages in blocks.

- (Global) Grover operator: $G_n = D_n U_t$

Grover's algorithm

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- For large database $N \to \infty$ Grover's algorithm finds the target state with high probability $1 - \mathcal{O}(1/N)$ after $j_{\max} = \lfloor \pi\sqrt{N}/4 \rfloor$ iterations. Here $N = 2^n$.
- Grover's algorithm is optimal in the number of queries to the oracle, but there are other complexity masures.
- Diffusion operator is also expensive (n-qubit Toffoli gate).
- Oracle is not the only operation in Grover's algorithm.
- Depth is better measure of computational resource. We work on lowering the depth of quantum search.

Local diffusion operator

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- Simplest version. Divide the database into blocks (2^{n-m} blocks):

$$D_m = \mathbb{1}_{2^{n-m}} \otimes (2|s_m\rangle\langle s_m| - \mathbb{1}_{2^m})$$

simultaneously in each block. Can be realised by Toffoli gates.

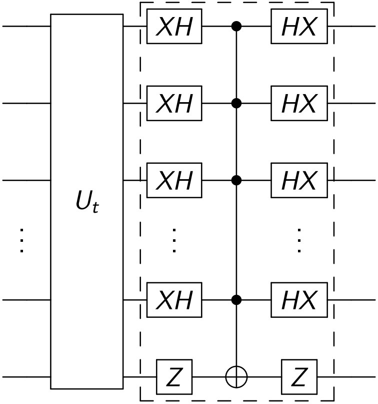
- Local Grover operator

$$G_m = D_m U_t$$

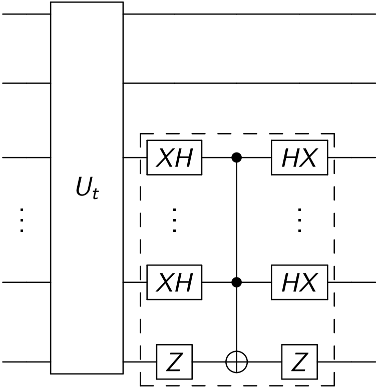
- Operators G_n and G_m can be represented as elements of $O(3)$ group. They do not commute if $m \neq n$.
- Over the course of the algorithm we can use different m.

Local diffusion operator

X is not-gate, Z is the phase-gate [Pauli matices]



D_n



D_m

Non-Abelian Grover search

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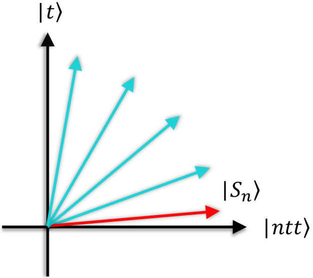
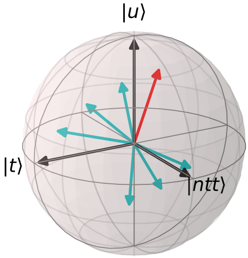


Grover's algorithm. $O(2)$ rotation.



Search with global and local diffusion operators. It is $O(3)$ rotation. Use of local diffusion operators of different sizes with increase the dimension of the sphere.

Three strategies to reduce noise

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Three strategies are both based on application of partial diffusion operators.

- Strategy I: hybrid classical-quantum search. Random guess partial bits, then run rescaled search.
- Strategy II: depth optimization by partial diffusion operator. $O(3)$ or higher dimensional search.
- Strategy III: divide-and-conquer. Each stage finds partial bits of target state. The input of each stage depends on the measurement results on last stage.

Critical depth ratio

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- Circuit/operator depth is the number of consecutive elementary operations required to run a circuit/operator on quantum hardware.

- Define the depth ratio

$$\alpha = \frac{\mathrm{d}(U_t)}{\mathrm{d}(D_n)}.$$

- Consider the critical ratio α_c, below which Grover's algorithm is not optimal in depth. We find $\alpha_c = \mathcal{O}(n^{-1}2^{n/2})$.

- If the circuit is divided into two stages, the critical ratio is a constant at large n: $\lim_{n\to\infty} \alpha_c = 1 + \sqrt{3}$.

Remarks

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- Different strategies can be jointly applied.
- Different problems (oracles), different quantum devices, different error rates, et al., all can give different optimal circuits. We can choose different local diffusion over the course of the algorithm [different m].
- Divide-and-conquer can also be applied to parallel running.

Setup for IBM Q

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- We choose the toy oracle which is single-qubit-gate equivalent to the n-qubit Toffoli gate.
- We choose the target states randomly.
- Probabilities are obtained on 30 trails with 8192 shots.

Notations

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For example, the 3-qubit search algorithm can have the circuits:

- D3M3: one 3-qubit diffusion operator followed by 3-qubit measurements.
- D2M3: one 2-qubit diffusion operator followed by 3-qubit measurements.
- G1D2M2: random guess one qubit then one 2-qubit diffusion operator followed by 2-qubit measurements.
- D2M1|D2M2: first stage is one 2-qubit diffusion operator followed by 1-qubit measurement; second stage is one 2-qubit diffusion operator followed by 2-qubit measurement.

