Approximation of π by rational numbers.

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Criterion of irrationality

Theorem

Let α be a real number and there exists a sequence $(q_n,\ p_n) \in \mathbb{Z}^2$ such that

$$0 < |q_n\alpha - p_n| \longrightarrow 0, \qquad n \to \infty.$$

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Доказательство.

Assume that $\alpha = \frac{a}{b} \in \mathbb{Q},\ \ b > 0$. Then

$$|q_n\alpha - p_n| = \frac{|q_n a - p_n b|}{b} \geq \frac{1}{b}.$$

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Example:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = 2,718281828459045235360287471352\ldots$$

Euler, 1737:

$$e = [2,1,2,1,1,4,1,1,6,1,1,\ldots] = 2 + \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\ddots}}}$$

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Fourier, 1815: $e = \sum_{k=0}^{\infty} \frac{1}{k!} \qquad q_n = n!, \qquad p_n = n! \sum_{k=0}^{n} \frac{1}{k!},$

$$0 < q_n e - p_n = n! \sum_{k>n} \frac{1}{k!} < \frac{2}{n+1}$$

Measure of irrationality

Davis (1978): 1. For any $\varepsilon > 0$ the inequality

$$\left| e - \frac{p}{q} \right| < \left(\frac{1}{2} + \varepsilon \right) \frac{\ln \ln q}{q^2 \ln q}$$

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2. For any $\varepsilon > 0$ the inequality

$$\left| e - \frac{p}{q} \right| < \left(\frac{1}{2} - \varepsilon \right) \frac{\ln \ln q}{q^2 \ln q}$$

has only finitely many solutions $\frac{p}{q}$.

Exponent of irrationality

Exponent of irrationality $\mu(\alpha)$ of a real number α is defined as supremum of the set of all $\varkappa$ such that the inequality

$$\left|\alpha - \frac{p}{q}\right| < q^{-\varkappa} \tag{1}$$

has infinitely many solutions in rational numbers $\frac{p}{q}$.

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3. For any integers a, b, c, d with $ad - bc \neq 0$ we have

$$\mu\left(\frac{a\alpha + b}{c\alpha + d}\right) = \mu(\alpha).$$

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- (Euler) $\mu(e) = 2$.

The number π

Mahler, 1953: $\mu(\pi) \leq 30$.
Mahler proved that for any $\kappa > 30$ the inequality $|\alpha - p/q| < q^{-\kappa}$ has only finitely many solutions and that for any
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G. Chudnovsky, 1982:

$$\mu(\pi) \leq 5 - 5 \cdot \frac{5 + 6\ln\left(2\cos(\pi/24)\right)}{5 + 6\ln\left(2\sin(\pi/24)\right)} = 19,88999444\ldots.$$

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$$|p + q\pi + r\ln 2| > H^{-\varkappa}, \; H = max(|q|, |r|) \ge H_0, \; \varkappa > 7,0160....$$

Corollary:
$\mu(\pi) \le 8,016045...$ $\quad q \ge q_0 \quad \Rightarrow \quad |\pi - p/q| \ge q^{-9}$.

The number π

M. Hata, 1993: $\mu(\pi) \leq 8,016045....$

V. Salikhov, 2008: $\mu(\pi) < 7,6063\ldots.$

D. Zeilberger, W. Zudilin, 2020: $\mu(\pi) < 7,1032\ldots.$

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Let α *be a real number and there exists a sequence* $(q_n,\ p_n) \in \mathbb{Z}^2$ *such that*

$$0 < |q_n\alpha - p_n| \longrightarrow 0, \qquad n \to \infty.$$

Then α *is irrational.*

Theorem (Hata, 1993)

Let α *be real irrational number and a sequence of pairs of integers* q_n, p_n *satisfies*

$$\lim_{n\to+\infty} \frac{1}{n} \ln |q_n| = \sigma > 0, \qquad \limsup_{n\to+\infty} \frac{1}{n} \ln |q_n\alpha - p_n| \le -\tau, \quad \tau > 0.$$

