BOGOLIUBOV'S CAUSAL QFT WITH HIDA OPERATORS

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ABSTRACT

We will present the axioms of Bogoliubov's causal perturbative QFT in which the creation-annihilation operators are interpreted as Hida operators. We will shortly present the results that can be achieved in this theory:

1. **Removal of UV and IR infinity in the scattering operator,**
2. **Existence of the adiabatic limit for interacting fields in QED,**
3. **Proof that charged particles have non-zero mass,**
4. **Existence of infrared and ultraviolet asymptotics for QED,**
5. **Simplification of the inductive step and**

we will provide some further perspectives.

We are using the Hida white noise operators

$$\partial_{\mathbf{p}}^{*},\ \ \partial_{\mathbf{p}}$$

which respect the canonical commutation or anticommutation relations

$$\left[\partial_{\mathbf{p}}, \partial_{\mathbf{k}}^{*}\right]_{\mp} = \delta(\mathbf{p}-\mathbf{k}),$$

as the creation-annihilation operators

$$a(\mathbf{p})^{+},\ \ a(\mathbf{p}),$$

of the free fields in the BOGOLIUBOV'S causal perturbative QFT, leaving all the rest of the theory completely unchanged. *I.e.* we are using the standard GELFAND triple

$$E \ \subset \ \mathcal{H} \ \subset \ E^{*} \ \ ,$$

over the single particle HILBERT space $\mathcal{H}$ of the total system of free fields determined by the corresponding standard self-adjoint operator A in $\mathcal{H}$ (with some negative power A^{-r} being nuclear), and its lifting to the standard GELFAND triple

$$(E) \ \subset \ \Gamma(\mathcal{H}) \ \subset \ (E)^{*} \ \ ,$$

over the total FOCK space $\Gamma(\mathcal{H})$ of the total system of free fields with the corresponding standard operator $\Gamma(A)$.

$$\begin{array}{ccccc} E & \subset & \mathcal{H} & \subset & E^* \\ \| & & \| & & \| \\ E_1 \oplus \ldots \oplus E_N & & \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_N & & E_1^* \oplus \ldots \oplus E_N^* \end{array},$$

$$\begin{array}{ccccc} \mathcal{S}_A(\sqcup\mathbb{R}^3;\mathbb{C}) & \subset & L^2(\sqcup\mathbb{R}^3;\mathbb{C}) & \subset & \mathcal{S}_A(\sqcup\mathbb{R}^3;\mathbb{C})^* \\ \downarrow\uparrow & & \downarrow\uparrow & & \downarrow\uparrow \\ E & \subset & \mathcal{H} & \subset & E^* \end{array}.$$

$$E = \bigcap_{k\in\mathbb{N}} \mathrm{Dom}\, A^k$$

$$E^* = \bigcup_{k\in\mathbb{N}} \mathrm{Dom}\, A^{-k}$$

with the Hilbertian defining norms

$$\big|\cdot\big|_k \overset{\mathrm{df}}{=} \big|A^k\cdot\big|_{L^2}, \quad \big|\cdot\big|_{-k} \overset{\mathrm{df}}{=} \big|A^{-k}\cdot\big|_{L^2}, \;\; k = 0,1,2,3,\ldots$$

For any Φ in (E) or in $(E)^*$ let

$$\Phi = \sum_{n=0}^{\infty} \Phi_n \quad \text{with} \ \Phi_n \in E^{\hat{\otimes}\, n} \ \text{or, respectively,} \ \ \Phi_n \in E^{*\hat{\otimes}\, n}$$

be its decomposition into n-particle states of an element Φ of the test HIDA space (E) or in its strong dual $(E)^*$, convergent, respectively, in (E) or in $(E)^*$. We define

$$a(w)\Phi_0 = 0, \quad a(w)\Phi_n = n\,\overline{w}\hat{\otimes}_1\Phi_n$$

$$a(w)^+\Phi_n = w\hat{\otimes}\Phi_n, \quad \text{for each fixed } w \in E^*.$$

DEFINITION. The HIDA operators are obtained when we put here the DIRAC delta functional for $w = \delta_{s,\mathbf{p}}$

$$\partial_{s,\mathbf{p}} = a_s(\mathbf{p}) = a(\delta_{s,\mathbf{p}}), \quad \partial_{s,\mathbf{p}}^+ = a_s(\mathbf{p})^+ = a(\delta_{s,\mathbf{p}})^+$$

