p-Adic Matter in the Closed Universe

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Volovich–75

Selected Topics in Mathematical Physics

Dedicated to 75-th Anniversary of I.V. Volovich

27-30.09. 2021

Online Conference

Our collaboration started in 1986.
Photo: Work at home 1988.

Photo: (Left)In Institute of Physics during 4th Summer School in Modern Mathematical Physics, Belgrade 2006. (Right)In Steklov Mathematical Institute during 3rd Int. Conf. on p-Adic Mathematical Physics, 2007

- “Volovich made important contributions to various areas of mathematical and theoretical physics and mathematics, including quantum field theory, integrable systems, superanalysis, quantum information and quantum computation, supergravity and superstrings, foundations of quantum mechanics, *p*-adic mathematical physics, method of stochastic limit in quantum dynamics, quantum chaos.”

  L.Accardi, B. Dragovich, M.O. Katanaev, A. Yu. Khrennikov, V.V. Kozlov, S. Kozyrev, F. Murtagh, M. Ohya, V.S. Varadarajan, and V.S. Vladimirov, “Igor V. Volovich”, *p*-Adic Num. Ultrametr. Anal. Appl. **4** (1), 1–4 (2012).

- Igor V. Volovich elected as foreign member of the Serbian Academy of Nonlinear Sciences, 2019.

I.V. Volovich impact on *p*-adic string theory and all that

- Introduced notion of *p*-adic string in 1987.
- There should be quantum fluctuations of geometry and number fields at the Planck scale.
- Space-time should be non-Archimedean at the Planck scale.
- Proposed the number field invariance principle: Fundamental physical laws should be invariant under changing number fields.
- Emphasized the role of rational numbers in measurements and construction of physical models.
- Pointed out the role of *p*-adic numbers in description of systems with hierarchical structure.

1. Introduction

I plan to

- give a brief review of basic properties of p-adic strings
- show how a matter can be derived from p-adic strings
- show that this p-adic matter is related to evolution of a closed universe
- discuss obtained results.

2. p-Adic Strings

Volovich, Vladimirov, Freund, Witten, Arefeva, B.D., ...
String amplitudes:

- standard crossing symmetric Veneziano amplitude

$$A_\infty(a,b) = g_\infty^2 \int_{\mathbb{R}} |x|_\infty^{a-1} \, |1-x|_\infty^{b-1} \, d_\infty x$$
$$= g_\infty^2 \, \frac{\zeta(1-a)}{\zeta(a)} \, \frac{\zeta(1-b)}{\zeta(b)} \, \frac{\zeta(1-c)}{\zeta(c)}$$

- p-adic crossing symmetric Veneziano amplitude

$$A_p(a,b) = g_p^2 \int_{\mathbb{Q}_p} |x|_p^{a-1} \, |1-x|_p^{b-1} \, d_p x$$
$$= g_p^2 \, \frac{1-p^{a-1}}{1-p^{-a}} \, \frac{1-p^{b-1}}{1-p^{-b}} \, \frac{1-p^{c-1}}{1-p^{-c}}$$

where $a = -s/2 - 1$ and $a, b, c \in \mathbb{C}$ and $a + b + c = 1$.

2. p-Adic Strings

1. Euler product formula for Riemann zeta function

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}, \quad \Re s > 1$$

2. Freund-Witten product formula for adelic strings

$$A(a, b) = A_\infty(a, b) \prod_p A_p(a, b) = g_\infty^2 \prod_p g_p^2 = const.$$

- amplitudes on equal footing
- various faces of an adelic string
- amplitude of ordinary strings can be regarded as product of p-adic inverses

3. Effective Field Theory for p-Adic Strings

- One of the main achievements in p-adic string theory is an effective field description of scalar open and closed p-adic strings. The corresponding Lagrangians are very simple and exact. They describe not only four-point scattering amplitudes but also all higher ones at the tree-level.
- The exact tree-level Lagrangian for effective scalar field φ which describes open p-adic string tachyon is

$$\mathcal{L}_p = \frac{m_p^D}{g_p^2}\,\frac{p^2}{p-1}\Big[-\frac{1}{2}\,\varphi\, p^{-\frac{\Box}{2m_p^2}}\,\varphi + \frac{1}{p+1}\,\varphi^{p+1}\Big]$$

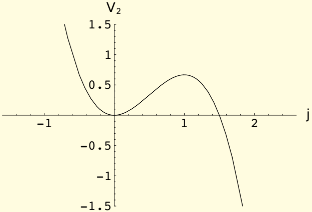
where p is any prime number, $\Box = -\partial_t^2 + \nabla^2$ is the D-dimensional d'Alembertian and metric with signature $(- + ... +)$ (Freund, Witten, Frampton, Okada, ...) .

