Some applications of ill-posed boundary value problems to geometry, analysis and mathematical physics

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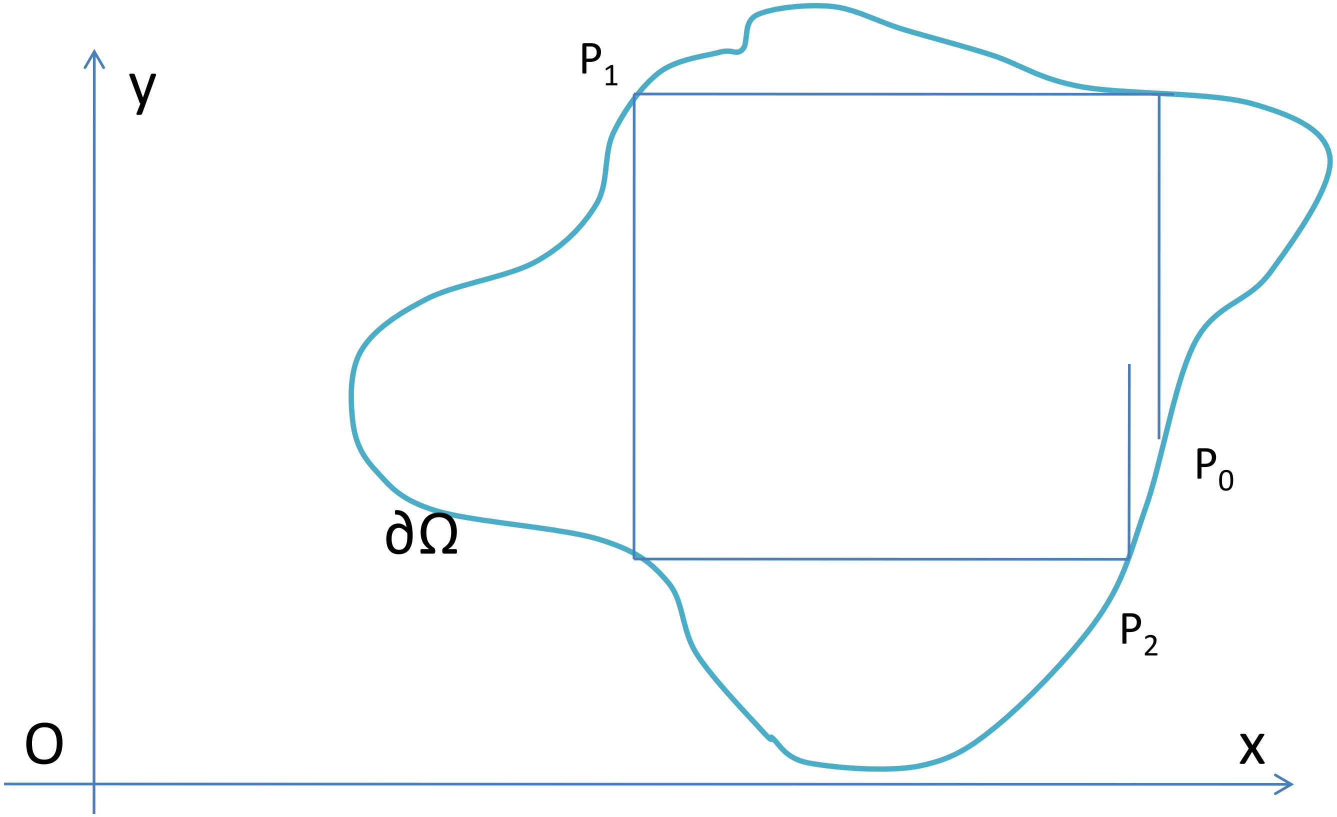
The communication is devoted to a connection between ill-posed boundary value problems in a bounded semialgebraic domain for partial differential equations and some problems as the Poncelet problem from geometry or the Abel equation from algebra, revealed by author and A.S. Zhedanov from Donetsk Institute of Physics and Technology. The Poncelet problem is one of famous problems of projective geometry and it by itself has numerous links with a set of different problems of analysis and physics. We will consider connections with the Dirichlet problem for the string equation $u_{xy} = 0 \quad \text{in} \quad \Omega, \; u|_C = \phi \quad \text{on} \quad C = \partial\Omega$ in a bounded domain Ω. The solution uniqueness of this Dirichlet problem is connected with properties of so-called the John automorphism $T : \partial\Omega \to \partial\Omega$. We consider this problem in a bounded semialgebraic domain, the boundary of which is given by so-called bi-quadratic algebraic curve

$$F(x, y) := \sum_{i,k=0}^{2} a_{ik}x^i y^k = a_{22}x^2y^2 + a_{21}x^2y + ... = 0. \;\; (C = \partial\Omega)$$