Then

$$\mu(\alpha) \le 1 + \frac{\sigma}{\tau}.$$

THEOREM 1.1. *For any $\varepsilon > 0$, there exists a positive integer $H_0(\varepsilon)$ such that*

$$|p + q\pi + r\log 2| \geq H^{-\mu-\varepsilon}$$

for any integers p, q, r with $H \equiv \max\{|q|, |r|\} \geq H_0(\varepsilon)$, where the exponent μ is given by

$$\mu = -\frac{2\log\alpha_0 + 6 - \pi/\sqrt{3}}{2\log\alpha_1 + 6 - \pi/\sqrt{3}}$$

with

$$\alpha_j = \frac{10}{9}\sqrt{18265}\cos\left(\theta_0 + \frac{4j\pi}{3}\right) + \frac{92}{3}\sqrt{6} \quad \text{and} \quad \theta_0 = \frac{1}{3}\arctan\left(\frac{23\sqrt{69}}{1209303}\right).$$

(Numerically one has $\mu = 7.016045\ldots$)

Lemma 2.1. *Let m be a fixed non-negative integer. Let γ_1 and γ_2 be real numbers satisfying*

$$q_n\gamma_1 - p_n = \varepsilon_n \qquad \text{and} \qquad q_n\gamma_2 - r_n = \delta_n$$

for some $p_n,\, q_n,\, r_n \in \mathbb{Z} + i\sqrt{m}\mathbb{Z}$ for all $n \geq 1$. Suppose that

$$\lim_{n\to\infty} \frac{1}{n} \log|q_n| = \sigma\,, \quad \lim_{n\to\infty} \frac{1}{n} \log|\varepsilon_n| = -\tau\,, \quad \lim_{n\to\infty} \frac{1}{n} \log|\delta_n| = -\tau'$$

for positive numbers σ, τ and τ' with $\tau' \geq \tau$. Suppose further that there exist infinitely many n's satisfying $\delta_n/\varepsilon_n \neq \varrho$ for any rational number ϱ. Then the numbers 1, γ_1 and γ_2 are linearly independent over $\mathbb{Q}$. More precisely, for any $\varepsilon > 0$, there exists a positive integer $H_0(\varepsilon)$ such that

$$|p + q\gamma_1 + r\gamma_2| \geq H^{-\sigma/\tau-\varepsilon}$$

for any integers p, q, r with $H \equiv \max\{|q|, |r|\} \geq H_0(\varepsilon)$.

LEMMA 2.2. *There exists a positive integer D_n such that*

$$\frac{D_n}{j+k+l-3n}\binom{2n}{j}\binom{2n}{k}\binom{2n}{l} \in \mathbb{Z} \tag{2.2}$$

for all integers $0 \leq j, k, l \leq 2n$ *with* $j + k + l \neq 3n$ *and that*

$$\lim_{n\to\infty} \frac{1}{n} \log D_n = \frac{1}{2}\left\{6 - \frac{\pi}{\sqrt{3}} + \log \frac{27}{16}\right\} \equiv \kappa, \quad \textit{say}. \tag{2.3}$$

(*Numerically one has* $\kappa = 2.35472\ldots$)

$$\Delta_1(n) = \prod_{\substack{p \text{ prime} \\ p \le \sqrt{3n}}} p^{\left[\frac{\log(3n)}{\log p}\right]}, \quad \Delta_2(n) = \prod_{\substack{p \text{ prime} \\ p \le 3n}} p, \quad \Delta_3(n) = \prod_{\substack{p \text{ prime} \\ p \in T_n}} p$$

for all $n \ge 1$. The above argument implies that the integer

$$D_n = \frac{\Delta_1(n)\Delta_2(n)}{\Delta_3(n)}$$

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1}{n} \log \Delta_3(n) &= \sum_{i=0}^{\infty} \lim_{n\to\infty} S_i(n) = \sum_{i=0}^{\infty} (c_i - b_i) \\
&= \frac{\Gamma'(2/3)}{\Gamma(2/3)} - \frac{\Gamma'(1/2)}{\Gamma(1/2)} = \frac{1}{2}\left\{ \frac{\pi}{\sqrt{3}} - \log \frac{27}{16} \right\},
\end{aligned}
$$

$$\log \Delta_1(n) = O(\sqrt{n}) \text{ and } \log \Delta_2(n) \sim 3n \text{ as } n \text{ tends to } +\infty,$$

$$\int_{\Gamma} (F(a_1, a_2, a_3; z))^n \frac{dz}{z} \tag{1.3}$$

where

$$F(a_1, a_2, a_3; z) = \frac{(z-a_1)^2(z-a_2)^2(z-a_3)^2}{z^3} \tag{1.4}$$

with non-zero distinct complex numbers a_1, a_2 and a_3. By taking $a_1 = 1$, $a_2 = 2$ and $a_3 = 1 + i$, the integral (1.3) enables us to obtain the following