3. Effective Field Theory for p-Adic Strings

The above Lagrangian is written completely in terms of real numbers and there is no explicit dependence on the p-adic world sheet. However, it can be rewritten as:

$$\mathcal{L}_p = \frac{m^D}{g^2}\,\frac{p^2}{p-1}\Big[\frac{1}{2}\,\varphi\int_{\mathbb{R}}\Big(\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}\chi_p(u)|u|_p^{\frac{k^2}{2m^2}}\,du\Big)\tilde{\varphi}(k)\,\chi(kx)\,d^Dk$$
$$+\frac{1}{p+1}\,\varphi^{p+1}\Big],$$

where $\chi(kx) = e^{-ikx}$. Since $\int_{\mathbb{Q}_p}\chi_p(u)|u|^{s-1}du = \frac{1-p^{s-1}}{1-p^{-s}} = \Gamma_p(s)$ and it is present in the scattering amplitude, one can say that variable u in $\int_{\mathbb{Q}_p\setminus\mathbb{Z}_p}\chi_p(u)|u|_p^{\frac{k^2}{2m^2}}du = -p^{\frac{k^2}{2m^2}}$ is related to the p-adic string world-sheet.



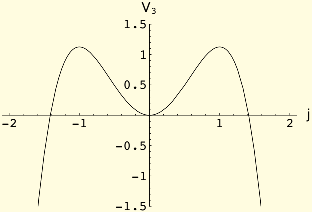


Figure: The 2-adic string potential $\mathcal{V}_2(\varphi)$ (on the left) and 3-adic potential $\mathcal{V}_3(\varphi)$ (on the right)

Potential .

$$\mathcal{V}_p(\varphi) = \frac{m_p^D}{g_p^2}\frac{p^2}{p-1}\left[\frac{1}{2}\varphi^2 - \varphi^{p+1}\right].$$

3. Effective Field Theory for p-Adic Strings

- The equation of motion is

$$p^{-\frac{\Box}{2m^2}}\,\varphi = \varphi^p, \quad \varphi = 0,\ \varphi = 1,\ (\varphi = -1,\ p \neq 2)$$

$$e^{A\partial_t^2}\, e^{Bt^2} = \frac{1}{\sqrt{1-4AB}}\, e^{\frac{Bt^2}{1-4AB}}, \quad 1 - 4AB > 0$$

There are also nontrivial solutions:.

$$\varphi(x^i) = p^{\frac{1}{2(p-1)}}\ \exp\Big(-\frac{p-1}{2m^2\, p\ln p}(x^i)^2\Big)$$

$$\varphi(t) = p^{\frac{1}{2(p-1)}}\ \exp\Big(\frac{p-1}{2p\ln p} m^2 t^2\Big)$$

$$\varphi(x) = p^{\frac{D}{2(p-1)}}\ \exp\Big(-\frac{p-1}{2\,p\ln p}\, m^2\, x^2\Big), \quad x^2 = -t^2 + \sum_{i=1}^{D-1} x_i^2 .$$

3. Effective Field Theory for p-Adic Strings

- When $p = 1 + \varepsilon \to 1$ there is the limit which is related to the ordinary bosonic string in the boundary string field theory (Gerasimov-Shatashvili):

$$\mathcal{L} = \frac{m^D}{g^2}\left[\frac{1}{2}\,\varphi\,\frac{\Box}{m^2}\,\varphi + \frac{\varphi^2}{2}\left(\ln\varphi^2 - 1\right)\right]$$

- arXiv:2105.00298: (M. Bocardo-Gaspar, H. García-Compeán, Edgar Y. López, W. A. Zú*ñ*iga-Galindo)
- Tachyon condensation (D. Ghoshal, A. Sen)
- AdS/CFT correspondence (S.S. Gubser, S. Parikh, M. Marcolli, I, Saberi, B. Stoica, ...)
- From these and some other developments it follows that some nontrivial features of ordinary strings are similar to p-adic ones and are related to the p-adic effective action.