Fritz John mapping for the Dirichlet boundary value problem in Ω

$$u_{xy} = 0, \; u\,|_{\partial\Omega} = 0 \qquad (1)$$



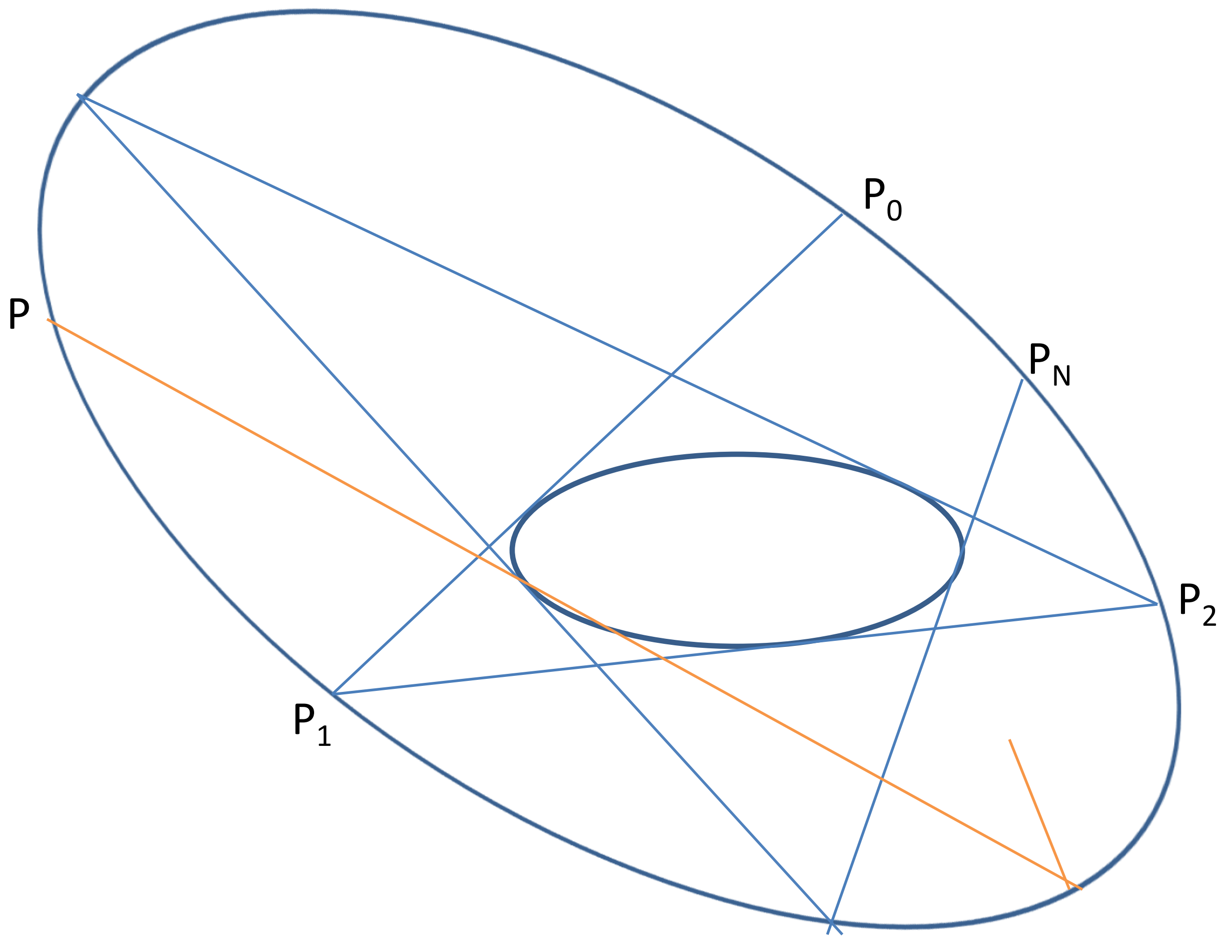
Domain Ω - convex for characteristics $X=C_1$, $Y=C_2$

Point P_0 is periodic with a period N, if $P_N = P_0$.

John theorems:

1. *If the boundary ∂Ω of the domain Ω is analytic, then only one of two possibilities is realized: 1) at each point P_0 its orbit $\{P_k, k \in Z\}$ is dense on ∂Ω,*
   2) any point P ϵ ∂Ω is periodic with one and the same the same period N.
2. *If on the boundary ∂Ω of the domain Ω the set of periodic points is empty, finite or countable, then the Dirichlet problem has only trivial solution.*

Poncelet's porism



Point P_0 is called periodic with a period N if $P_N = P_0$. The Poncelet's theorem:

If there is one periodic point P_0 on the outer ellipse, then any another point P of this ellipse will be periodic with the same period N.

Other projectively different ways of placing conics

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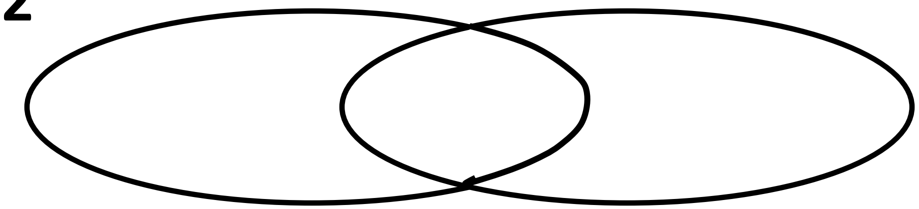

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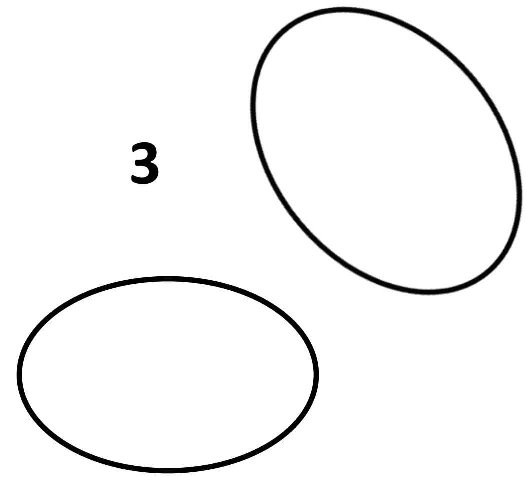