For each fixed spin-momentum point $(s, \mathbf{p})$ the HIDA operators are well-defined (generalized) operators

$$a_s(\mathbf{p}) \in \mathscr{L}\big((E),(E)\big) \subset \mathscr{L}\big((E),(E)^*\big),$$
$$a_s(\mathbf{p})^+ \in \mathscr{L}\big((E)^*,(E)^*\big) \subset \mathscr{L}\big((E),(E)^*\big),$$

with the last "$\subset$" by topological inclusion $(E) \subset (E)^*$. Let $\phi \in \mathscr{E}$ (here $\mathscr{E}$ is the space-time test space $\mathcal{S}$ or $\mathcal{S}^{00}$) and let $\kappa_{l,m}$ be any $\mathscr{L}(\mathscr{E}, \mathbb{C}) = \mathscr{E}^*$-valued distribution

$$\kappa_{l,m} \in \mathscr{L}(E^{\hat{\otimes}(l+m)}, \mathscr{E}^*) = \mathscr{L}(\mathscr{E}, E^{*\hat{\otimes}(l+m)}) = E^{*\hat{\otimes}(l+m)} \otimes \mathscr{E}^*,$$

then we put

$$\begin{array}{ccc} \Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi) & \overset{\mathrm{df}}{=} & \sum_{n=0}^{\infty} \kappa_{l,m} \otimes_m (\Phi_{n+m} \otimes \phi) \\ \| & & \| \\ \Xi_{l,m}\big(\kappa_{l,m}(\phi)\big)\,\Phi & & \sum_{n=0}^{\infty} \kappa_{l,m}(\phi) \otimes_m \Phi_{n+m} \end{array}.$$

which for any $\mathscr{E}^*$-valued distribution $\kappa_{l,m}$ is a well-defined (generalized) operator

$$\Xi_{l,m}(\kappa_{l,m}) \in \mathscr{L}\big((E)\otimes\mathscr{E}, (E)^*\big) \cong \mathscr{L}\big(\mathscr{E}, \mathscr{L}((E),(E)^*)\big).$$

$\Xi_{l,m}(\kappa_{l,m})$ defines integral kernel operator $\Xi_{l,m}(\kappa_{l,m})$ which is uniquely determined by the condition

$$\langle\langle \Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi), \Psi \rangle\rangle = \langle \kappa_{l,m}(\eta_{\Phi,\Psi}), \phi \rangle, \quad \Phi, \Psi \in (E), \phi \in \mathscr{E}$$

or, respectively,

$$\langle\langle \Xi_{l,m}(\kappa_{l,m})(\Phi \otimes \phi), \Psi \rangle\rangle = \langle \kappa_{l,m}(\phi), \eta_{\Phi,\Psi} \rangle, \quad \Phi, \Psi \in (E), \phi \in \mathscr{E},$$

depending on $\kappa_{l,m}$ is regarded as element of

$$\mathscr{L}(E^{\hat{\otimes}(l+m)}, \mathscr{E}^*) \text{ or, respectively, of } \mathscr{L}(\mathscr{E}, E^{*\hat{\otimes}(l+m)}).$$

Here

$$\eta_{\Phi,\Psi}(s_1, \mathbf{p}_1, \ldots, s_l, \mathbf{p}_l, s_{l+1}, \mathbf{p}_{l+1}, \ldots, s_{l+m}, \mathbf{p}_{l+m}) \\ \overset{\mathrm{df}}{=} \langle\langle a_{s_1}(\mathbf{p}_1)^+ \ldots a_{s_l}(\mathbf{p}_l)^+ a_{s_{l+1}}(\mathbf{p}_{l+1}) \ldots a_{s_{l+m}}(\mathbf{p}_{l+m})\Phi, \Psi \rangle\rangle$$

is the function which always belongs to $E^{\hat{\otimes}(l+m)}$. (HIDA, OBATA, SAITÔ)