To avoid tachyon, consider transition $m^2 \to -m^2$ in D = 4 dimensions. Also change sign to lagrangian to avoid ghost. Then the related new Lagrangian is

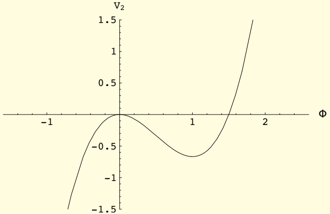
$$L_p = \frac{m^4}{g^2}\,\frac{p^2}{p-1}\Big[\frac{1}{2}\,\phi\, p^{\frac{\Box}{2m^2}}\,\phi - \frac{1}{p+1}\,\phi^{p+1}\Big] \tag{1}$$

with the corresponding potential

$$V_p(\phi) = \frac{m^4}{g^2}\,\frac{p^2}{p-1}\Big[\frac{1}{p+1}\,\phi^{p+1} - \frac{1}{2}\,\phi^2\Big].$$

and equation of motion

$$p^{\frac{\Box}{2m^2}}\,\phi = \phi^p \tag{2}$$



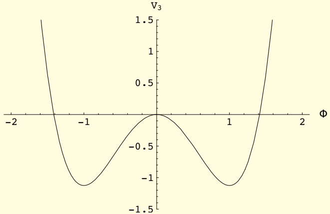


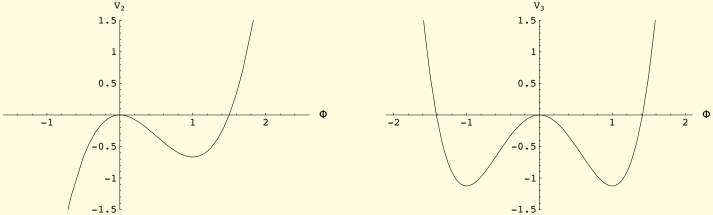
Figure: New potentials $V_2(\phi)$ and $V_3(\phi)$, which are related to new Lagrangian.

Trivial solutions

$$p^{\frac{\Box}{2m^2}}\,\phi = \phi^p, \quad \phi = 0,\ \phi = 1,\ (\phi = -1,\ p \neq 2)$$

and also previous nontrivial solutions with $m^2 \to -m^2$.

4. p-Adic Matter in Minkowski space



Consider weak field approximation $\phi = 1 + \theta, \quad |\theta| \ll 1$.

$$p^{\frac{\Box}{2m^2}}\,(1+\theta) = (1+\theta)^p, \quad \Rightarrow \quad p^{\frac{\Box}{2m^2}}\,\theta = p\,\theta.$$

EoM $p^{\frac{\Box}{2m^2}}\,\theta = p\,\theta$ has solution since the following Klein-Gordon equation $(\Box - 2m^2)\,\theta = 0$, is satisfied and
$\theta \sim a\,e^{i(-Et+\vec{k}\vec{x})} + \bar{a}\,e^{-i(-Et+\vec{k}\vec{x})}$ is a scalar field with $E^2 = 2m^2 + \vec{k}^2$.

5. A Closed Universe with p-Adic Matter

A 4-dimensional gravity with a nonlocal scalar field ϕ and cosmological constant Λ, given by the EH action

$$S = \gamma \int d^4x \sqrt{-g}\, (R - 2\Lambda) + S_m,$$

where $\gamma = \frac{1}{16\pi G}$, R is the Ricci scalar and

$$S_m = \sigma \int d^4x \sqrt{-g}\, \Big(\frac{1}{2}\phi F(\Box)\phi - U(\phi)\Big),$$

where $F(\Box) = \sum_{n=0}^{\infty} f_n \, \Box^n$ and $U(\phi)$ is a part of the potential. Note that now

$$\Box = \nabla_\mu \nabla^\mu = \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu$$

.