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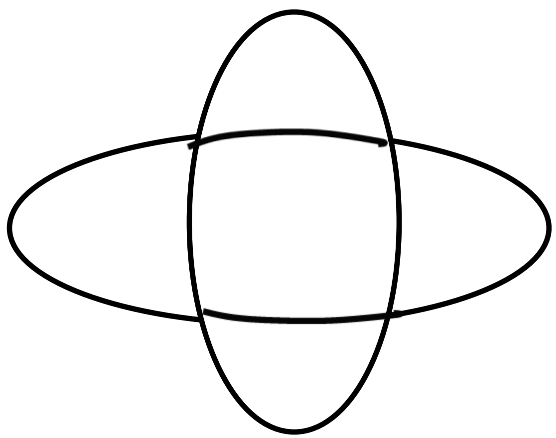

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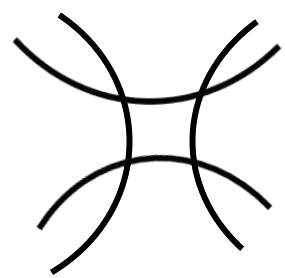
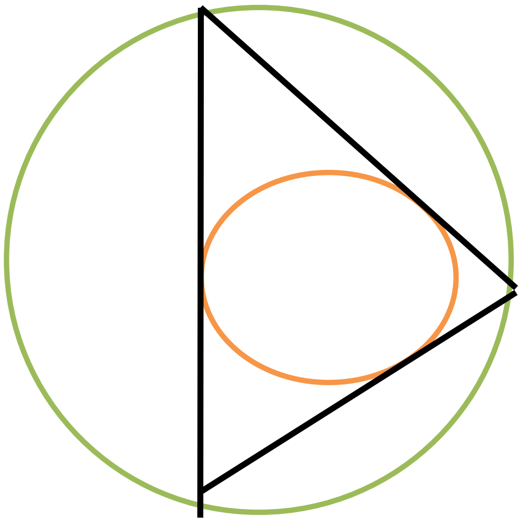
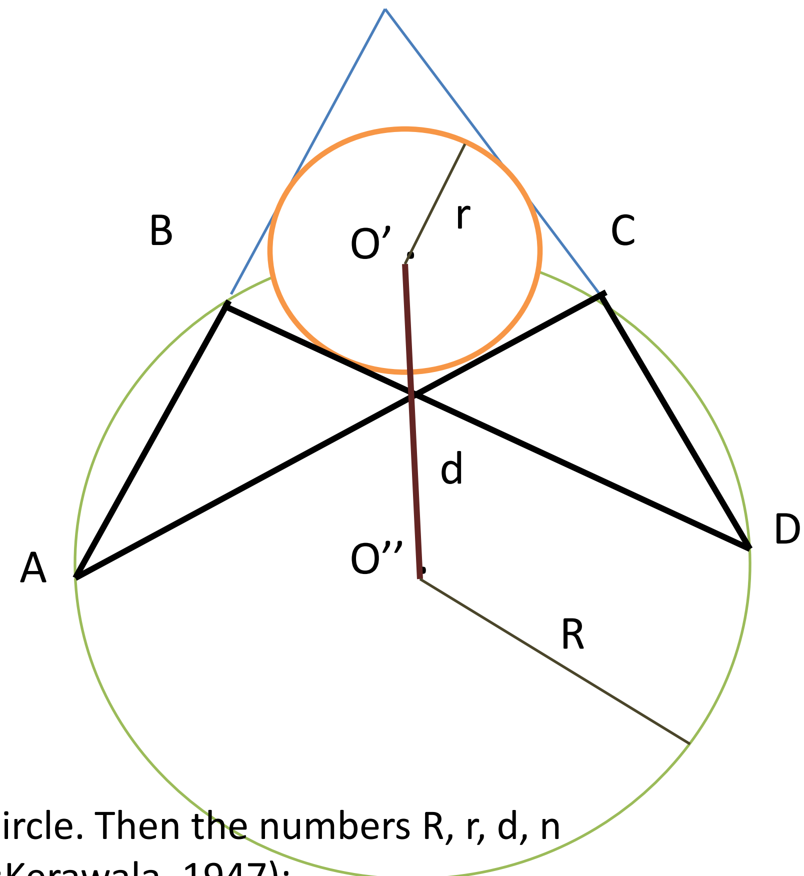


Poncelet porism in the form of two circles (bicentric polygons)





An n-gon is called bicentric if it has a circumcircle and an incircle. Then the numbers R, r, d, n are related to each other by the relations(Richelot, 1830; …;Kerawala, 1947):

$$\lambda = 1 + \frac{2c^2(a^2 - b^2)}{a^2(b^2 - c^2)}, \quad \omega = \cosh^{-1}\lambda, \quad k^2 = 1 - e^{-2\omega}, \quad \mathrm{sc}\left(\frac{K(k)}{n}, k\right) = \frac{c\sqrt{b^2 - a^2} + b\sqrt{c^2 - a^2}}{a(b + c)},$$

here $a = \frac{1}{R+d}, b = \frac{1}{R-d}, c = \frac{1}{r}$, sc (α)=sn (α) /cn (α), K(k)- elliptic integral of the first kind.