EXAMPLE. Free fields A

$$A(\phi) = A^{(-)}(\phi) + A^{(+)}(\phi) = \Xi_{0,1}(\kappa_{0,1}(\phi)) + \Xi_{1,0}(\kappa_{1,0}(\phi))$$

with the integral kernels $\kappa_{l,m}$ represented by ordinary functions:

$$\kappa_{0,1}(\nu, \mathbf{p}; \mu, x) = \frac{g_{\nu\mu}}{(2\pi)^{3/2}\sqrt{2p^0(\mathbf{p})}} e^{-ip\cdot x}, \quad p = (|p_0(\mathbf{p})|, \mathbf{p}),\, p\cdot p = 0,$$

$$\kappa_{1,0}(\nu, \mathbf{p}; \mu, x) = \frac{g_{\nu\mu}}{(2\pi)^{3/2}\sqrt{2p^0(\mathbf{p})}} e^{ip\cdot x}, \quad p\cdot p = 0,$$

for the free e.m.potential field A (in the Gupta-Bleuler gauge) and

$$\kappa_{0,1}(s, \mathbf{p}; a, x) = \begin{cases} (2\pi)^{-3/2} u_s^a(\mathbf{p}) e^{-ip\cdot x}, \;\; p = (|p_0(\mathbf{p})|, \mathbf{p}),\, p\cdot p = m^2 & \text{if } s = 1,2 \\ 0 & \text{if } s = 3,4 \end{cases},$$

$$\kappa_{1,0}(s, \mathbf{p}; a, x) = \begin{cases} 0 & \text{if } s = 1,2 \\ (2\pi)^{-3/2} v_{s-2}^a(\mathbf{p}) e^{ip\cdot x}, \;\; p\cdot p = m^2 & \text{if } s = 3,4 \end{cases}$$

for the free Dirac spinor field $A = \boldsymbol{\psi}$, and which are in fact the respective complete systems of plane wave solutions of d'Alembert and of Dirac equation.

The standard Wick theorem decomposition holds for the (tensor) product operator

$$:A^{(1)}(x)\dots A^{(n)}(x)::A^{(n+1)}(y)\dots A^{(n+k)}(y):$$

with the kernels of the decomposition given by the contractions

$$\kappa_{l,m}(\phi\otimes\varphi) = \sum_{\kappa'_{l',m'},\kappa''_{l'',m''},k} \kappa'_{l',m'}(\phi)\otimes_k \kappa''_{l'',m''}(\varphi)$$

where in this sum $\kappa'_{l',m'}, \kappa''_{l'',m''}$ range over the kernels respectively of the operators

$$:A^{(1)}(x)\dots A^{(n)}(x): \ \text{and} \ :A^{(n+1)}(y)\dots A^{(n+k)}(y):$$

and

$$l'+l''-k = l, \ \ m'+m''-k = m$$

and where the contractios $\otimes_k$ are performed upon all k pairs of spin-momenta variables in which the first variable in the pair corresponds to an annihilation operator variable and the second one to the creation operator variable or *vice versa*. All these contractions are given by absolutely convergent sums/integrals with respect to the contracted variables. After the contraction, the kernels should be symmetrized in Boson spin-momentum variables and antisymmetrized in the Fermion spin-momentum variables in order to keep one-to-one correspondence between the kernels and operators. ■

This converts the free fields $\mathbb{A}$ and the n-th order contributions

$$S_n(g^{\otimes n}) \qquad \text{and} \qquad \mathbb{A}^{(n)}_{\text{int}}(g^{\otimes n}, \phi)$$

written frequently as

$$S_n(g) \qquad \text{and} \qquad \mathbb{A}^{(n)}_{\text{int}}(g, \phi),$$

to the scattering operator

$$S(g) = \mathbf{1} + \sum_{n=1}^{\infty} \tfrac{1}{n!} S_n(g^{\otimes n}), \quad S(g)^{-1} = \mathbf{1} + \sum_{n=1}^{\infty} \tfrac{1}{n!} \overline{S_n}(g^{\otimes n}),$$
$$\text{or} \quad S(g,h) = \mathbf{1} + \sum_{n=1}^{\infty} \sum_{p=0}^{n} \tfrac{1}{n!} S_{n-p,p}(g^{\otimes (n-p)} \otimes h^{\otimes p})$$

