(1)

Bismut's way of the Malliavin Calculus for non-Markovian semigroups: an introduction

I Introduction

M a compact manifold.

L a densely defined generator on $L^2(M)$

$$\int_M g(x)\, L h(x)\, dx = \int_M h(x)\, L g(x)\, dx$$

≥ 0

$$\int_M h(x)\, L h(x)\, dx \geq 0$$

Semigroup

$$\frac{\partial P_t}{\partial t} h = -L P_t h$$

$$P_0 h = h$$

__Question__: is there an heat kernel

$$P_t h(x) = \int_M p_t(x,y)\, h(y)\, dy$$

(2)

Several way

→ Micro-local analysis. Fourier transform as a tool.

→ Harmonic analysis: Functional inequalities as a tool.

→ Malliavin calculus. Markovian semi-groups: $P_t f \geq 0$ if $h \geq 0$.

Uses a stochastic representation of the semi group.

For Markovian generators, we have adapted Malliavin integration by parts for diffusions

$$E[h^{(\alpha)}(x_t(x))] = E[h(x_t(x))\, Q^{(\alpha)}]$$

For a system SDE in cascade

Bismut's approach a system of SDE in cascade. Avoids the heavy apparatus of differential operators on Wiener space (Malliavin calculus: plenty of precursors motivated by mathematical physics. Albeverio, Hoegh-Krohn, Mida, Berezanskii)

Bismut's approach: a system in cascade

(3)

(4) of parabolic equation. Leandre

$\Rightarrow$ extended to Non_Markovian semi-groups

(II) <u>The case of a formal stochastic</u>

<u>differential equation</u>

G a compact Lie group $g \in G$ dg Haar measure

e_i a basis of its Lie algebra. considered

$\to$ First order differential operator

$\to$ right invariant vector fields

$$L_0 = \sum e_i^{2k}$$

$$L = L_0 + Q$$

Q polynomial of order $< 2k$ of e_i

$$L = L_0 + Q.$$

L is an elliptic operator ~~positive~~

$\Rightarrow$ perturbation of L_0. Main term L_0

elliptic operator on $G \times \mathbb{R}$

$$\sum_i e_i^{2k} + \sum \lambda_{i,t} e_i \frac{\partial}{\partial u} + \frac{\partial^{2k}}{\partial u^{2k}} = \tilde{L}_t$$

generates by elliptic theory a semi-group

on $C_b(G \times \mathbb{R})$

<u>Theorem</u> (elementary integration by parts)

$$\int_0^t P_{t-s} \sum h_{s,i} b_i P_s[h] \, ds \qquad (1)$$
$$= \tilde{P}_t^{h} [uh](\cdot, 0)$$

<u>Proof</u>: the two sides satisfy the same parabolic element with a second member

$V = G \times M_d$. M_d is the space of symmetric matrices on Lie G $(x, m) \in V$

m is called Malliavin matrix

$$\hat{X}_0 = \left(0, \sum \langle g^{-1} b_i, \cdot \rangle^2\right)$$

$$\hat{L} = L - \hat{X}_0$$

<u>Theorem</u> $\hat{L}$ spans a semi-group called the Malliavin semi-group on $C_b(V)$

<u>Proof</u>: Volterra expansion

<u>Theorem (Malliavin)</u> If the Malliavin condition holds, $(\hat{P}_t | [m^{-1}])[g, 0] < \infty$

$\Rightarrow P_t$ has an heat kernel

(5)

Proof : we extremize the integration by parts formula (1) in order to get

$$|P_t\{dh, b_i\}| \leq C\|h\|_\infty$$

Remark : (without e)

set of paths in $\mathbb{R}^d$ r_t^i

$$\sum \frac{\partial^2}{\partial u_i^2}$$

$$dx_t(u)(e) = \sum b_i \, dr_t^i$$

issued from e.

