Linear damping around inhomogeneous states of the Vlasov-HMF model

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Vlasov-HMF

Variables $(x, v) \in \mathbb{T} \times \mathbb{R}$. $\{f, g\} = \partial_x f \partial_v g - \partial_x g \partial_v f$.

$$
\left| \begin{array}{l}
\partial_t f(t,x,v) + \{f, H[f]\}(t,x,v) = 0 \\
H[f] = \dfrac{v^2}{2} - \phi[f](t,x) \\
\phi[f](x) = \displaystyle\int_{\mathbb{T}\times\mathbb{R}} P(x-y) f(t,y,v) \mathrm{d}v
\end{array} \right.
$$

- Vlasov-Poisson : P kernel of the Laplace operator

$$
P(x) = \sum_{k \neq 0} \frac{1}{|k|^2} e^{ikx}
$$

- Vlasov-HMF (Hamiltonian-Mean-Field). $P(x) = \cos(x)$.

Derivation of Vlasov equations : BBGKY hierarchy

N-particles dynamics $(X, V) := (x_i, v_i)_{i=1}^N$. $N >> 1$.

$$H(X, V) = \sum_{i=1}^{N} \frac{|v_i|^2}{2} + \frac{1}{2} \sum_{i \neq j} P(x_i - x_j).$$

Probability density function

$$\begin{aligned} f^N(t, X, V) &= \prod_{i=1}^{N} f_i(t, x_i, v_i) && \text{(Propagation of chaos)} \\ &= \prod_{i=1}^{N} f(t, x_i, v_i) && \text{(Indiscernibility)} \end{aligned}$$

Then at the limit $N \to +\infty$, $f(t, x, v)$ solves Vlasov with kernel P .

Vlasov-HMF

$$H[f] = \frac{v^2}{2} - \mathcal{C}[f]\cos(x) - \mathcal{S}[f]\sin(x).$$

$$\mathcal{C}[f] = \int_{\mathbb{T}\times\mathbb{R}} \cos(y) f(t,y,v) \mathrm{d}y\mathrm{d}v, \qquad \mathcal{S}[f] = \int_{\mathbb{T}\times\mathbb{R}} \sin(y) f(t,y,v) \mathrm{d}y\mathrm{d}v,$$

- Lagrangian description :$\varphi_t^{H[f]}$ flow of the Hamiltonian

$$\left| \begin{array}{ll} \dot{x} = \partial_v H[f] & = v \\ \dot{v} = -\partial_x H[f] & = -\mathcal{C}[f](t)\sin(x) + \mathcal{S}[f](t)\cos(x) \end{array} \right.$$

  then $f(t,x,v) = f(0, (\varphi_t^{H[f]})^{-1}(x,v))$.
- Flow of a time dependent pendulum.
  Coupling between the transport and the Hamiltonian flow.

Stationary states

$$\boxed{\begin{array}{ll} \eta(x,v) = G(h_0(x,v)) & G : \mathbb{R} \to \mathbb{R}_+ \\ h_0(x,v) = \dfrac{v^2}{2} - M_0 \cos(x) & \\ M_0 = \mathcal{C}[G(\frac{v^2}{2} - M_0 \cos(x))] & \end{array}}$$

- Indeed : $H[\eta] = \frac{v^2}{2} - \mathcal{C}[\eta]\cos(x) = h_0(x,v)$. And
$$\{\eta, H[\eta]\} = G'(h_0)\{h_0, h_0\} = 0.$$
- Example : Maxwellian $G(z) = \alpha e^{-\beta z}$,
$M_0 = \alpha\sqrt{\frac{2\pi}{\beta}} \int_{\mathbb{T}\times\mathbb{R}} e^{\beta M_0 \cos x} \cos x dx = \alpha\sqrt{\frac{2\pi}{\beta}} I_1(\beta M_0)$
$M_0 = 0$ solution. Non trivial solution $M_0 > 0$ if $\alpha\sqrt{\beta} \leq \frac{2}{\sqrt{2\pi}}$.
- For Vlasov-Poisson : $h_0 = \frac{v^2}{2} - V(x)$ and
$V(x)$ solves a nonlinear PDE.