(denoted also by $S(g\mathcal{L})$, $S(g\mathcal{L})^{-1}$ or, respectively, $S(g\mathcal{L} + h\mathbb{A})$) and to the interacting fields

$$\mathbb{A}_{\text{int}}(g, \phi) = \int \frac{i\delta}{\delta h(x)} S(g\mathcal{L} + h\mathbb{A})^{-1} S(g\mathcal{L})\Big|_{h=0} \phi(x) dx,$$

into the finite sums of generalized integral kernel operators

$$\Xi(\kappa_{\ell m}) = \int \kappa_{\ell m}(\mathbf{p}_1, \ldots, \mathbf{p}_\ell, \mathbf{k}_1, \ldots, \mathbf{k}_m)\ \partial^*_{\mathbf{p}_1} \ldots \partial^*_{\mathbf{p}_\ell} \partial_{\mathbf{k}_1} \ldots \partial_{\mathbf{k}_m} d\mathbf{p}_1 \ldots d\mathbf{p}_\ell d\mathbf{k}_1 \ldots d\mathbf{k}_m,$$

e.g. for the contributions S_n:

$$S_n(g^{\otimes n}) = \sum_{\ell, m} \int \kappa_{\ell m}(\mathbf{p}_1, \ldots, \mathbf{p}_\ell, \mathbf{k}_1, \ldots, \mathbf{k}_m; g^{\otimes n})\ \partial^*_{\mathbf{p}_1} \ldots \partial^*_{\mathbf{p}_\ell} \partial_{\mathbf{k}_1} \ldots \partial_{\mathbf{k}_m} d\mathbf{p}_1 \ldots d\mathbf{p}_\ell d\mathbf{k}_1 \ldots d\mathbf{k}_m$$

$$= \int \mathrm{d}^4 x_1 \ldots \mathrm{d}^4 x_n\, S_n(x_1, \ldots, x_n)\, g(x_1) \ldots g(x_n),$$

with vector-valued distributional kernels $\kappa_{\ell m}$ in the sense of Obata, with the values in the distributions V^* over the test nuclear space

$$V = \mathscr{E} \ni \phi, \quad \text{or} \quad V = \mathscr{E}^{\otimes\,(n-p)} \otimes (\oplus_1^d \mathscr{E})^{\otimes\,p} \ni g^{\otimes\,(n-p)} \otimes h^{\otimes\,p},$$

$$\text{or} \quad V = \mathscr{E}^{\otimes\,n} \ni g^{\otimes\,n} \quad \text{or, respectively,} \quad V = \mathscr{E}^{\otimes\,n} \otimes (\oplus_1^d \mathscr{E}) \ni g^{\otimes\,n} \otimes \phi$$

with

$$\mathscr{E} = \mathcal{S}(\mathbb{R}^4; \mathbb{C}).$$

Each of the 3-dim Euclidean integration $d\mathbf{p}_i$ with respect to the spatial momenta $\mathbf{p}_i$ components $\mathbf{p}_{i1}, \mathbf{p}_{i2}, \mathbf{p}_{i3}$, also includes here summation over the corresponding discrete spin components $s_i \in (1, 2, \ldots)$ hidden under the symbol $\mathbf{p}_i$.

$$\Xi(\kappa_{\ell m}) \in \begin{cases} \mathscr{L}\big(V, \mathscr{L}((E),(E)^*)\big), \\ \mathscr{L}\big(V, \mathscr{L}((E),(E))\big), \end{cases}$$

if and only if

$$\begin{cases} \kappa_{\ell m} \in \mathscr{L}\big(E^{\widehat{\otimes}\,(\ell+m)}, V^*\big), \\ \kappa_{\ell m} \text{ can be extended to a separately cont. map} : E^{*\widehat{\otimes}\ell} \times E^{\widehat{\otimes}m} \longrightarrow V^*, \end{cases}$$

respectively (HIDA, OBATA).