We have shown the John mapping in this case is the same as Ponselet mapping in some rational parametrizations of conics. From it we obtain

Link 1. For generic bi-quadratic curve the Dirichlet problem has non-unique solution if and only if corresponding Poncelet problem has periodic trajectory.

The second our observation shows a connection between the Poncelet problem and algebraic the Pell-Abel equation

$$P^2(t) + R(t)Q^2(t) = 1.$$

Link 2. The Poncelet problem is periodic with an even period iff corresponding the Pell-Abel equation is solvable.

Then we obtain a list of equivalent problem:
1) **Existence of a nontrivial solution of the homogeneous Dirichlet problem**

$$u_{xy} = 0 \quad \text{in} \quad \Omega, \qquad u|_C = 0 \quad \text{on} \quad C = \partial\Omega. \qquad (1)$$

2) **Existence of a nonconstant solution of the homogeneous Neumann problem** (u'_{ν_*} is conormal derivative)

$$u_{xy} = 0 \quad \text{in} \quad \Omega, \qquad u'_{\nu_*}|_C = 0 \quad \text{on } C = \partial\Omega. \qquad (2)$$

$$\frac{\partial}{\partial \nu_*} = l(\nu)\frac{\partial}{\partial \nu} - \frac{1}{2k}[l(\nu(s))]'_s \cdot \frac{\partial}{\partial s}, \quad l(\xi) = a\xi_1^2 + b\xi_1\xi_2 + c\xi_2^2$$

is the symbol of the operator L, ν is an unit vector of normal, s is natural parameter on $\partial\Omega$, $k = \pm|\nu'_s|$ is the kurvature, more exactly, $\nu'_s = k\tau$, where $\tau = (-\nu_2, \nu_1)$ is the tangent vector.
3) Consider following a moment problem: $\forall N = 0, 1, 2, ...$

$$\int_C \alpha(s)(x(s))^N ds = \mu_N^1; \int_C \alpha(s)(y(s))^N ds = \mu_N^2; \qquad (MP)$$

where for two given sequences of numbers μ_N^j one should find a function α.

It is obviously that for the case when $\partial\Omega$ is the unit circle this moment problem is similar to well-known trigonometric moment problem if we change $(x(s))^N, (y(s))^N$ to $\cos Ns, \sin Ns$. Therefore this moment problem will be called by generalized trigonometric.

Existence of a nontrivial solution of the homogeneous generalized trigonometrical moment problem

Among a lot of problems connected with the moment problem (MP) we will consider the problem of indeterminacy (uniqueness): for what curve $\partial\Omega$ there exists a function $\alpha \neq 0$ of some functional class such that

$$\forall k = 0, 1, ... \int\limits_C [x(s)]^k \alpha(s)\, ds = \int\limits_C [y(s)]^k \alpha(s)\, ds = 0, \qquad (3)$$

The problems (1), (2), (3) are equivalent in the following sense.

Statement 1. **Let** *$m \geq k \geq 3$ and let we have three sets of assertions formulated for Sobolev spaces:*
1_m) The homogeneous moment problem (3) has a nontrivial solution $\alpha \in H^{m-3/2}(\partial\Omega)$.
2_k) The Dirichlet problem $u|_{\partial\Omega} = 0$ for the string equation has a nontrivial solution $u \in H^k(\Omega)$.
3_k) The Neumann problem $u'_{\nu_*}|_{\partial\Omega} = 0$ for the string equation has a nonconstant solution $u \in H^k(\Omega)$.
Then $1_m) \Rightarrow 2_{m-q})$; $1_m) \Rightarrow 3_{m-q})$; $2_m) \Rightarrow 1_m)$; $3_m) \Rightarrow 1_m)$ *with $q = 1 + 0$ (by definition, for bounded domain $H^{k+0}(\Omega) = \bigcup_{\epsilon>0} H^{k+\epsilon}(\Omega)$).*

Note also, that the case when **the domain Ω is an ellipse** has examined in the works by A. Huber, R. Alexandrjan, V.I. Arnold and the author. An answer to the question about properties of such the Dirichlet problem is the following. Reduce by means of a linear transform our problem in an ellipse to the same problem in the unit disk for the equation $(\nabla \cdot a^1)(\nabla \cdot a^2)u = 0$. Find slope angles φ_1, φ_2 of characteristics and an angle $\varphi_0 = \varphi_1 - \varphi_2$ between them.

Statement 2. *The problem (1) has a nontrivial solution in the ellipse in a space $H^k(\Omega), k \geq 2$ if and only if*

$$\varphi_0/\pi \in \mathbb{Q}. \qquad (\mathbb{Q}_1)$$

If the condition $\mathbb{Q}_1$ is fulfilled then there is a denumerable set of linear independent polinomial solutions of the problem (1), a therefore the same for the problems (2) and (3).