Three-qubit search

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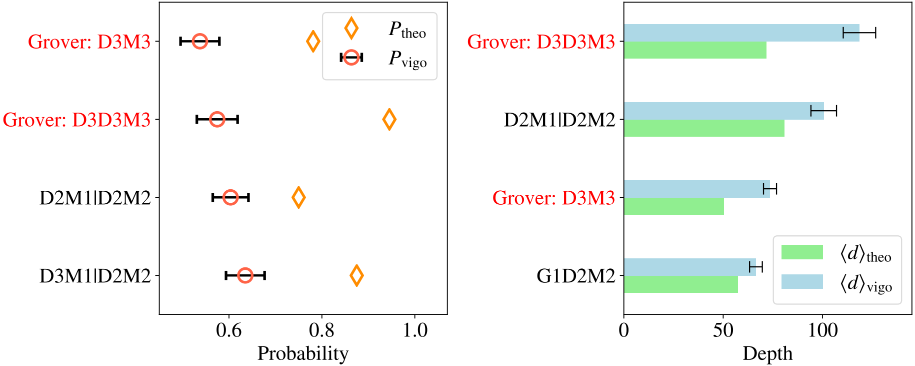

Four-qubit search

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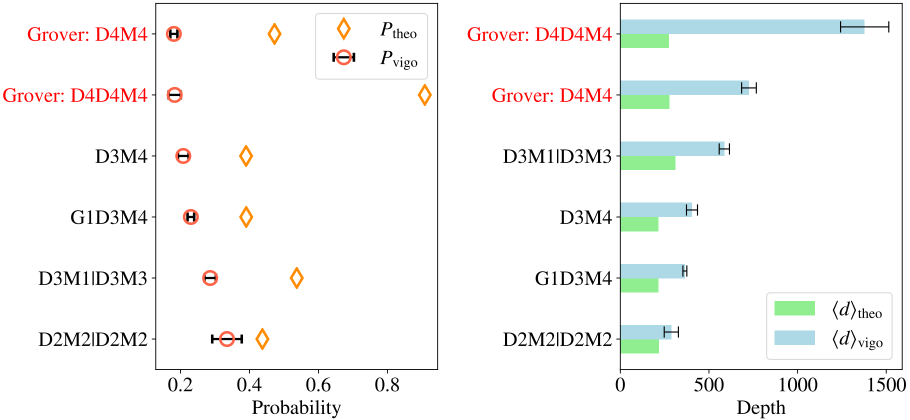
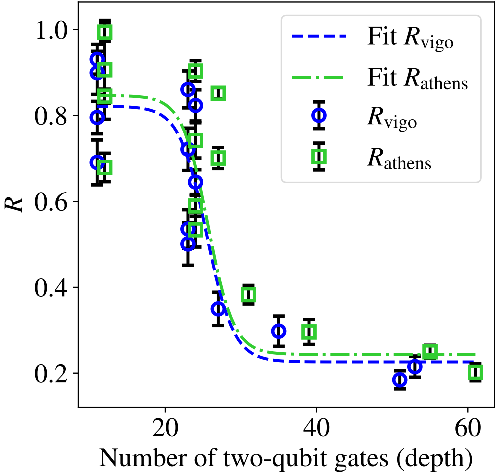


Five-qubit search

Circuits	P_{theo}	P_{casa}	$\langle d_{\text{casa}} \rangle_{\text{theo}}$	$\langle d_{\text{casa}} \rangle$
Grover: D5M5	0.258	0.0278 ± 0.0115	460.94	4723 ± 1236
G3D2M2	0.125	0.0971 ± 0.0036	398.88	514.35 ± 60.86

Degraded ratio

NISQ devices favors shallow depth circuits. $R = P_{\text{real}}/P_{\text{theo}}$



Conclusions

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- There is a gap between the standard theoretical computational complexity and the physical computation resources.

- Reducing the depth can mitigate the noises on NISQ devices. Reducing the depth also save the running time [in nano-seconds] of quantum circuit.

- Grover's algorithm is not optimal in depth. We reduce the depth. We suggest hardware efficient modification of the search. We make the search less vulnerable to noise.

Conclusions

- Local diffusion operator can be used for depth optimization, multi-stage search and parallel quantum search.

- Connectivity of the future quantum computers will be complicated [loops]. We can adjust our search: different size of the blocks and different order of application [now diffusion operators do not commute]. Higher dimensional search : we replace $O(2)$ of Grover by $O(M+1)$, here M is the number of different diffusion operators [sizes of the blocks] used in the course of the algorithm.

Acknowledgment

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Discailmer

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The material of the lecture is in the papers:

- K. Zhang and V. Korepin, *Depth optimization of quantum search algorithms beyond Grover's algorithm*, Phys. Rev. A, **101**, 032346 (2020).
- K. Zhang, P. Rao, K. Yu, H. Lim, and V. Korepin, *Implementation of improved quantum search algorithms on IBM quantum processors*, Quantum Inf. Process. 20, 233 (2021).