Let E, E_1, F, F_1 be t.v.s. such that E, F are, respectively dense in E_1, F_1. Suppose that $\mathfrak{S}$ is a family of bounded subsets of E with the property that $\mathfrak{S}_1$ covers E_1 , where $\mathfrak{S}_1$ denotes the family of the closures, taken in E_1 , of all subsets in $\mathfrak{S}$; analogously let $\mathfrak{F}, \mathfrak{F}_1$ be such families in F, F_1; finally, let G be a quasi-complete Hausdorff t.v.s. Under these assumptions, the following extension theorem holds:

Every $(\mathfrak{S}, \mathfrak{F})$-hypocontinuous bilinear mapping of $E \times F$ into G has a unique extension to $E_1 \times F_1$ (and into G) which is bilinear and$(\mathfrak{S}_1, \mathfrak{F}_1)$-hypocontinuous.

The class to which the operators S_n and $\mathbb{A}_{\text{int}}^{(n)}$ belong, expressed in terms of the Hida test space, depend on the fact if there are massless free fields present in the interaction Lagrange density operator $\mathcal{L}$ or not. Namely:

$$S_n \in \begin{cases} \mathscr{L}\big(\mathscr{E}^{\otimes n}, \mathscr{L}((E),(E))\big), & \text{if all fields in } \mathcal{L} \text{ are massive,} \\ \mathscr{L}\big(\mathscr{E}^{\otimes n}, \mathscr{L}((E),(E)^*)\big), & \text{if there are massless fields in } \mathcal{L}. \end{cases}$$

and the same holds for $S_{n,p}$ with $\mathscr{E}^{\otimes n}$ repaced by $\mathscr{E}^{\otimes n} \otimes (\oplus_1^d \mathscr{E})^{\otimes p}$ if both g and h, are, respectively, $\mathbb{C}$ and $\mathbb{C}^d$-valued Schwartz test functions. But the same distribution valued kernels of $S_{n,p}$ can be evaluated at the *Grassmann-valued test functions* h, in the sense of Berezin, so that in this case

$$S_{n,p} \in \begin{cases} \mathscr{L}\big((E),(E)\big) \otimes \mathscr{L}(\mathscr{E}^{\otimes n} \otimes \mathscr{E}^p, \mathcal{E}^{p\,*}), & \text{if all fields in } \mathcal{L} \text{ are massive,} \\ \mathscr{L}\big((E),(E)^*\big) \otimes \mathscr{L}(\mathscr{E}^{\otimes n} \otimes \mathscr{E}^p, \mathcal{E}^{p\,*}), & \text{if there are massless fields in } \mathcal{L}, \end{cases}$$

with $\mathcal{E}^{p\,*}$ being the subspace of grade p of the *abstract Grassmann algebra* $\oplus_p \mathcal{E}^{p\,*}$ *with inner product and involution* in the sense of Berezin. $\mathscr{E}^p$ denotes the space of Grassmann-valued test functions h^p of grade p due to Berezin, and replacing ordinary test functions $h^{\otimes p}$. Here $\mathscr{L}(E_1, E_2)$ denotes the linear space of linear continuous operators $E_1 \longrightarrow E_2$ endowed with the natural topology of uniform convergence on bounded sets.

BOGOLIUBOV'S causality axioms (I)-(V) read

(I) $$S(g_1 + g_2) = S(g_2)S(g_1), \text{ whenever } \operatorname{supp} g_1 \preceq \operatorname{supp} g_2.$$

(II) $$U_{a,\Lambda}S(g)U^+_{b,\Lambda} = S(T_{b,\Lambda}g), \; T_{b,\Lambda}g(x) = g(\Lambda x + b).$$

(III) $$\eta S(g)^+\eta = S(g)^{-1}.$$

(IV) $$S_1(x_1) = i\mathcal{L}(x_1)$$

where $\mathcal{L}(x_1)$ is the interaction Lagrangian density operator.