Below **we will consider more complex domain** Ω that an ellipse, namely, domains which boundary is **a biquadratic curve**

$$F(x,y) := \sum_{i,k=0}^{2} a_{ik} x^i y^k = a_{22} x^2 y^2 + a_{21} x^2 y + ... = 0. \qquad (C)$$

We will need the following John mapping. Let Ω be arbitrary bounded domain, which is convex with respect to characteristic directions, i.e. it has the boundary C intersected in at most two points by each straight line that is parallel to x- or y-axes. We start from arbitrary point M_1 on C and consider a vertical line passing through M_1. Obviously, there are two points of intersection with the curve C: M_1 and some M_2, which may be coinsided with M_1. We denote I_1 an involution which transform M_1 into M_2. Then, starting from M_2, we consider a horizontal line passing through M_2. Let M_3 be the second point of intersection with the curve C. Let I_2 be corresponding involution: $I_2 M_2 = M_3$. We then repeat this process, applying step-by-step involutions I_1 and I_2. Denote $T = I_2 I_1, T^{-1} = I_1 I_2$. This transformation $T : C \to C$ gives us a discrete dinamical system on C, i.e. an action of group $\mathbb{Z}$ and each point $M \in C$ generates an orbit $\{T^n M | n \in \mathbb{Z}\}$. This orbit can be finite or denumerable set. The point M with finite orbit is called a periodic point and smallest n, for which $T^n M = M$, is called a period of the point M. In the paper by John the uniqueness breakdown in the problem have studied in connection with topological properties of the mapping T for the case of even mapping T. The mapping T is called to be even or preserving an orientation if each positive oriented arc (P, Q) with points $P, Q \in C$ transforms into positive oriented arc (TP, TQ).

Fritz John have proved several useful assertions for even T, among of which we extraxt the following one.

Sufficient condition of uniqueness. The homogeneous Dirichlet problem for the string equation in the bounded domain has only a trivial solution in the space $C^2(\overline{\Omega})$ if the set of periodic points on C is finite or denumerable.

Below we give several problem settings from different areas of mathematics, **each of them proves to be equivalent to each of above given in a domain with a biquadratic boundary**.

4) **Integration by elementary functions.**

Let us consider the problem (N.H. Abel, 1826):

for what polinomial $R(t)$ of degree $2m$ and of one variable t exist there polinomials $\rho(t), P(t), Q(t)$ such that

$$\int \frac{\rho}{\sqrt{R}} dt = A \ln \frac{P + \sqrt{R}Q}{P - \sqrt{R}Q} + C \qquad (4)$$

Later Liouville, Golubev and others have proved that if the primitive from left-side part of this equality is an elementary function (i.e. composition of polinomials, exponents, roots,

trigonometric functions) then it must be a function from the right-side part.
Abel proved two citeria for this problem:
Statement 3. *The equality (4) holds if and only if the expansion in a continous fraction*

$$\sqrt{R} = r_0(t) + \frac{1}{r_1(t) + \frac{1}{r_2(t)+\dots}}$$

is periodic, that is $\exists N, \exists k_0, \forall k > k_0, r_k = r_{k+N}$.
Statement 3. *The equality (4) holds if and only if the algebraic Pell-Abel equation*

$$P^2(t) + R(t)Q^2(t) = 1 \tag{5}$$

is solvable.
Here for given polinomial R of even order one should find polinomials P, Q such that the equality (5) is valid, then $\rho = 2P'/Q$. Below we will deal with the case $m = 2$, $\operatorname{ord} R = 4$.
Thus we have also the setting of
5) **Problem of solvability of Pell-Abel equation.**

As it was shown in a works by Sodin-Yuditskii the last setting is equivalent to the following:

6) **Problem of maximal set with least deviation by Chebyshev-Akhiezer.**

Consider the Chebyshev problem of finding a polinomial of least deviation on a closed set in the real axe. Let $I = [-1,1] \setminus \bigcup_{j=1}^{l-1} (a_j, b_j)$ − a system of l closed intervals and one should find a polinomial of a given order n with leading coefficient 1 which gives a least deviation on the set I, i.e. to find a minimum of the functional

$$\|t^n - P_{n-1}(t)\|_{C(I)} \to min.$$

The general polinomial $P_{n-1}(t)$ is running a finite-dimensional subspace and we deal with a problem of functional minimization on the nonreflexive Banach space. In 30 years of XX century Borel, based on Chebyshev ideas, proved that such polinomial P exist there on each set I. But if the polinomial P is minimal on I then, possible, it will be minimal on a more large set $\tilde{I}$ which is an expansion of I. Such maximal among of $\tilde{I}$ a set

$E = [-1,1] \setminus \bigcup_{j=1}^{m-1} (\alpha_j, \beta_j)$, which is called n-correct. If one take the polinomial R in the form $R = (t^2-1) \prod_{j=1}^{m-1} (t-\alpha_j)(t-\beta_j)$ then

Statement 3. *The solvability of the Pell-Abel equation*

$$P^2(t) + R(t)Q^2(t) = L^2 \tag{6}$$

where the constant L is unknown also, is equivalent to that the set E is n-correct. In addition, the polinomial P gives us a solution of extremal problem and the number L is the minimal deviation.

7) **Specrum of infinite Jakoby matrix.** It is iteresting that the set E is a continuous spectrum of some infinite selfajoint real Jacoby (three-diagonal) matrix in the space l^2 if and only if the set E is n-correct set, that is if and only if the equation (6) is solvable. For our aims the case of two intervals (the Akhiezer problem) is appropriated. A connection of the problem (6) with boundary value problems is realized by means of the Poncelet problem.