$$P_t[h](e) = \text{"}E\text{"}[h(x_t(e))]$$

the Itô map $r_0 \to x_t(e)$ is a submersion

theorem : under the previous elliptic assumption

$$|\hat{P}_t|[\nu^{-1}](g_0, 0) < \infty$$

Theorem. if $t > 0$, the semi group P_t has a smooth heat kernel

$$P_t(h](g) = \int r_t(g, g')\, h(g')\, dg'$$

III The full use of the symmetry of the group

Pseudo-diff on $\mathbb{R}^d \times$ ~~$\mathbb{R}$~~

$\mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{C}$

$a(x, \xi)$

$$\sup_{x \in \mathbb{R}^d} |D_x^r D_\xi^l (a(x, \xi))| \leq C |\xi|^{m-l}$$

$$\operatorname{Inf} |a(x, \xi)| \geq C |\xi|^{m'}$$

$\hat{h}$ Fourier transform of h

$$\hat{i}\, h(x) = \int_{\mathbb{R}^d} a(x, \xi)\, \hat{h}(\xi)\, d\xi$$

Pseudo-diff ellipitic of order m'

Invariant if we do a diffeomorphism on $\mathbb{R}^d$

$\Rightarrow$ Elliptic pseudo differential operators

ON A MANIFOLD

$$R_{g_0} : h \longrightarrow (g \to h(g g_0)).$$

$$R_{g_0}(f^i h) = f^i (R_{g_0} h).$$

Right invariant operator

$$L R_{g_0} = R_{g_0} L$$

Theorem : if L elliptic positive operator of order larger than $2k$ on G

$$P_t h(g_0) = \int_G p_t(g_0, g)\, h(g)\, dg.$$

$g \to p_t(g_0, g)$ is smooth.

Algebraic scheme of the proof : Malliavin integration by parts.

$C^\infty(G \times \mathbb{R}^m)$

$$\tilde{L}^m_t = L + \sum_{i=1}^m b^{ii} \frac{\partial}{\partial u_i} \alpha^i_t + \sum_{i=1}^m \frac{\partial^{2k}}{\partial u_i^{2k}} \quad . \quad \tilde{P}_t \text{ semi-group}$$

Elementary integration by parts

$$\tilde{P}^{m+1}_t [h(g)\, h^m(u)\, v] (0, 0, 0) =$$

$$\int_0^t \tilde{P}_{t,s} [b^{d+1} \alpha_s^{m+1} \tilde{P}^m_s [h(g)\, h^m(u)]](0,0)$$

From this elementary integration by parts, we deduce by hypothesis recursion

$$\left|\tilde{P}_t^m\left[b^{(\alpha)} h \prod_{i=1}^m v_i \right](g,u)\right| \le C\, t^{-r_i} \|h\|_{\mathcal{D}} \left(1 + \prod_{i=1}^m |u_i|\right)$$

$\Rightarrow$ Therefore

$$\left|P_t\left[b^{(\alpha)} h \right](g_0)\right| \le C_{(\alpha)} \|h\|_{\mathcal{D}}$$

<u>Estimates: the Davies gauge transform</u>

$$\tilde{P}_t\left[h \prod_{j=1}^m v_j\right](\cdot, 0)$$

the test functions are not bounded!

$\tilde{P}_t^m$ acts only on $C_b(G \times \mathbb{R}^m)$

$g(0) = C(|u|)$. The gauge transform

$$\prod g(v_i)^{-1}\, \tilde{L}_t^m\left(\prod_{i=1}^m g(u_i)\, \cdot\ \right) \cancel{[illegible]}$$

$\Rightarrow$ semi-group is transformed in

9

$$\prod_{i=1}^{n} g_i(\cdot)^{-1} \tilde{P}_t^{m} \left[\prod_{i=1}^{n} g(v_i) h(\cdot) h^{m}(v_i)\right](\cdot,\cdot)$$

on the other hand

$$g(v_i)^{-1} \frac{\partial}{\partial v_i} (g(v_i)\cdot) = \frac{\partial}{\partial v_i} + C(v_i)$$

The gauge transformed semi-group acts

on $C_b(G \times \mathbb{R}^n)$

<u>Example</u>: $\xi^{(\alpha)} = \xi^{\alpha_1} \cdots \xi^{\alpha_{|\alpha|}}$

$f^{(\alpha)} = f^{(\alpha_1)} \cdots f^{a_{|\alpha|}}$.

symmetric matrix $a_{\alpha,\beta}$

$$L = \sum_{\substack{(\alpha),(\beta) \\ |\alpha| = l \ |\beta| = l}} f^{(\alpha)} a_{(\alpha),(\beta)} f^{(\beta)}$$

$(-1)^l L$ is a strictly positive symmetric densely defined operator on L^2. P_t has an heat kernel

no immediate formal SDE representing it

IV The case of an intrinsic variation

L pseudo-diff elliptic positive strict on a compact manifold $l \geq 2k$

<u>Theorem</u> : There is an heat kernel $p_t(x,y)$ associated to P_t. if $t>0$

$$P_t f(x) = \int_M p_t(x,y) f(y) dy$$

<u>Proof</u> : algebraic scheme. Malliavin integration by part

L^α $\quad \alpha \in]0,1[$.