3. Proof of Theorem 1.1. Let $\Gamma_{z,w}$ be a smooth oriented path departing from z, arriving at w, and contained in $\mathbb{C}-\{0\}$. We then consider the integral

$$\begin{aligned} I_n(\Gamma_{z,w}) &\equiv \int_{\Gamma_{z,w}} (F(a_1,a_2,a_3;z))^n \frac{dz}{z} \\ &= \sum_{j=0}^{2n}\sum_{k=0}^{2n}\sum_{l=0}^{2n} B_{j,k,l}\binom{2n}{j}\binom{2n}{k}\binom{2n}{l}\int_{\Gamma_{z,w}} z^{j+k+l-3n-1}dz, \end{aligned}$$

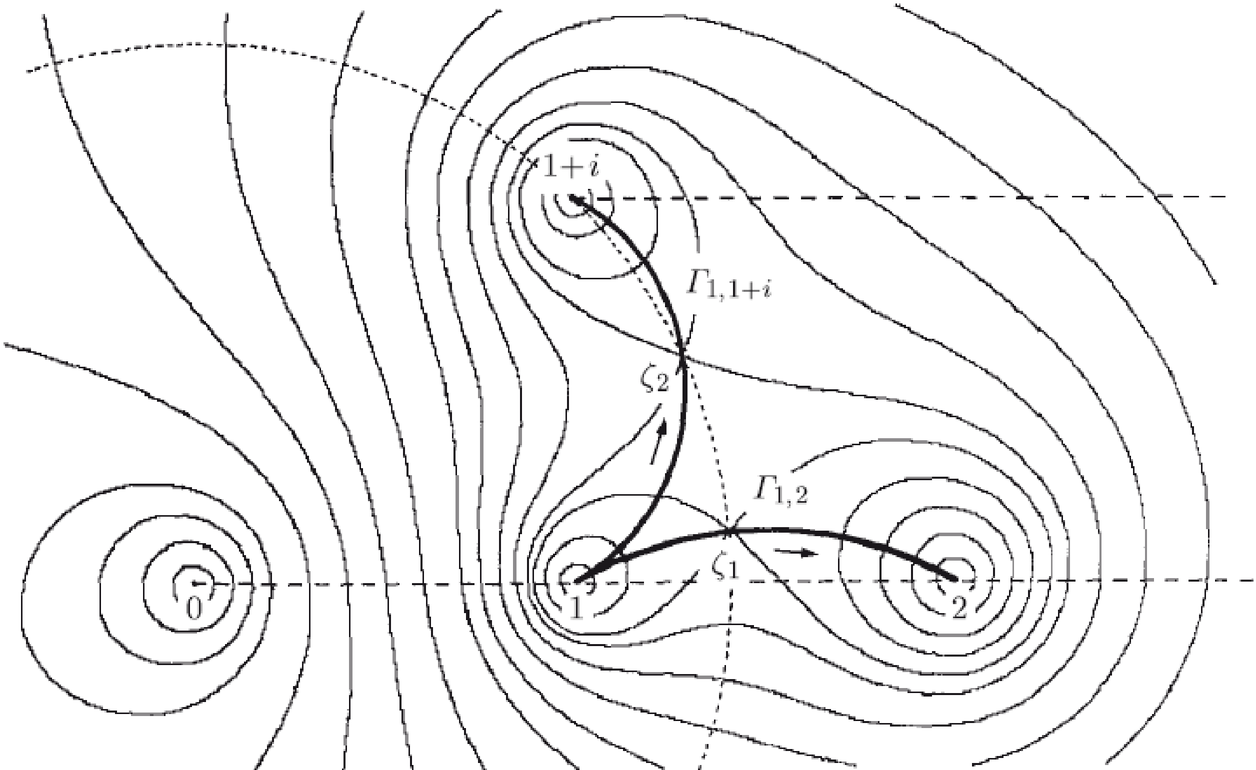
where

$$B_{j,k,l} = (-1)^{j+k+l}a_1^{2n-j}a_2^{2n-k}a_3^{2n-l}.$$

$$
\begin{aligned}
(3.1)\qquad & I_n(\Gamma_{z,w}) \\
&= \sum_{j+k+l\neq 3n} \frac{B_{j,k,l}}{j+k+l-3n}\binom{2n}{j}\binom{2n}{k}\binom{2n}{l}(w^{j+k+l-3n}-z^{j+k+l-3n}) \\
&\quad + \sum_{j+k+l=3n} B_{j,k,l}\binom{2n}{j}\binom{2n}{k}\binom{2n}{l}\int_{\Gamma_{z,w}}\frac{dz}{z} \\
&\equiv u_n(z,w)+v_n\int_{\Gamma_{z,w}}\frac{dz}{z}, \quad \text{say.}
\end{aligned}
$$