8) **Setting of the Poncelet problem.** Recall the Poncelet problem for the case of two ellipses, for simplicity and as it was introduced by Jean-Victor Poncelet himself. We take two arbitrary ellipses A and B, A inside B in the plane $\mathbb{R}^2$ of variables ξ, η. Let us have an arbitrary point Q_1 on the ellipse A and pass a tangent stright line to A at the point Q_1. This tangent crosses the ellipse B at two points P_1 and P_2, P_1 before P_2 with respect to a standard orientation. Then we take the point P_2 on B and pass the second tangent to the ellipse A. We denote as Q_2 the point on A where this tangent contacts with A. This tangent meets the ellipse B in two points P_2 and P_3. Take the point P_3 and repeat this procedure. Then we obtain a mapping $U_B : B \to B$ which acts by the rule $U_B : P_k \to P_{k+1}$ that will be called the Poncelet mappings below. The point P_1 will be named a periodic point and of a period N if $U_B^N P_1 = P_{N+1}$ and N is minimal with this property. The big Poncelet theorem says: if there is a periodic point P then each point is periodic with the same period. This construction is projective so that in general we may build it for a pair of conics (conic sections).

For any conic A it is possible to find polynomials $E_0(x)$, $E_1(x), E_2(x)$ with $\deg(E_i(x)) \leq 2$ such that $\xi = \frac{E_1(x)}{E_0(x)}, \quad \eta = \frac{E_2(x)}{E_0(x)}$. Quite analogously, the conic B can be parametrized as $\xi = \frac{G_1(y)}{G_0(y)}, \quad \eta = \frac{G_2(y)}{G_0(y)}$, where $G_i(y)$ are some other polynomials of most degrees 2. Our observations (in coautorship with A.S. Zhedanov) shows that in the plane of variables x, y the Poncelet mapping U_B turn into the John mapping on a biquadratic curve (C) which is given by conics A and B.

This gives us the followimg

Theorem 1. *A given the Poncelet problem is periodic if and only if the John mapping for corresponding biquadratic curve is periodic.*

Conversely, any generic biquadratic curve generated a projective class of a conics pair and we obtain.

Theorem 2. *For generic biquadratic curve the Dirichlet problem has non-unique solution if and only if corresponding John mapping has a periodic trajectory and if and only if corresponding Poncelet problem has a periodic trajectory.*

Note that for the case of bounded domain with a biquadratic boundary we have proved that John's sufficient uniqueness condition is also necessary, moreover, it will be so even for cases when the curve C is unbounded but then we should change the setting of the problem. Namely, along with the usual setting of the uniqueness property: *The examined bounded domain such that the homogeneous Dirichlet problem (1) has only trivial solution in the space* $C^2(\overline{\Omega})$

for cases when the curve C is unbounded we examine the following modification of uniqueness property for the homogeneous Dirichlet problem :

The examined curve C *is such that each analytic in real sense solution in* $\mathbb{R}^2$ *of the string equation with the property* $u|_C = 0$ *is only zero solution.*

Note, the assumption "analytic" is introduced in order that we can consider such curves C for them there exist characteristic lines which are not intersect C and are between of curve branchs, because without this assumption for such curve and e.g. with an assumtion of infinite smoothness one may build a simple example of a smooth nontrivial solution of the problem in sense (1).

The Poncelet porism in form of two circles

Let a circle A lies inside another circle B. From any point on B, draw a tangent to A and extend it to B. From the point, draw another tangent, etc. For n tangents, the result is called an n-sided Poncelet transverse. This Poncelet transverse can be closed for one point of origin, i.e. there exists one circuminscribed (simultaneously inscribed in the outer and circumscribed on the inner) n-gon. We could begin with a polygon that is understood as the union of a set of stright lines sequentialy joint a given cyclic sequence of points (vertices) on the plane. If there exist two circles, inscribed and circumscribed for this polygon, then this polygon is called a bicentric polygon. Note that sides of the polygon can intersect and the intersection point is not obligatory to be a vertex. Furthermore, the inscribed circle does not obligatory touch a segment between vertices, the contact point can lie on extension of the side and therefore the circles can intersect. Bicentric polygons are popular objects of investigations in geometry. This is most known form of the Poncelet porism. If we denote by r the radius of the inscribed circle, by R the radius of the circumscribed circle and by d a distance between the circumcenter and incenter for a bicentric polygon then these three numbers can not be arbitrary and together with n they satisfy some relations. So, for the case of triangle the relation is sometimes known as the Euler triangle formula $R^2 - 2Rr - d^2 = 0$. One of popular notations for such relations (necessary and sufficient for existence of a bicentric polygon) is given in terms of additional quantities

$$a = \frac{1}{R+d}\ , \quad b = \frac{1}{R-d}\ , \quad c = \frac{1}{r}\ .$$

So, for a triangle above the Euler formula has the view: $a + b = c$, for a bicentric quadrilateral, the radii and distance are connected by the equation $a^2 + b^2 = c^2$. The relationship for a bicentric pentagon is $4(a^3 + b^3 + c^3) = (a + b + c)^3$. In a general case one introduses numbers

$$\lambda = 1 + \frac{2c^2(a^2 - b^2)}{a^2(b^2 - c^2)}\ , \quad \omega = \cosh^{-1}\lambda\ , \quad k^2 = 1 - e^{-2\omega},$$

$$K = K(k) = \int_0^1 \frac{dt}{\sqrt{(1-k^2t^2)(1-t^2)}} \tag{7}$$

and then the relationship can be written by means of elliptic functions in the form

$$\operatorname{sc}\left(\frac{K}{n}, k\right) = \frac{c\sqrt{b^2 - a^2} + b\sqrt{c^2 - a^2}}{a(b + c)} \tag{8}$$

(Richelot (1830) – the first edition of the criterion,
Kerawala (1947) – the above criterion).