L^α is still a strictly positive pseudo-diff $C^\infty(M \times \mathbb{R}^n$

$$\tilde{L}_s^m = L + s^r L^\alpha \frac{\partial}{\partial u_m} + \sum_{i=1}^{m} \frac{\partial^{2k}}{\partial u_i^{2l}}$$

$\tilde{P}_t^m$ L^α commutes with $\tilde{L}_s^m$

$$L^\alpha \tilde{P}_t^m = \tilde{P}_t^m L^\alpha$$

(11)

elementary integration by parts

$$\tilde{P}_t^{m+1}\left[f \prod_{i=1}^{m} v_i v\right](x, v_i, v) =$$

$$\int_0^t \tilde{P}_{t-s}^{m}\left[s^r L^\alpha \tilde{P}_s^m\left[f \prod_{i=1}^{m} v_i\right]\right](x, v_i)$$

$$= \tilde{P}_t^m\left[L^\alpha f \prod_{i=1}^{m} v_i\right](x, v_i) \int_0^t s^r ds$$

By induction on l.

$$\tilde{P}_t\left[(L^\alpha)^l f \prod_{i=1}^{m} v_i\right](x, v_i) \leq$$

$$C t^{-r(l)}\left(1 + \prod_{i=1}^{m} |v_i|\right).$$

$$\Rightarrow \quad |P_t|\left[(L^\alpha)^l f\right](x) \leq C t^{-r(l)} \|f\|_\infty$$

~~results~~ holds from Sobolev lemma

because L is elliptic

V Wentzell–Freidlin

estimates for the semi-group only

$$L = \sum_{j=0}^{2k} \sum_{i=0}^{r(j)} (X_{i,j})^j$$

elliptic of order $2k$.

ellipticity Hamiltonian

$$\sum_{i=0}^{r(2k)} \langle X_{i,2k}, \xi \rangle^{2k} = H(x, \xi) \geq C|\xi$$

The Lagrangian

$$L(x,p) = \sup_{\xi} \left(\langle p, \xi \rangle - H(x,\xi) \right)$$

Estimates

$$-C + C|p|^{\frac{2k}{2k-1}} \leq L(x,p) \leq C + C|p|^{\frac{2k}{2k-1}}$$

If ϕ is a continuous ~~pa~~ differentiable path on M

$$S(\phi) = \int_0^1 L(\phi(t), d/dt\, \phi(t))\, dt$$

$$\ell(x,y) = \inf_{\phi(0)=x,\ \phi(1)=y} S(\phi)$$

(13)

Theorem (Wentzel-Freidlin) if O is an open ball of M, we have when $t \to 0$

$$\overline{\lim}\ t^{\frac{1}{2l-1}} \log |P_t|(1_O)(x) \leq - \inf_{y \in O} l(x,y)$$

Proof: $\varepsilon = t^{\frac{1}{2l-1}}$ Maslov school

$$L_\varepsilon = \varepsilon^{2l-1} L \qquad P_s^\varepsilon$$

$$P_t = P_1^t$$

conv. if we show

$$\overline{\lim}\ \varepsilon \log |P_1^\varepsilon|(1_O)(x) \leq - \inf_{y \in O} l(x,y)$$

Lemma: $|P_s^\varepsilon|[1_{B(x,\delta)^c}](x) \leq \exp[-C/\varepsilon]$

if $s < s_\delta$

In stochastic analysis, proof by martingale exponential. Here by ~~do~~ doing algebraic transformation on the generator

After lemma: exactly as in the proof of Wentzel Freidlin

Large deviation for stochastic processes: for the whole process. here only on the ~~...~~ semi group

Remark: In WKB analysis (ala Maslov)

$1/\varepsilon \; a(x, \varepsilon \xi)$

precise estimates

$\varepsilon^{-d} \exp\left[-\ell(y)/\varepsilon\right] \sum \varepsilon^{i} c_i(y)$

