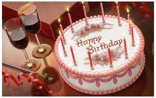

On anti–Kähler (Norden–Kähler) structures

Andrzej Borowiec
Wroclaw University

based on a joint work with Marco Ferraris, Mauro Francaviglia, and Igor Volovich,

Dedicated to Igor Volovich on his 75th birthday

Selected Problems in Mathematical Physics

Steklov Mathematical Institute, Moscow 2021

Figure: Torino from Mole Antonelliana

Ricci squared gravity & differential geometric structures

A. B., M. Ferraris, M. Francaviglia, I. Volovich,
Universality of Einstein equations for the Ricci squared Lagrangians,
Class.Quant.Grav. 15 (1998) 43-55
e-Print: gr-qc/9611067 [gr-qc]

A. B., M. Ferraris, M. Francaviglia, I. Volovich,
Almost complex and almost product Einstein manifolds from a variational principle,
J.Math.Phys. 40 (1999) 3446-3464
e-Print: dg-ga/9612009 [math.DG]

A. B., M Francaviglia, I Volovich,
Anti-Kählerian manifolds,
Differential Geom. Appl. 12 (2000), no. 3, 281–289
arXiv:math-ph/9906012

$f(S)$ -Palatini gravity – source free case

Metric and torsionless connections are independent fields, n=dim ST ($n > 2$)

$$A_{\rm grav} = \int \sqrt{\det g}\, f(S)\, d^n x \qquad (1)$$

where $S \equiv S(g, \Gamma) = g^{\mu\alpha} R_{(\alpha\nu)}(\Gamma) g^{\nu\beta} R_{(\beta\mu)}(\Gamma)$ is the so called Ricci squared invariant (in short $S = R_{(\mu\nu)} R^{\mu\nu}$) and Γ is a torsion free connection.

$$2f'(S) g^{\alpha\beta} R_{(\mu\alpha)}(\Gamma) R_{(\beta\nu)}(\Gamma) - \frac{1}{2} f(S) g_{\mu\nu} = 0 \qquad (2)$$

$$\nabla^{\Gamma}_{\sigma}(\sqrt{\det g} f'(S) g^{\mu\alpha} R_{(\alpha\beta)}(\Gamma) g^{\beta\nu}) = 0 \qquad (3)$$

In order to solve (3) one defines a new metric h by

$$\sqrt{\det h}\, h^{\mu\nu} = \sqrt{\det g}\, f'(S) g^{\mu\alpha} R_{(\alpha\beta)}(\Gamma) g^{\beta\nu}$$

We firstly find S_0 as a solution of the structural equation

$$f'(S) S - \frac{n}{4} f(S) = 0 \qquad (4)$$

and then algebraic equation (2) by ($P = P^{\nu}_{\mu} = g^{\nu\alpha} R_{(\alpha\mu)}(\Gamma)$)

$$P^2 = \frac{f(S_0)}{4f'(S_0)} \, I \tag{5}$$

Now, the solution for h takes the form

$$h_{\mu\nu} = h_{\mu\nu}(S_0) = f'(S_0)\sqrt{\det P(S_0)} \, g_{\mu\alpha} \left(P^{-1}\right)^{\alpha}_{\nu} \tag{6}$$

i.e. in algebraic terms the metric h is conformal to

$$h \simeq (g^{-1} R g^{-1})^{-1} = P^{-1} g \tag{7}$$

The generalized Einstein equation reads

$$R_{\mu\nu}(h) = P^{\alpha}_{\nu} g_{\mu\alpha} = h_{\mu\nu} \tag{8}$$

Universality of Einstein equations: $f(S)$-Palatini

Theorem *Let M be a n-dimensional manifold, $n > 2$, with a metric g and a symmetric connection Γ and let us consider the Euler-Lagrange equations (2)-(3) for the action (1). Let us assume that the analytic function $f(S)$ is such that eq. (4) has an isolated root S_0 with $f'(S_0) \neq 0$; setting then $h_{\mu\nu} = R_{(\mu\nu)}(\Gamma)$, the Euler-Lagrange equations imply the relations $(g^{-1}h)^2 = \epsilon I$, $Ric(h) = h$. Therefore:*

(i) *if $\epsilon > 0$ after rescaling and denoting $P = g^{-1}h$ one gets an almost-product Einstein manifold (M, h, P), i.e.*

$$Ric(h) = \gamma h$$

$$P^2 = I, \quad h(PX, PY) = h(X, Y), \quad X, Y \in \chi(M) \quad ;$$

(ii) *if $\epsilon < 0$ after rescaling and denoting $J = g^{-1}h$ one gets instead an anti-Hermitian Einstein Norden manifold (M, h, J), i.e.*

$$Ric(h) = \gamma h$$

$$J^2 = -I, \quad h(JX, JY) = -h(X, Y), \quad X, Y \in \chi(M)$$

Here $\chi(M)$ is the Lie algebra of vector fields on M.