Connection with the Pell-Abel equation comes from the well-known Cayley criterion for the Poncelet problem. Recall that the Cayley criterion can be formulated as follows. Let $f(\lambda) = det(A - \lambda B)$ be a characteristic determinant for the one-parameter pencil of conics A and B presented in the projective form. In more details, assume that the conic A has an affine equation $\phi_A(x, y) = 0$. We then pass to the projective co-ordinates ξ: $x = \xi_1/\xi_0, y = \xi_2/\xi_0$ and present the equation of the conic A in the form

$$\sum_{i,k=0}^{2} A_{ik}\xi_i\xi_k = 0$$

with some 3×3-matrix A. Similarly, the projective equation for the conic B has the form

$$\sum_{i,k=0}^{2} B_{ik}\xi_i\xi_k = 0$$

with some 3×3-matrix B. Then we define the polynomial $f(\lambda) = det(A - \lambda B)$ of the third degree. Note that $f(\lambda)$ is a characteristic polynomial for the generalized eigenvalue problem for two matrices A, B. Calculate the Taylor expansion

$$\sqrt{f(\lambda)} = c_0 + c_1\lambda + \ldots c_n\lambda^n + \ldots$$

and compute the Hankel-type determinants from these Taylor coefficients:

$$H_p^{(1)} = \begin{vmatrix} c_3 & c_4 & \ldots & c_{p+1} \\ c_4 & c_5 & \ldots & c_{p+2} \\ \ldots & \ldots & \ldots & \ldots \\ c_{p+1} & c_{p+2} & \ldots & c_{2p-1} \end{vmatrix}, \quad p = 2, 3, 4, \ldots$$

and

$$H_p^{(2)} = \begin{vmatrix} c_2 & c_3 & \dots & c_{p+1} \\ c_3 & c_4 & \dots & c_{p+2} \\ \dots & \dots & \dots & \dots \\ c_{p+1} & c_{p+2} & \dots & c_{2p} \end{vmatrix}, \quad p = 1, 2, 3, \dots$$

Then the Cayley criterion is: **the trajectory of the Poncelet problem is periodic with the period N if and only if $H_p^{(1)} = 0$ for $N = 2p$, and $H_p^{(2)} = 0$ for $N = 2p + 1$.** Moreover, we have done the following observation: the Cayley condition coincides with a solvability criterion of the Pell-Abel equation by V.A. Malyshev (2001, Algebra and Analysis, S-Petersburg, No.6).

$$A^2(\lambda) + \tilde{f}(\lambda) B^2(\lambda) = 1$$

with $\deg \tilde{f} = 4$, $f(0) = 0$, (for details concerning solvability of the Pell-Abel equation and its relations with other problems of mathematics see, e.g. papers by Malyshev) if one takes $f(x) = x^4 \tilde{f}(x^{-1})$. We have the following proposition :
Theorem 3. *The Poncelet problem is periodic with an even period iff corresponding the Pell-Abel equation is solvable.*

Methods of the work are based on the theory of elliptic functions. Such functions are generated by the biquadratic curve from the Dirichlet problem, to which each of above problem can be reduced. The John mapping of the Dirichlet problem acts on a biquadratic curve generated of the Poncelet problem for two conics which are built by data of the Pell-Abel equation. This biquadratic curve, posiible after a pojective transformation of the plane, may be parametized by an elliptic function $\phi(z)$ of the second order: $x = \phi(z)$, $y = \phi(z + \eta)$. Then the John mapping on the complex biquadratic curve of the complex space $\mathbb{C}^2$ (that is a Riemann survace of a genus 1, i.e. a torus) may be given as a shift $z \to z + 2\eta$. Because the periods ω_1, ω_2 of the elliptic function ϕ (as η also) are counted up by data of the corresponding biquadratic curve then we obtain our periodicity criterion of the John mapping in the view:

$$2\eta N = m_1\omega_1 + m_2\omega_2 \tag{9}$$

with some integer N, m_1, m_2. Note that first the complex characteristic billiard was observed by E.A. Burjachenko for the case of a complex circle.

Futher we have done an analysis on reality of the functions that gives us a criterion for each of the problem in the form:

$$\frac{\theta}{2K} = \frac{m}{n} \in \mathbb{Q} \qquad (\mathbb{Q}_2)$$

where the number K is given in the formula (7) and the number θ is counted up by data of the corresponding problem as, for instance, the number k may be counted up by the formula (8).
Theorem 4. *The condition ($\mathbb{Q}_2$) is the criterion of uniqueness breakdown for the Dirichlet problem, the criterion of periodicity for the Poncelet process, and the criterion of solvability for the Pell-Abel equation.*
We see that this criterion is similar to the criterion ($\mathbb{Q}_1$) of uniqueness breakdown for the Dirichlet priblem in an ellipse.
9) **Discrete periodicity of solutions of the Toda chain.** Note that with complex characteristic billiard, for our case of biquadratic boundary is closely connected a question on solution periodicity of famous the Toda chain, that is a discrete dynamical system consisting of two sets $u_n(t), b_n(t)$ of complex variables depending on continuous parameter t and discrete parameter