(V) The value of the retarded part of a vector valued kernel should coincide with the natural formula given by the multiplication by the step theta function on a space-time test function, whenever the natural formula is meaningful for this test function.

The axiom (V) has been added by EPSTEIN and GLASER.

BOGOLIUBOV'S AXIOMS

$$\text{(I)} \qquad S_n(x_1,\ldots,x_n) = S_k(x_1,\ldots,x_k)S_{n-k}(x_{k+1},\ldots,x_n), \quad \text{whenever } \{x_{k+1},\ldots,x_n\} \preceq \{x_1,\ldots,x_k\},$$

$$\text{(II)} \qquad U_{b,\Lambda}S_n(x_1,..,x_n)U^+_{b,\Lambda} = S_n(\Lambda^{-1}x_1 - b,..,\Lambda^{-1}x_n - b),$$

$$\text{(III)} \qquad \overline{S}_n(x_1,\ldots,x_n) = \eta S_n(x_1,\ldots,x_n)^+\eta,$$

$$\text{(IV)} \qquad S_1(x_1) = i\mathcal{L}(x_1),$$

(V) The singularity degree of the retarded part of a kernel should coincide with the singularity degree of this kernel, for the kernels of the generalized integral kernel and causal operators $D_{(n)}$ which are equal to linear combinations of products of the generalized operators S_k,

INDUCTIVE STEP (EPSTEIN-GLASER)

$$\{S_k\}_{k\leq n-1} \longrightarrow S_n(Z,x_n) = S(Z,x_n) \qquad (X\cup Y = \{x_1,\dots,x_n\} = Z,\ X\cap Y = \emptyset)$$

$$A'_{(n)}(Z,x_n) = \sum_{X\sqcup Y=Z, X\neq\emptyset} \overline{S}(X)S(Y,x_n),\quad R'_{(n)}(Z,x_n) = \sum_{X\sqcup Y=Z, X\neq\emptyset} S(Y,x_n)\overline{S}(X),$$

$$A_{(n)}(Z,x_n) = \sum_{X\sqcup Y=Z} \overline{S}(X)S(Y,x_n) = A'_{(n)}(Z,x_n) + S(x_1,\dots,x_n),$$

$$R_{(n)}(Z,x_n) = \sum_{X\sqcup Y=Z} S(Y,x_n)\overline{S}(X) = R'_{(n)}(Z,x_n) + S(x_1,\dots,x_n),$$

$$D_{(n)} = R'_{(n)} - A'_{(n)} = R_{(n)} - A_{(n)} \qquad \text{causally supported}$$

$R_{(n)}$ – retarded part of $D_{(n)}$ $\qquad$ $A_{(n)}$ – advanced part of $D_{(n)}$

$$\boxed{S(x_1,\dots,x_n) = R_{(n)}(x_1,\dots,x_n) - R'_{(n)}(x_1,\dots,x_n)}$$

INDUCTIVE STEP (A SIMPLIFICATION)

$$
\begin{aligned}
&S(x_1, \ldots, x_n) \\
&= \theta(x_1 - x_n) \ldots \theta(x_{n-1} - x_n) D_{(n)}(x_1, \ldots, x_n) \Omega(x_1 - x_n) \ldots \Omega(x_{n-1} - x_n) \\
&\qquad - R'_{(n)}(x_1, \ldots, x_n).
\end{aligned}
$$

The scalar distributions in $D_{(n)}$ have the form of the product

$$
d_1(x_1 - x_n) \ldots d_{n-1}(x_{n-1} - x_n)
$$

in which d_k are equal to product of pairings (enetring Wick product decomposition of $\mathcal{L}(x)\mathcal{L}(y)$) or their advanced and retarded parts, in the space-time variables $x_k - x_n$, $k = 1, \ldots, n-1$.