Linearized equation

- Perturbation : $f(t,x,v) = \eta(x,v) + r(t,x,v)$

$$\partial_t r + \{\eta + r, H[\eta + r]\} = 0$$
$$\partial_t r + \{r, H[\eta]\} - \{\eta, \phi[r]\} - \cancel{\{r, \phi[r]\}} = 0$$

- Linearized equation

$$\boxed{\partial_t r + \{r, h_0\} - \{\eta, \phi[r]\} = 0}$$

- Homogeneous case $M_0 = 0$.

$\eta = \eta(v)$. $h_0(x,v) = \frac{v^2}{2}$. Flow $\psi_t \left| \begin{array}{l} \dot{x} = v \\ \dot{v} = 0 \end{array} \right.$

$$\partial_t r + \{r, h_0\} = 0 \quad \Longleftrightarrow \quad r(t,x,v) = r(0, x - tv, v) = r^0 \circ \psi_{-t}$$

Homogeneous case

Solution of $\partial_t r + \{r, h_0\} = \partial_t r + v\partial_x r = 0$.

- $\|r(t,x,v)\|_{H^s} = \|r(0, x-tv, v)\|_{H^s} \sim t^s$.
- Fourier coefficients $\hat{r}_k(t,\xi) = \int_{\mathbb{T}\times\mathbb{R}} r(t,x,v) e^{-ikx-i\xi v} \mathrm{d}x\mathrm{d}v$, $r(0,x,v) = r^0(x,v) \in H^s$.

$$\partial_t \hat{r}_k - k\partial_\xi \hat{r}_k = 0 \Longrightarrow \hat{r}_k(t,\xi) = \hat{r}_k^0(\xi + kt) = \mathcal{O}(\frac{1}{1+|\xi+kt|^s})$$

- Landau damping : r converges weakly to

$$v \mapsto \int r^0(x,v)\mathrm{d}x = \int \hat{r}_0^0(\xi) e^{i\xi v} \mathrm{d}\xi$$

- Strong topology : $r(t, x+tv, y) = r^0(x,v)$. (bounded in H^s !)

Homogeneous case

- Linearized equation $\partial_t r + \{r, h_0\} - \{\eta, \phi[r]\} = 0$.
- $g(t,x,v) = r(t, x+tv, v) = r \circ \psi_t$ solution of

$$\partial_t g - \{\eta, \phi[g \circ \psi_{-t}]\} \circ \psi_t = 0$$

$$\partial_t g - \{\eta, \phi(t,g)\} = 0 \quad \text{with}$$

$$\begin{aligned}\phi(t,g)(x,v) &= \int_{\mathbb{T}\times\mathbb{R}} (\cos(x-y+t(v-u)))g(t,y,u)\mathrm{d}u\mathrm{d}y \\ &= \sum_{\ell\neq 0} \hat{P}_\ell \hat{g}_k(t,kt)e^{ikx}e^{iktv}\end{aligned}$$

- Closed equation on the quantities $\zeta_k(t) := g_k(t,kt)$.

Homogeneous case

$K(n,t) = -nP_n nt\hat{\eta}(nt)\mathbb{1}_{t\geq 0}$

$$\zeta_n(t) = \hat{g}_n(0,nt) + \int_0^t K(n,t-\sigma)\zeta_n(\sigma)\mathrm{d}\sigma$$

Penrose criterion : $\boxed{\inf_{\mathrm{Im}\,\tau\leq 0} |1-\hat{K}(n,\tau)| \geq \kappa > 0}$

$$\hat{g}_n(0,nt) \sim \frac{1}{t^p} \quad \text{or} \quad e^{-\rho t} \quad \Longrightarrow \quad \zeta_n(t) \sim \frac{1}{t^{p'}} \quad \text{or} \quad e^{-\rho' t}$$

(Paley-Wiener)

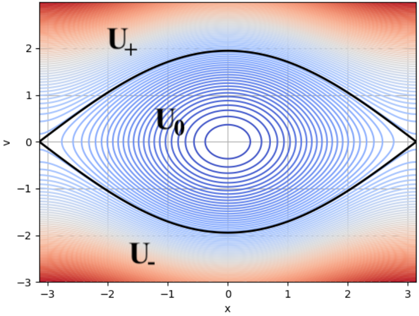
- Mouhot-Villani 2011. Analytic. Proof of a nonlinear result.
- Bedrossian-Masmoudi-Mouhot 2013. Nonlinear, Gevrey
- Faou-Rousset 2016. HMF Nonlinear, Sobolev.