5. A Closed Universe with p-Adic Matter

The equations of motion for $g_{\mu\nu}$ and ϕ are

$$\gamma(G_{\mu\nu} + \Lambda g_{\mu\nu}) - \frac{\sigma}{4}\, g_{\mu\nu}\, \phi F(\Box)\phi + g_{\mu\nu}\, \frac{\sigma}{2}\, U(\phi) + \frac{\sigma}{4}\, \Omega_{\mu\nu}(\phi) = 0,$$
$$F(\Box)\phi - U'(\phi) = 0,$$

where

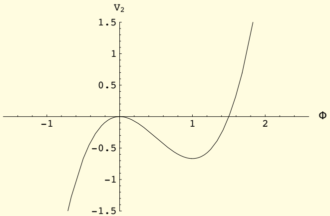
$$\Omega_{\mu\nu}(\phi) = \sum_{n=1}^{\infty} f_n \sum_{\ell=0}^{n-1} \Big[g_{\mu\nu} \left(\nabla^\alpha \Box^\ell \phi \nabla_\alpha \Box^{n-1-\ell} \phi + \Box^\ell \phi \Box^{n-\ell} \phi \right) - 2 \nabla_\mu \Box^\ell \phi \nabla_\nu \Box^{n-1-\ell} \phi \Big].$$

5. A Closed Universe with p-Adic Matter

Matter of interest is p-adic scalar field

$$S_p = \sigma_p \int d^4x \sqrt{-g} \left(\frac{1}{2} \phi \, p^{\frac{1}{2m^2}\Box} \, \phi - \frac{1}{p+1} \, \phi^{p+1} \right),$$

where $\sigma_p = \frac{m_p^D}{g_p^2} \frac{p^2}{p-1}$.



EoM for this p-adic field ϕ is $p^{\frac{1}{2m^2}\Box}\phi \equiv e^{\frac{\ln p}{2m^2}\Box}\phi = \phi^p$, It has the same trivial solutions as in the Minkowski space-time.

5. A Closed Universe with p-Adic Matter

We are interested in cosmological solutions of EoM in the homogeneous and isotropic space given by the FLRW metric

$$ds^2 = -dt^2 + a^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\Big),$$

where $a(t)$ is the cosmic scale factor, and $k = 0, +1, -1$ for the flat, closed and open universe, respectively. Owing to symmetries, there are only two independent EoM: trace

$$4\Lambda - R - \sigma\phi F(\Box)\phi + 2\sigma U(\phi) + \frac{\sigma}{4}\Omega = 0$$

and 00-component

$$\gamma(G_{00} - \Lambda) + \frac{\sigma}{4}\,\phi F(\Box)\phi - \frac{\sigma}{2}\,U(\phi) + \frac{\sigma}{4}\,\Omega_{00}(\phi) = 0,$$

where $\Omega = g^{\mu\nu}\Omega_{\mu\nu}$.

5. A Closed Universe with p-Adic Matter

We look for a solution of EoM in a weak field approximation $\phi = 1 + \theta$, where $|\theta| \ll 1$.

$$p^{\frac{\Box}{2m^2}}(1+\theta) = (1+\theta)^p, \quad \Rightarrow \quad p^{\frac{\Box}{2m^2}}\,\theta = p\,\theta,$$

where now

$$\Box = -\frac{\partial^2}{\partial t^2} - 3H\frac{\partial}{\partial t}, \quad H = \frac{\dot a}{a}.$$

$$p^{\frac{\Box}{2m^2}}\,\theta = p\,\theta$$

has solution if there is solution of $\Box\theta = 2m^2\theta$, i.e.

$$\frac{\partial^2\theta}{\partial t^2} + 3H\frac{\partial\theta}{\partial t} + 2m^2\theta = 0,$$

where $H = \dot a/a$ is the Hubble parameter.

5. A Closed Universe with p-Adic Matter

The simplest case is $H = constant$ and it corresponds to the scale factor $a(t) = Ae^{Ht}$. There is solution in the form $\theta(t) = C\ e^{\lambda t}$, where λ must satisfy quadratic equation

$$\lambda^2 + 3H\lambda + 2m^2 = 0.$$

Simple solutions $\lambda_{1,2} = \pm m$ and the general solution can be written as

$$\theta(t) = C_1\ e^{-mt} + C_2\ e^{mt} = \theta_1(t) + \theta_2(t),$$

where C_1 and C_2 are integration constants. Note that H and λ must have opposite sign. We have pairs:
$\theta_1(t) = C_1 e^{-mt}, \;\; a_1(t) = A_1 e^{mt}$ and
$\theta_2(t) = C_2 e^{mt}, \;\; a_2(t) = A_2 e^{-mt}$.