INDUCTIVE STEP (A SIMPLIFICATION)

Computation of the retarded part

$$\theta(x_1 - x_n)\ldots\theta(x_{n-1} - x_n)D_{(n)}(x_1,\ldots,x_n)\Omega(x_1 - x_n)\ldots\Omega(x_{n-1} - x_n)$$

of the causal $D_{(n)}$ factorizes and can be done factor-by-factor in one space-time variable, separately for each of the basic distributions d_k, provided the splitting of the product d_k of parings, equal to the scalar contraction of the product of the free field kernels, is well defined, which indeed is the case, as I have shown.

$$D_0^{\rm av} * \Pi_\mu^{\rm av\ \mu_k} * \ldots * D_0^{\rm av} * \Pi_{\mu_1}^{\rm av\ \nu} * D_0^{\rm av} * :\psi^\sharp \gamma_\nu \psi:,$$

$$\Big(D_{0\,\epsilon}^{\rm av} * \Pi_\mu^{\rm av\ \mu_k} * \ldots * D_{0\,\epsilon}^{\rm av} * \Pi_\nu^{\rm av\ \mu_1} * D_{0\,\epsilon}^{\rm av} * \big[\kappa_{\ell_1,m_1}^\sharp(\xi_1)\gamma^\nu \dot{\otimes} \kappa_{\ell_2,m_2}(\xi_2)\big]\Big)(x)$$
$$= \sum_{s_1,s_2} \int \mathrm{d}^3\mathbf{p}_1 \mathrm{d}^3\mathbf{p}_2 \frac{\xi_1(s_1,\mathbf{p}_1)\xi_2(s_2,\mathbf{p}_2)u_{s_1}^{\pm}(\mathbf{p}_1)^\sharp \gamma^\nu u_{s_2}^{\mp}(\mathbf{p}_2)}{[(\pm p_1 \pm p_2)^2 + i\epsilon\,(\pm p_{10} \pm p_{20})]^{k+1}} \times$$
$$\times\ \widetilde{\Pi_\mu^{\rm av\ \mu_k}}(\pm p_1 \pm p_2)\widetilde{\Pi_{\mu_k}^{\rm av\ \mu_{k-1}}}(\pm p_1 \pm p_2)\ldots\widetilde{\Pi_\nu^{\rm av\ \mu_1}}(\pm p_1 + \pm p_2)e^{i(\pm p_1 \pm p_2)\cdot x}$$

$$p_1, p_2 \in \mathscr{O}_{m,0,0,0} = \{p:\ \ p\cdot p = m^2,\ p_0 > 0\}.$$

For the "natural" normalization in the Epstein-Glaser splitting, and in case $m \neq 0$, the singularity appearing in the limit

$$\frac{1}{[p^2+\epsilon p_0]^{k+1}} \overset{\epsilon\to 0}{\longrightarrow} \frac{1}{(p^2)^{k+1}} - \mathrm{sgn}\,(p_0)\,\frac{i\pi(-1)^k}{k!}\delta^{(k)}(p^2),$$

is cancelled by the Fourier transform $\widetilde{\Pi^{\rm av\,\mu\nu}}$ of $\Pi^{\rm av\,\mu\nu}$, as $\widetilde{\Pi^{\rm av\,\mu\nu}} = (\frac{p^\mu p^\nu}{p^2} - g^{\mu\nu})\widetilde{\Pi}(p)$ with a regular $\widetilde{\Pi}$ in the vicinity of the cone $p^2 = 0$, and equal there to $\widetilde{\Pi}(p) = [p^2]^2 g_0(p)$ with still regular g_0 there.

THEOREM

For QED with massive charged field the higher order contributions $\psi^{(n)}_{int}(g^{\otimes n})$ *and* $A^{(n)}_{int}(g^{\otimes n})$ *to interacting fields* ψ_{int} *and* A_{int} *in the adiabatic limit* $g \to 1$ *are well-defined as sums of generalized integral kernel operators with vector valued kernels in the sense of OBATA,*

$$\lim_{g\to 1} \psi^{(n)}_{\text{int}}(g^{\otimes n}), \lim_{g\to 1} A^{(n)}_{\text{int}}(g^{\otimes n}) \in \mathscr{L}\big(\oplus_1^d \mathscr{E}, \mathscr{L}((E),(E)^*)\big),$$

and this is the case only for the "natural" choice in the EPSTEIN-GLASER splitting in the construction of the scattering operator.