Inhomogenous case $M_0 > 0$

$$h_0 : \left| \begin{array}{l} \dot{x} = v \\ \dot{v} = -M_0 \sin x \end{array} \right. \qquad \text{symplectic flow} \quad \psi_t$$

$$\partial_t r + \{r, h_0\} = 0 \quad \Longleftrightarrow \quad r(t,x,v) = r \circ \psi_{-t}$$

Dimension 1 in space : h_0 completely integrable



Homocline : $h_0 = M_0$.

Inhomogenous case $M_0 > 0$

- Action-angle variables : There exist symplectic changes of variable

$$U_\circ \ni (x,v) \mapsto (\theta, a) \in \mathbb{T} \times J_0 = \mathbb{T} \times [0, \frac{8}{\pi}\sqrt{M_0}]$$

$$U_\pm \ni (x,v) \mapsto (\theta, a) \in \mathbb{T} \times J_\pm = \mathbb{T} \times [\frac{4}{\pi}\sqrt{M_0}, +\infty]$$

such that the flow of h_0 is $\left| \begin{array}{l} \dot\theta = \omega(a) \\ \dot a = 0 \end{array} \right.$

$$f(\psi_t(x,v)) = f(\theta + t\omega(a), a) = \sum_{k\in\mathbb{Z}} f_k(a) e^{ik\theta} e^{ikt\omega(a)}.$$

- *Explicit* in terms of Jacobi elliptic functions.
- Heuristics by Barré, Bouchet, Yamaguchi 2010.

Damping effect of the pendulum flow

Theorem

$f(x,v)$, $g(x,v)$ smooth ($\langle v\rangle^\mu \partial^\alpha_{x,v} f \in L^\infty$).

$$p = \max\{n \geq 1,\ \partial^\alpha_{x,v} f(0,0) = 0,\ \forall \alpha,\ 1 \leq |\alpha| \leq n\},$$
$$q = \max\{n \geq 1,\ \partial^\alpha_{x,v} g(0,0) = 0,\ \forall \alpha,\ 1 \leq |\alpha| \leq n\},$$

Then

$$\left| \int_{\mathbb{T}\times\mathbb{R}} f(x,v) g(\psi_t(x,v)) \mathrm{d}x\mathrm{d}v - \sum_{*\in\{\pm,\circ\}} \int_{J_*} f_0^*(a) g_0^*(a) \mathrm{d}a \right| \leq \frac{C}{\langle t\rangle^{\frac{p+q}{2}+2}}.$$

Damping effect of the pendulum flow

$$\int_{\mathbb{T}\times\mathbb{R}} f(x,v)g(\psi_t(x,v))\mathrm{d}x\mathrm{d}v = \sum_{*\in\{\pm,\circ\}}\sum_{\ell\in\mathbb{Z}}\int_{J_*} f_\ell^*(a)g_{-\ell}^*(a)e^{it\ell\omega_*(a)}\mathrm{d}a.$$

$\ell = 0$ constant in time. $\ell \neq 0$ integration by parts

$$\begin{aligned}\int_J f_\ell(a)g_{-\ell}(a)e^{-it\ell\omega(a)}\mathrm{d}a &= -\int_\alpha^\beta \frac{f_\ell(a)g_{-\ell}(a)}{i\ell t\partial_a\omega(a)}\frac{\mathrm{d}}{\mathrm{d}a}e^{-it\ell\omega(a)}\mathrm{d}a \\ &= -\frac{1}{t}\left[\frac{f_\ell(a)g_{-\ell}(a)}{i\ell\partial_a\omega(a)}e^{-it\ell\omega(a)}\right]_\alpha^\beta + \frac{1}{t}\int_\alpha^\beta e^{-it\ell\omega(a)}\frac{\mathrm{d}}{\mathrm{d}a}\frac{f_\ell(a)g_{-\ell}(a)}{i\ell\partial_a\omega(a)}\mathrm{d}a\end{aligned}$$

Problems : *Singularities* at the boundary and *critical points* $\partial_a\omega(a) = 0$.