$n = 0, \pm 1, \pm 2, \dots$. The moving equations are

$$\dot{b}_n = u_{n+1} - u_n, \quad \dot{u}_n = u_n(b_n - b_{n-1}).$$

Theorem 4.
1) Each elliptic solution for the unrestricted Toda chain can be presented in the from

$$b_n = \omega\Big(\zeta(\omega t - p(n+1) + r) - \zeta(\omega t - pn + r)\Big) - \lambda,$$

$$u_n = \omega^2\Big(\wp(p) - \wp(\omega t - pn + r)\Big) =$$

$$\omega^2 \, \frac{\sigma(\omega t - p(n+1) + r)\,\sigma(\omega t - p(n-1) + r)}{\sigma^2(p)\,\sigma^2(\omega t - pn + r)}$$

with arbitrary parameters ω, p, r, λ *and arbitrary invariants* g_2, g_3. *Here* $\wp(z), \sigma(z), \zeta(z)$ *are standard Weierstrass functions.*
2) The function $u_n(t)$ *is periodic with respect to n (i.e.* $u_n(t) = u_{n+N}(t)$*) iff the condition*

$$pN = 2m_1\omega_1 + 2m_2$$

is fufilled.

We see that the last criterion coincides with the periodicity criterion of the John mapping of complex characteristic billiard that is explicit built for this case.
Along of above writing problems there are another problems that allows their investigation by means of our approach. We hope to do it at least for some of them when it will be possible.
Note that a part of these results and references one can find in the works:
10. Classical Heisenberg XY spin chain. There is an interesting relation to the classical Heisenberg XY spin chain, which is a system of two-dimensional unit vectors (”spins”) $\vec{r}_n = (x_n; y_n),\ |\vec{r}_n| = 1$ with the energy of interaction given as

$$E = \sum_{n=0}^{N-1} (\vec{r}_n, J\vec{r}_{n+1}) \to \min, \tag{10}$$

where $J = diag(J_1, J_2)$ is a 2-by-2 diagonal matrix. The problem here is to find static solutions that provide a local extremum to the energy E. As it can be shown, this problem is equivalent to finding solutions of the systems of non-linear vector equations in the form:

$$(\vec{r}_n, J(\vec{r}_{n-1} + \vec{r}_{n+1})) = 0, \quad n = 1, 2, \dots, N-1. \tag{11}$$

Below we will select **cases of a closed chain, in which $\vec{r}_0 = \vec{r}_N$ for some** N. Based on this, it will be assumed that
$J_1 = 1, J_2 = j > 1$. Among all the solutions of (11), we choose the so-called regular solutions [4], satisfying the condition:

$$\vec{r}_{n-1} + \vec{r}_{n+1} \neq 0.$$

Then it is possible to show that the scalar product

$$W = (\vec{r}_n, J\vec{r}_{n+1})$$

does not depend on n and, hence, it can be considered as an integral of the system (11). It is sufficient to construct general regular solutions [4], which will be of two types. The choice of the solutions depends on the value of the integral W. If $|W| < 1/j$ then

$$x_n = cn(q(n-\theta); k), \quad y_n = sn(q(n-\theta); k),$$

where parameters k, q can be found from

$$dn(q;k) = 1/j, \quad k^2 = \frac{1 - j^{-2}}{1 - W^2}.$$

If $1/j < |W| < 1$ then

$$x_n = dn(q(n-\theta);k), \quad y_n = k\, sn(q(n-\theta);k),$$

where

$$cn(q;k) = 1/j, \quad k^2 = \frac{1 - W^2}{1 - j^{-2}}.$$

In both cases, the parameter θ is an arbitrary real number depending on an initial condition. If the chain is periodic, then we will have

$$qN = 4Km_1 + 2iK'm_2$$

which coincides with (9) . The reason for such coincidence can be described as follows.

Let us consider an equation of the following integral

$$x_n x_{n+1} + j^{-1} y_n y_{n+1} = W$$

with a fixed value W. This equation can be reduced to the algebraic form by a standard substitution (stereographic projection from a unit circle to a line):

$$x_n = \frac{1-u_n^2}{1+u_n^2}, \quad y_n = \frac{2u_n}{1+u_n^2}$$

It is easy to see that the variables u_n, u_{n+1} lie on the Euler-Baxter biquadratic curve

$$u_n^2 u_{n+1}^2 + 1 + a(u_n^2 + u_{n+1}^2) + b u_n u_{n+1} = 0$$

with parameters a, b simply related to the "physical" parameters j, W. Then it can be easily verified that finding solutions (step-by-step) of equations (11) for the regular solutions is equivalent to finding points $M_2, M_3, \dots M_N$ for the John algorithm. Note that some arguments allow concluding that the parameters q and θ must be real; hence, the condition takes the form

$$\frac{q}{4K} = \frac{m_1}{N}.$$

Thus, static regular solutions for the closed finite classical Heisenberg XY-chain are equivalent to periodic solutions of the John algorithm for the Euler-Baxter biquadratic curve, or, equivalently, to finding periodicity condition of the Poncelet process for a unit circle $x^2 + y^2 = 1$ and a concentric conic $x^2/\xi_1 + y^2/\xi_2 = 1$. This choice of conics corresponds to the Bertrand model of the Poncelet process. This means that there exists equivalence between static periodic solutions of the Heisenberg XY-chain and the Bertrand model of the Poncelet process.

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