This extends well-know (classical) result about Universality of Einstein equations for $f(R)$-Palatini gravity from

M. Ferraris, M. Francaviglia, I. Volovich,
The Universality of vacuum Einstein equations with cosmological constant,
Class.Quant.Grav. 11 (1994) 1505-1517 e-Print: gr-qc/9303007 [gr-qc]

New element is the presence of algebraic relations (constraints) for the Einstein metric.

Twin metrics

It turns out that solution of the algebraic constraints can be done in a form of twin metrics:
Given $(1,1)$-tensor K and a metric h ($(0,2)$-tensor) such that $K^2 = \epsilon I$ and $h(KX, KY) = \epsilon h(X, Y)$, where $\epsilon = \pm 1$.
Then $K = g^{-1}h$ where $g(X, Y) = h(KX, Y)$ is called twin metric.
Moreover $g(KX, KY) = \epsilon g(X, Y)$, and $h(X, Y) = g(KX, Y)$.

For $\epsilon = 1$, $K = P$ at any point the corresponding matrices can be simultaneously diagonalized with entries ± 1. It provides $G = O(r, s; \mathbb{R}) \times O(k, l; \mathbb{R})$ structure with $n = r + s + k + l$.

For $\epsilon = -1$, $K = J$ instead at any point the corresponding matrices can reach the following canonical form:

$$h = \begin{pmatrix} I_m & 0 \\ 0 & -I_m \end{pmatrix}, \quad g = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}.$$

In particular, the dimension of spacetime $n = 2m$, and both metrics have neutral signature. $G = O(m; \mathbb{C})$.

Integrability

It is known (e.g. Kobayashi, Nomizu) that $\nabla K = 0$ for a torsion free connection then the corresponding K-structure is integrable, i.e. Nijenhuis tensor vanishes. If K is integrable then there exists an atlas of adapted coordinate charts on M in which K takes a canonical form.

Definition A metric g on a (K, h)-manifold is called a *Kähler-like metric* if $\nabla^h K = 0$, i.e.

$$\nabla^h_X(KY) = K\nabla^h_X Y, \quad X, Y \in \chi(M) \tag{4.6}$$

where ∇^h is the Levi-Civita connection of h.
Kähler-like structures are integrable. Their Ricci tensors satisfy $S^h(KX, KY) = \epsilon S^h(X, Y)$. Particularly, twin of Einstein metric is Einstein.

Anti-Kähler manifold

Norden-Kähler manifold is a triple (M, h, J) which consists of a smooth manifold M, an almost complex structure J and a metric h such that $\nabla^h J = 0$, and the metric h is anti-Hermitian (Norden):

$$h(JX, JY) = -h(X, Y)$$

Theorem There is one-to one correspondence between real anti-Kähler manifolds and complex manifolds with holomorphic metrics. Moreover, any such real metric is the real part of the corresponding holomorphic metric. In adapted complex coordinates $x^\mu = (x^a, y^a \equiv x^{m+a})$, $z^a = x^a + iy^a$ this correspondence is given by the following fromula

$$h_{\mu\nu} dx^\mu dx^\nu = 2 \text{ Re } [h_{ab} dz^a dz^b]$$

where $\mu = 1, \ldots, 2m$, $a = 1, \ldots, m$.

Complex Riemannian manifold

A complex metric on a complex manifold is said to be holomorphic iff

$$\partial_{\bar{a}} h_{bc} = 0$$

for any $a, b, c = 1, \ldots, m$.
We can define now (in complex coordinates) the complex Christoffel symbols (holomorphic connection) Γ^c_{ab} by

$$\nabla_{\partial_a}\partial_b = \Gamma^c_{ab}\partial_c = \frac{1}{2}h^{cd}(\partial_a h_{bd} + \partial_b h_{da} - \partial_d h_{ab})$$

holomorphic Riemann R^d_{abc} and Ricci R_{ab} tensors by standard classical formulae. Finally, we are in position to define holomorphic Einstein condition

$$R_{ab} = \lambda h_{ab}$$

Thus the relation between complex and real Einstein equations is given

Anti-Kähler Einstein manifold

Theorem A holomorphic complex metric $h_{ab}(z)$ is holomorphic Einstein with real constant λ iff the corresponding real metric $h_{\mu\nu}(x)$ satisfies Einstein equations too. More exactly, we have:

$$R_{\mu\nu}(h_{\alpha\beta}) = \lambda h_{\mu\nu} \quad \text{iff} \quad R_{ab}(h_{cd}) = \lambda h_{ab}$$

For complex constant $\lambda = \lambda_1 + i\lambda_2$ the corresponding real equations take the form

$$R_{\mu\nu}(h_{\alpha\beta}) = R_{\mu\nu}(g_{\alpha\beta}) = \lambda_1 h_{\mu\nu} + \lambda_2 g_{\mu\nu}$$

of generalized Einstein equations, where $g_{\mu\nu}$ denotes the twin metric.