Damping effect of the pendulum flow

- Good news : $\partial_a \omega(a) \neq 0$.
- $a \to +\infty$. No problem (like the homogeneous case). $\omega(a) \sim a$
- Homocline :
  - $f_\ell(a)$ and $g_\ell(a)$ Logarithmic singularities.
  - But so does $\partial_a \omega(a)$ as $\omega(a) \sim \frac{1}{\log(a - a_0)}$
  - Compensation ! No problem
- Center of the eye : more delicate.
  - Close to the Harmonic oscillator : no damping !
  - But $\partial_a \omega|_{a=0} \neq 0$ (explicit formula)
  - $x \sim \sqrt{a} \cos(\theta)$, $y \sim \sqrt{a} \sin(\theta)$.

Problems come from the singularities $f_\ell(a) g_{-\ell}(a) \sim \sqrt{a}^{p+q}$.

Linear damping

- Linearized equation $\partial_t r + \{r, h_0\} - \{\eta, \phi[r]\} = 0$.
- $g = r \circ \psi_t$ satisfies the equation ($X(x, v) = x$)

$$\partial_t g = \mathcal{C}(t)\{\eta, \cos(\mathrm{X} \circ \psi_t)\} + \mathcal{S}(t)\{\eta, \sin(\mathrm{X} \circ \psi_t)\}.$$

$$\mathcal{C}(t) = \mathcal{C}[r(t)] = \mathcal{C}[g \circ \psi_{-t}] = \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}(y, w))g(t, \psi_{-t}(y, w))\mathrm{d}y\mathrm{d}w$$

$$\mathcal{S}(t) = \mathcal{S}[r(t)] = \mathcal{S}[g \circ \psi_{-t}] = \int_{\mathbb{T}\times\mathbb{R}} \sin(\mathrm{X}(y, w))g(t, \psi_{-t}(y, w))\mathrm{d}y\mathrm{d}w.$$

- $\mathcal{C}(t)$ and $\mathcal{S}(t)$ satisfy the following Volterra integral equations

$$\mathcal{C}(t) = F_{\mathcal{C}}(t) + \int_0^t \mathcal{C}(s)K_{\mathcal{C}}(t-s)\mathrm{d}s$$

$$\mathcal{S}(t) = F_{\mathcal{S}}(t) + \int_0^t \mathcal{S}(s)K_{\mathcal{S}}(t-s)\mathrm{d}s,$$

Linear damping

with

$$
\begin{aligned}
F_{\mathcal{C}}(t) &= \int_{\mathbb{T}\times\mathbb{R}} \cos(\mathrm{X}\circ\psi_t(y,w))r^0(y,w)\mathrm{d}y\mathrm{d}w,\\
F_{\mathcal{S}}(t) &= \int_{\mathbb{T}\times\mathbb{R}} \sin(\mathrm{X}\circ\psi_t(y,w))r^0(y,w)\mathrm{d}y\mathrm{d}w,\\
K_{\mathcal{C}}(t) &= \mathbb{1}_{\{t\geq 0\}}\int_{\mathbb{T}\times\mathbb{R}} \{\eta,\cos(\mathrm{X})\}\cos(\mathrm{X}\circ\psi_t), \quad \text{and}\\
K_{\mathcal{S}}(t) &= -\mathbb{1}_{\{t\geq 0\}}\int_{\mathbb{T}\times\mathbb{R}} \{\eta,\sin(\mathrm{X})\}\sin(\mathrm{X}\circ\psi_t).
\end{aligned}
$$

Previous result : decay of $F_{\mathcal{C}}(t)$ and $F_{\mathcal{S}}(t)$. But non zero term in $F_{\mathcal{C}}(t)$.

$$F_{\mathcal{C}}(t) = \sum_{\ell}\int C_\ell(a)r^0_{-\ell}(a)e^{-it\ell\omega(a)}\mathrm{d}a.$$

and weak decay because of the singularities at the center.