For the proof of Theorem 1.1 we choose $a_1 = 1$, $a_2 = 2$ and $a_3 = 1 + i$.

$$\int_{\Gamma_{1,2}} \frac{dz}{z} = \log 2 \quad \text{and} \quad \int_{\Gamma_{1,1+i}} \frac{dz}{z} = \frac{1}{2}\log 2 + \frac{\pi}{4} i\,.$$



$$p_n = -2iD_n\{u_n(1,2) - 2u_n(1,1+i)\}, \qquad q_n = D_n v_n, \qquad r_n = -D_n u_n(1,2),$$

we obtain

$$q_n \pi - p_n = 2iD_n\{I_n(\Gamma_{1,2}) - 2I_n(\Gamma_{1,1+i})\} \equiv \varepsilon_n$$

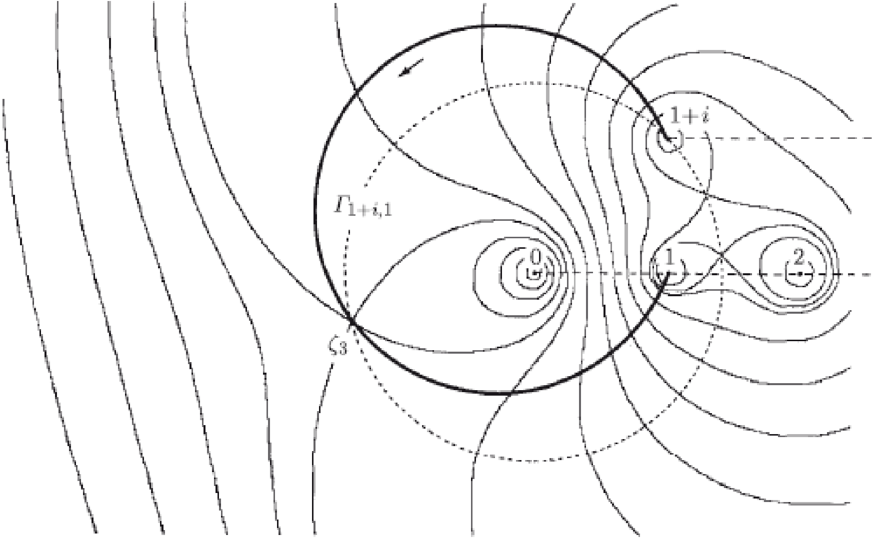
and

$$q_n \log 2 - r_n = D_n I_n(\Gamma_{1,2}) \equiv \delta_n, \quad \text{say},$$

where $p_n, q_n, r_n \in \mathbb{Z} + i\mathbb{Z}$. Then it follows from Lemmas 2.2 and 2.4 that

$$v_n = \sum_{j+k+l=3n} B_{j,k,l} \binom{2n}{j} \binom{2n}{k} \binom{2n}{l} = \frac{1}{2\pi i} \int_C (F(z))^n \frac{dz}{z},$$

where $C = \Gamma_{1,1+i} \cup \Gamma_{1+i,1}$ is a closed oriented curve enclosing the origin and $\Gamma_{1+i,1}$ is the path illustrated in Figure 2 through the saddle ζ_3. Hence



1+i
Γ1+i,1
0
1
2
ζ3

Salikhov, 2008: $\mu(\pi) < 7,6063\ldots.$

$$R(x) = \frac{(x^2 - 8x + 20)^{3n}(x-5)^{3n}(x^2 - 12x + 40)^{3n}}{x^{5n+1}(10-x)^{5n+1}}$$

The number π

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$$R(10-x) = R(x)$$

$$R(x) = P(x) + \sum_{j=1}^{5n+1} \left(\frac{a_j}{x^j} + \frac{a_j}{(10-x)^j} \right), \; P(x) = \sum_{\nu=0}^{5n-2} b_\nu x^\nu, \tag{3}$$

где все $a_j \in \mathbb{Q}$, все $b_\nu \in \mathbb{Z}$.