$f(S)$ -Palatini gravity: the case with matter source

$$A = A_{\rm grav} + A_{\rm mat} = \int (\sqrt{\det g}\, f(S) + 2\kappa L_{\rm mat}) d^4 x \tag{9}$$

$$2f'(S) g^{\alpha\beta} R_{(\mu\alpha)}(\Gamma) R_{(\beta\nu)}(\Gamma) - \frac{1}{2} f(S) g_{\mu\nu} = \kappa T_{\mu\nu} \tag{10}$$

$$\nabla^{\Gamma}_{\sigma} (\sqrt{\det g} f'(S) g^{\mu\alpha} R_{(\alpha\beta)}(\Gamma) g^{\beta\nu}) = 0 \tag{11}$$

Again we try to solve (11) defining a new metric h

$$\sqrt{\det h}\, h^{\mu\nu} = \sqrt{\det g}\, f'(S) g^{\mu\alpha} R_{(\alpha\beta)}(\Gamma) g^{\beta\nu}$$

With the same strategy as before we first find $S(\tau)$ as a solution of the structural equation

$$f'(S) S - \frac{n}{4} f(S) = \frac{\kappa}{2} g^{\alpha\beta} T_{\alpha\beta} \equiv \frac{\kappa}{2} \tau \tag{12}$$

and then algebraic equation (11) by ($P = P_\mu^\nu$)

$$P^2 = \frac{f(S(\tau))}{4f'(S(\tau))} I + \frac{\kappa}{2f'(S(\tau))} \hat{T} \tag{13}$$

Now, the solution for h takes the form

$$h_{\mu\nu} = h_{\mu\nu}(\tau) = f'(\tau)\sqrt{\det P(\tau)}\, g_{\mu\alpha} \left(P^{-1}\right)_\nu^\alpha \tag{14}$$

i.e. in algebraic terms the metric h is conformal to

$$h \simeq (g^{-1} R g^{-1})^{-1} = P^{-1} g \tag{15}$$

The generalized Einstein equation for the system (10-11) is

$$R_{\mu\nu}(h) = P_\nu^\alpha g_{\mu\alpha} \tag{16}$$

The last equation has been used later on for constructing FLRW type cosmological models: $\tau = 3P - \rho$.

Generalized Ricci type Lagrangians

$$L_F(g,\Gamma) = \sqrt{g}\, F(p_1, \dots, p_{n-1})$$

Let us consider a $(1,1)$ tensor valued concomitant of a metric g and a linear torsion free connection Γ defined by

$$\mathbf{P}^\mu_\nu \equiv \mathbf{P}^\mu_\nu(g,\Gamma) = g^{\mu\lambda} R_{(\lambda\nu)}(\Gamma)$$

One can use it to define a family of scalar concomitants of the Ricci type

$$p_k = tr\, \mathbf{P}^k$$

for $k = 1, \dots n-1$. We can eliminate the higher order Ricci scalars p_k with $k \geq n \doteq \dim M$, by using a characteristic polynomial equation for the $n \times n$ matrix $\mathbf{P}$. One immediately recognizes that $R \equiv p_1 = tr\, \mathbf{P}$ and $S \equiv p_2 = tr\, \mathbf{P}^2$, etc..

Missed/opened opportunity

Theorem The universality property extends also to this class of Lagrangians. Moreover, one gets more complicated polynomial structure

$$w_F(\mathbf{P}) = 0$$

with some polynomial $w_F(t) = \sum_{k=1}^{n} a_k t^k$ of degree n, where

$$a_k = k\frac{\partial F}{\partial s_k}(c_1, \ldots, c_n)$$

and $(c_1, \ldots, c_n)$ are numerical soltions of the functional equation $\sum_{k=1}^{n} k\frac{\partial F}{\partial s_k} = 1/2F$. However their differential-geometric structures (integrability, twin metrics, Einstein metrics, etc..) remains unexplored.

A. B. *Metric-Polynomial Structures and Gravitational Lagrangians*, GROUP 24: Physical and Mathematical Aspects of Symmetries: Proceedings of the 24th International Colloquium on Group Theoretical Methods in Physics, Paris, 15-20 July 2002, Ed.: J-P. Gazeau, R. Kerner, J-P. Antoine, S. Metens, J-Y. Thibon, Institute of Physics Conference Series CS 173:241-244 , 2003.

e-Print version: *Nonlinear Lagrangians of the Ricci Type*, gr-qc/9906043

I would like to thank Igor for 10 years of fruitful collaboration and wish you all the best

Spasibo za plodotvornoye sotrudnichestvo. S Dnem Rozhdeniya!

Happy Birthday Igor Vasill'evich !

Thank you for your attention !