For Schroedinger equation

$\ell(x,y)$ solves an highly non-linear equation H.J.B. Hamilton-Jacobi-Bellman equation, which is difficult to solve

Another example which comes from stochastic analysis:

$$L f(x) = (-1)^{\ell+1} \int_{\mathbb{R}^d} \Big(f(x+y) - f(x) - \sum_{i=1}^{2\ell} \langle y^{\otimes i}, f^{\otimes i}(x) \rangle \Big) \frac{h(x,y)}{|y|^{2\ell+1+\alpha}} \, dy$$

(1)

$\frac{h(x,y)}{|y|^{2\ell+d+\alpha}}\,dy$ is called the Lévy measure $\alpha \in]-1,0[$

<u>Theorem</u>: Suppose that L is of Lévy type and that $h(y) = h(-y) = h(x,y)$ and therefore L is positive symmetric and admits by ellipticity a selfadjoint extension on $L^2(\mathbb{R}^d)$ which generates a contraction semi-group on $L^2(\mathbb{R}^d)$

<u>Remark</u>: The symbol $a(x,\xi)$ of the generator is

$$(-1)^{\ell+1} \int_{\mathbb{R}^d} \left(\exp(\sqrt{-1}\langle y,\xi\rangle) - \sum_{i=1}^{2\ell} \frac{(\sqrt{-1}\langle \xi, y\rangle)^i}{i!} \right) \frac{h(x,y)}{|y|^{2\ell+d+\alpha}}\,dy$$

16

The HAMILTONIAN is given by

$$(-1)^{\ell+1} \int_{\mathbb{R}^d} \left[\exp\left(\langle y,\xi\rangle\right) - \sum_{i=1}^{\ell} \left(\frac{\langle \xi, y\rangle^{2i}}{2i!}\right)\right] \frac{h(y)\,dy}{|y|^{2\ell+1+\alpha}}$$

LAGRANGIAN

$$L(p) = \sup_{\xi} \left(\langle \xi, p\rangle - H(\xi)\right)$$

$$t \to \phi_t$$

$$\int_0^t dt\, L\left(\phi_t, \frac{d}{dt}\phi_t\right) = S(\phi)$$

$$\ell(x,y) = \inf_{\phi_0 = x,\ \phi_1 = y} S(\phi)$$

MASLOV rescaling $\frac{1}{\varepsilon} a(x, \varepsilon \xi)$ P_s^{ε}

<u>Theorem</u> (Wentzel – Freidlin)

$$\overline{\lim}\, \varepsilon \log P_1^{\varepsilon} [1_O](x) \leq$$

$$- \inf_{y \in O} \ell(x,y)$$

<u>Remark</u>

LARGE deviation not related to SMALL time asymptotics

(17) (VI)

Application: some Varadhan estimates

Hamiltonian

$$H(g,\xi) = \sum_{|\alpha| = l,\ |\beta| = l} \langle b^{\alpha_1}, \xi\rangle \cdots \langle b^{\alpha_l}, \xi\rangle\, a_{(\alpha),(\beta)} \langle b^{\beta_1}, \xi\rangle \cdots\cdots \langle b^{\beta_l}, \xi\rangle$$

$a_{(\alpha),(\beta)} \geq 0$ symbol

Lagrangian

$$L(g,\xi) = \sup_p \left(\langle \xi, p\rangle - H(g,\xi) \right)$$

$$l(g_0, g_1) = \inf_{\phi_0 = g_0;\ \phi_1 = g_1} S(\phi)$$

semi group P_t associated to

$$L = \sum b^{(\alpha)} a_{(\alpha),(\beta)} b^{(\beta)}. \qquad \text{semi-group } P_t$$

Theorem

$$\overline{\lim_{t\to 0}}\ t^{1/2l-1} \log |P_t(g_0, g_1)| \leq - l(g_0, g_1)$$

Proof

$$P_t[\chi f^{(\alpha)} h](g_0) \leq$$

$$C t^{-r|\alpha|} \exp\left[-\frac{l(g_0, g_1) + \delta}{t^{1/2l-1}}\right] \|h\|_D$$

Bibliography

R. LÉANDRE: Far East Jour Math Sciences 30. 2008

R. LÉANDRE: Russian Journal of Mathematical Physics. 22 2015

R. LÉANDRE. Bismut's way of the Malliavin Calculus for non-markovian semigroups: An introduction

To appear Analysis of Pseudo differential operators