$$J = J_1 + J_2 + J_3, \tag{4}$$

где

$$J_1 = \int\limits_{4-2i}^{4+2i} P(x)dx,\ r_1 = \frac{1}{i} J_1 \in \mathbb{Q}, \tag{5}$$

$$J_2 = \int\limits_{4-2i}^{4+2i} \left(\sum_{j=2}^{5n+1} \left(\frac{a_j}{x^j} + \frac{a_j}{(10-x)^j} \right) \right) dx,\ r_2 = \frac{1}{i} J_2 \in \mathbb{Q}, \tag{6}$$

$$
\begin{aligned}
J_3 &= \int_{4-2i}^{4+2i} \left(\frac{a_1}{x} + \frac{a_1}{10-x} \right) dx = a_1 \left(\log x - \log(10-x) \right) \Big|_{4-2i}^{4+2i} \\
&= a_1 \left(\log(4+2i) - \log(4-2i) + \log(6+2i) - \log(6-2i) \right) \\
&= a_1 i \left(2 \operatorname{arctg} \frac{1}{2} + 2 \operatorname{arctg} \frac{1}{3} \right) = \frac{1}{2} a_1 i \pi,
\end{aligned}
$$

$$
\frac{1}{i} J_3 = \frac{1}{2} a_1 \pi, \tag{7}
$$

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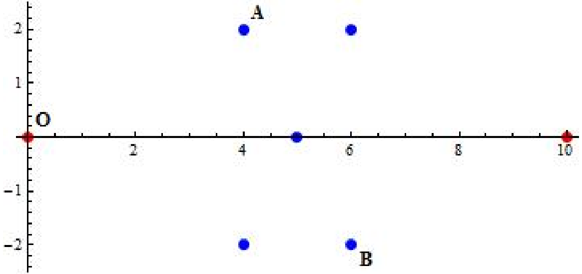

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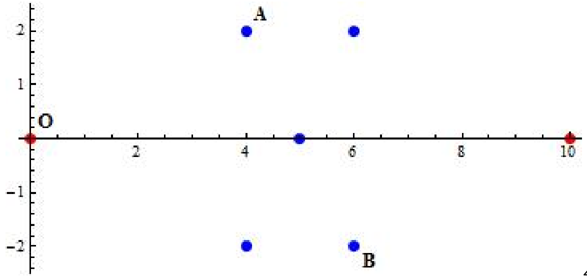

$$\angle AOB = \frac{\pi}{4}$$

1. **Salikhov, 2008:** $\mu(\pi) < 7,6063\ldots$.

$$I_1(n) = i\int_{4-2i}^{4+2i}\left(\frac{(x^2-8x+20)^3(x-5)^3(x^2-12x+40)^3}{x^5(x-10)^5}\right)^n \cdot$$

$$\cdot\frac{dx}{x(x-10)} = u_n\pi - v_n, \quad u_n, v_n \in \mathbb{Q}.$$

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$$\cdot\frac{dx}{x(x-10)} = u_n\pi - v_n, \quad u_n, v_n \in \mathbb{Q}.$$

2. **Zeilberger and Zudilin, 2020:** $\mu(\pi) < 7,1032\ldots.$

$$I_2(n) = 5i\int_{4-2i}^{4+2i}\left(\frac{(x^2-8x+20)^2(x-5)^2(x^2-12x+40)^2}{x^3(10-x)^3}\right)^n\cdot$$

$$\frac{dx}{x(x-10)} = b_n\pi - a_n, \quad a_n, b_n \in \mathbb{Q}.$$

The number π.

Denote D_m — least common multiple of numbers $1, 2, 3, \ldots, m$,

$$P_n = \{p \,|\, \max(5, \sqrt{3n}) < p \leq 2n, \ \frac{1}{2} \leq \left\{\frac{n}{p}\right\} < \frac{2}{3}\},$$

$$\Phi_n = \prod_{p \in P_n} p, \qquad L_n = \frac{D_{4n}}{\Phi_n} \in \mathbb{Z}.$$

Then

$$2^{-[5n/2]+2} \cdot L_n \cdot I_2(n) = B_n\pi - A_n \in \mathbb{Z} + \mathbb{Z}\pi.$$

The number π

There are two possibilities to finish the proof of the upper bound

$$\mu(\pi) \leq 7,103205\ldots$$

with Hata's theorem.

1. One can find with computer a linear recurrence equation for $I_2(n)$ with polynomial in n coefficients. The sequences A_n and B_n are solutions of this equation. After that one can use Poincare's theorem for calculation of asymptotics of $I_2(n)$ and B_n.

2. One can find an integral representation for B_n and after that with saddle point method fo calculate the asymptotic for two sequences $I_2(n)$ and B_n.