5. A Closed Universe with p-Adic Matter

The next step is to explore how solution for $\theta(t)$ satisfies EoM for gravitational field. The EH action with θ field is

$$S = \gamma \int d^4x\, \sqrt{-g}\,(R - 2\Lambda) + \sigma_p \int d^4x\, \sqrt{-g}\, \Big(\frac{1}{2}\,\theta p^{\frac{\Box}{2m^2}}\,\theta - \frac{p}{2}\theta^2 + \alpha_p\Big),$$

where $\alpha_p = \frac{p-1}{2(p+1)}$.
The potential $V_p(\theta) = -L_p(\Box = 0)$ is

$$V_p(\theta) = \sigma_p\Big(\frac{p-1}{2}\theta^2 - \alpha_p\Big) \tag{3}$$

and it has the form resembling that of the harmonic oscillator.

5. A Closed Universe with p-Adic Matter

With relevant replacements

$$\phi \to \theta, \quad \sigma \to \sigma_p, \quad U(\theta) = \frac{p}{2}\theta^2 - \alpha_p,$$

$$\gamma(4\Lambda - R) - \sigma_p\, \theta F(\Box)\theta + 2\sigma_p\, (\frac{p}{2}\, \theta^2 - \alpha_p) + \frac{\sigma_p}{4}\, \Omega = 0,$$
$$\gamma(G_{00} - \Lambda) + \frac{\sigma_p}{4}\, \theta F(\Box)\theta - \frac{\sigma_p}{2}\, (\frac{p}{2}\theta^2 - \alpha_p) + \frac{\sigma_p}{4}\, \Omega_{00}(\theta) = 0.$$

$$F(\Box) = p^{\frac{\Box}{2m^2}} = \sum_{n=0}^{\infty} \Big(\frac{\ln p}{2m^2}\Big)^n \frac{1}{n!}\, \Box^n = \sum_{n=0}^{\infty} f_n \Box^n .$$

Since $p^{\frac{\Box}{2m^2}}\, \theta = p\, \theta$, it simplifies the above equations

$$\gamma(4\Lambda - R) - 2\sigma_p\alpha_p + \frac{\sigma_p}{4}\, \Omega = 0,$$
$$\gamma(G_{00} - \Lambda) + \frac{\sigma_p}{2}\, \alpha_p + \frac{\sigma_p}{4}\, \Omega_{00}(\theta) = 0.$$

5. A Closed Universe with p-Adic Matter

Recall that in the FLRW metric

$$R_{00} = -3\frac{\ddot{a}}{a}, \quad G_{00} = 3\Big(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\Big), \quad R = 6\Big(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\Big),$$

computation for $a_1(t) = A_1 e^{mt}$ and $a_2(t) = A_2 e^{-mt}$ gives

$$R_{00}^{(1)} = R_{00}^{(2)} = -3m^2,$$
$$G_{00}^{(1)} = 3\Big(m^2 + \frac{k}{A_1^2}\, e^{-2mt}\Big), \quad G_{00}^{(2)} = 3\Big(m^2 + \frac{k}{A_2^2}\, e^{2mt}\Big),$$
$$R_1 = 6\Big(2m^2 + \frac{k}{A_1^2}\, e^{-2mt}\Big), \quad R_2 = 6\Big(2m^2 + \frac{k}{A_2^2}\, e^{2mt}\Big).$$

Direct calculation of $\Omega = g^{\mu\nu}\, \Omega_{\mu\nu}(\theta)$ and $\Omega_{00}(\theta)$ gives

$$\Omega_1(\theta) = 3p\ln p\, \theta_1^2, \quad \Omega_2(\theta) = 3p\ln p\, \theta_2^2,$$
$$\Omega_{00}^{(1)} = -\frac{3}{2}p\ln p\, \theta_1^2, \quad \Omega_{00}^{(2)} = -\frac{3}{2}p\ln p\, \theta_2^2.$$