But:

THEOREM

For QED with massless charged field the higher order contributions to interacting fields are not well-defined, even as sums of generalized integral kernel operators in the sense of OBATA, and for no choice in the EPSTEIN-GLASER splitting in the construction of the scattering operator.

- Each higher order contribution $A_{\text{int}}^{(n)}$ to interacting exists as a generalized integral kernel operator $\Xi(\kappa_{\ell m})$ in the adiabatic limit $g \to 1$ in BOGOLIUBOV'S causal perturbative QED with HIDA operators.

- The UV and IR asymptotics should be $SL(2,\mathbb{C})$ invariant.

- The direct integral decomposition $\int U_\chi \, d\chi$ of the repreesentation U of $SL(2,\mathbb{C}) \subset T_4 \ltimes SL(2,\mathbb{C})$ determines naturally direct integral decomposition $\int \Xi(\kappa_{\chi\, \ell m}) \, d\chi$ of $\Xi(\kappa_{\ell m}) = A_{\text{int}}^{(n)}$.

- Decomposition components

$$A_{\chi\,\text{int}}(x) = \sum \kappa_{\chi\, \ell m}(l_1, m_1, \ldots, l_{\ell+m}, m_{\ell+m}; x) a^{+}_{\chi\, l_1, m_1} \ldots a_{\chi\, l_{\ell+m}, m_{\ell+m}}$$

of A_{int} act in the FOCK spaces over the UV-asymptotcally homogeneous states of UV-asymptotic homogeneity degree determined by the decompositon parameter χ.

$$\mathcal{S}(\mathbb{R};\mathbb{C}) \subset L^2(\mathbb{R};\mathbb{C}) \subset \mathcal{S}(\mathbb{R};\mathbb{C})^*$$

$$\begin{aligned}\langle F, f\rangle = \int\limits_{\mathrm{Spec}=\mathbb{R}} F_\chi \mathscr{F} f(\chi)\,\mathrm{d}\chi &= \int\limits_{\mathrm{Spec}=\mathbb{R}} \langle F_\chi, \mathscr{F} f(\chi)\rangle\,\mathrm{d}\chi \\ &= \int\limits_{\mathrm{Spec}=\mathbb{R}} \big\langle \mathscr{F} F(\chi), (f)_\chi \big\rangle\,\mathrm{d}\chi, \quad f \in \mathcal{S}(\mathbb{R};\mathbb{C}).\end{aligned}$$

DEFINITION

Let

$$\Phi, \Psi \in (E), \quad \Phi = \int \Phi_\chi\,\mathrm{d}\chi, \quad \Psi = \int \Psi_\chi\,\mathrm{d}\chi$$

be any two elements of the test Hida space with their direct integral decompositions. Let $\mathrm{d}\chi$ be a σ-measure on the spectrum of the decomposition of the representation of $SL(2,\mathbb{C})$ acting in the single particle Hilbert space $\mathcal{H}'$. We say that the generalized integral kernel operator $\Xi(\kappa_{lm})$ is equal to the direct integral

$$\Xi(\kappa_{lm}) = \int \Xi_\chi(\kappa_{\chi\, lm})\,\mathrm{d}\chi$$

of (discrete-) integral kernel operators $\Xi_\chi(\kappa_{\chi\, lm})$, acting in the Fock spaces over the single particle Gelfand triples $E_\chi \subset \mathcal{H}'_\chi \subset E^*_\chi$, if

$$\begin{aligned}\int \langle\langle \Xi_\chi(\kappa_{\chi\, lm}(\phi))\Phi_\chi, \Psi_\chi\rangle\rangle\,\mathrm{d}\chi &= \int \Big\langle \kappa_{\chi\, lm}(\phi), (\eta_{\Phi,\Psi})_\chi \Big\rangle\,\mathrm{d}\chi \\ &= \langle \kappa_{lm}(\phi), \eta_{\Phi,\Psi}\rangle = \big\langle\big\langle \Xi(\kappa_{lm}(\phi))\Phi, \Psi \big\rangle\big\rangle\end{aligned}$$

for all

$$\Phi, \Psi \in (E), \phi \in \mathscr{E}.$$