One can easily verify that EoM are satisfied in both cases

$$\gamma(4\Lambda - R) - 2\sigma_p\alpha_p + \frac{\sigma_p}{4}\,\Omega = 0,$$
$$\gamma(G_{00} - \Lambda) + \frac{\sigma_p}{2}\,\alpha_p + \frac{\sigma_p}{4}\,\Omega_{00}(\theta) = 0,$$
$$p^{\frac{\Box}{2m^2}}\,\theta = p\,\theta$$

with conditions $6\gamma m^2 + \sigma_p\alpha_p - 2\gamma\Lambda = 0, \quad p\ln p\,\sigma_p\,A_i^2 C_i^2 - 8\gamma\,k = 0, \quad (i = 1, 2), \quad k = +1$, or in the more explicit form

$$\Lambda = 3m^2 + \frac{4\pi G}{g^2}\frac{p^2}{p-1}m^4, \qquad \frac{1}{\left(A_1 C_1\right)^2} = \frac{1}{\left(A_2 C_2\right)^2} = \frac{2\pi G}{g^2}\frac{p^3\ln p}{p-1}m^4.$$

6. Conclusion

- p-Adic strings are nonlocal, nonlinear and non-Archimedean objects with several ways related to ordinary strings.
- By slight modification of Lagrangian for p-adic strings follows scalar matter that makes sense.
- In a closed universe with p-adic matter and cosmological constant, there is exponential expansion (contraction)

$$\theta(t) = Ce^{\mp mt}, \qquad a(t) = Ae^{\pm mt}$$

Some references

1)- I.V. Volovich, *p*-Adic string, *Class. Quant. Grav.* **4** (1987) L83–L87.
2)- P.G.O. Freund and M. Olson, Non-archimedean strings, *Phys. Lett. B* **199** (1987) 186–190.
3)- L. Brekke and P.G.O. Freund, *p*-Adic numbers in physics, *Phys. Rep.* **233** (1) (1993) 1–66.
4)- V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, *p-Adic Analysis and Mathematical Physics* (World Scientific, Singapore, 1994).
5)- B. Dragovich, A.Yu. Khrennikov, S.V. Kozyrev and I.V. Volovich, On *p*-adic mathematical physics, *p-Adic Numbers Ultrametric Anal. Appl.* **1** (1) (2009) 1–17.
6)- S. S. Gubser, J. Knaute, S. Parikh, A. Samberg and P. Witaszczyk, *p*-adic AdS/CFT, *Commun. Math. Phys.* **352**, no. 3, 1019 (2017).
7)- M. Heydeman, M. Marcolli, I. Saberi and B. Stoica, Tensor networks, *p*-adic fields, and algebraic curves: arithmetic and the AdS_3/CFT_2 correspondence, *Adv. Ther. Math. Phys.* **22** (2018) 93–176.
8)- L. Brekke, P.G.O. Freund, M. Olson and E. Witten, Nonarchimedean string dynamics, *Nucl. Phys. B* **302** (3) (1988) 365–402.

Some references

9)- P.H. Frampton and Y. Okada, Effective scalar field theory of p-adic string, *Phys. Rev. D* **37** (1988) 3077–3084.
10)- D. Ghoshal and A. Sen, Tachyon condensation and brane descent relations in p-adic string theory, *Nucl. Phys. B* **584** (2000) 300–312.
11)- A.A. Gerasimov and S.L. Shatashvili, On exact tachyon potential in open string field theory, JPEP **10** (2000) 034.
12)- V.S. Vladimirov, On some exact solutions in p-adic open-closed string theory, *p-Adic Numbers Ultram. Anal. Appl.* **4** (2012) 57–63.
13)- M. Bocardo-Gaspar, H. Garcia-Compean and W.A. Zu˜niga-Galindo, On p-adic string amplitudes in the limit p approaches to one, *JHEP* 08 (2018) 043.
14)- P.G.O. Freund and E. Witten, Adelic string amplitudes, *Phys. Lett. B* **199** (1987) 191–194.
15)- B. Dragovich, Towards p-adic matter in the Universe, Springer Proc. Math. Stat. **36** (2013) 13–24.
16)- I. Dimitrijevic, B. Dragovich, Z. Rakic and J. Stankovic, Variations of infinite derivative modified gravity, Springer Proc. Math. Stat. **263** (2018) 91–111.

THANK YOU FOR YOUR ATTENTION!