Linear damping

Theorem

Let $\eta(x,v) = G(h_0(x,v))$*, G decreasing and smooth,* r^0 *smooth. Assume*

$$\min_{\mathrm{Im}(\xi)\leq 0} |1 - \hat{K}_{\mathcal{C}}(\xi)| \geq \kappa \quad \text{and} \quad \min_{\mathrm{Im}(\xi)\leq 0} |1 - \hat{K}_{\mathcal{S}}(\xi)| \geq \kappa.$$

$$p = \max\left\{k \geq 1, \quad \partial^{\alpha}_{x,v} r^0(0,0) = 0, \quad \forall\, 1 \leq |\alpha| \leq k\right\}.$$

Then, if r^0 *satisfies the orthogonality condition*

$$\int_J C_0(a) r_0^0(a) \mathrm{d}a = 0,$$

$$|\mathcal{C}(t)| \leq \frac{C}{\langle t\rangle^{\max(3,\frac{p+5}{2})}} \quad \text{and} \quad |\mathcal{S}(t)| \leq \frac{C}{\langle t\rangle^2}.$$

Linear damping

Corollary

- *$\exists\, g_\infty(x,v)$ such that when $t \to +\infty$,*

$$\|g(t) - g_\infty\|_{L^1_{x,v}} \leq \frac{C}{\langle t \rangle}.$$

- *$\exists\, r_\infty(x,v)$ that depends only on a, $r_\infty(x,v) = r^*_\infty(a)$ for $(x,v) \in U_*$ and $* \in \{\pm, \circ\}$, such that for every test function ϕ, we have that*

$$\int_{\mathbb{T}\times\mathbb{R}} r(t,x,v)\phi(x,v)\,\mathrm{d}x\mathrm{d}v \to_{t\to+\infty} \int_{\mathbb{T}\times\mathbb{R}} r_\infty(x,v)\phi(x,v)\,\mathrm{d}x\mathrm{d}v.$$

Stability criterion

Theorem

Let $\eta = G(h_0)$ stationary state and G smooth and decreasing. Then if

$$1 - \hat{K}_{\mathcal{C}}(0) > 0 \quad \textit{and} \quad 1 - \hat{K}_{\mathcal{S}}(0) > 0.$$

the Penrose criterion holds true.

True for $\xi = 0 \Longrightarrow$ true for all ξ, $\mathrm{Im}\xi \leq 0$.

Idea of Proof :

$$\hat{K}_{\mathcal{C}}(\xi) = \int_0^{+\infty} e^{-it\xi} \frac{\mathrm{d}}{\mathrm{d}t} Q_{\mathcal{C}}(t)\mathrm{d}t = \hat{K}_{\mathcal{C}}(0) + i\xi \int_0^{+\infty} e^{-it\xi} Q_{\mathcal{C}}(t)\mathrm{d}t,$$

$$Q_{\mathcal{C}}(t) = \sum_{\ell \neq 0} \int_J G'(a)|C_\ell(a)|^2 e^{it\ell\omega(a)}\mathrm{d}a,$$

No problem when ξ small or large, and when $\mathrm{Im}\xi < -\delta$.

Stability criterion

$$\hat{K}_{\mathcal{C}}(\xi) = \sum_{\ell \neq 0} \int_J G'(h_0(a)) |C_\ell(a)|^2 \left(\frac{\ell \omega(a)}{\xi - \ell \omega(a)} \right) \mathrm{d}a.$$

When $\mathrm{Im}\xi \to 0$ the limit can be identified.
And this limit is non zero because $C_\ell(a)$ is never zero !

$$\cos(x(\theta,h)) = 1 - 2k(h)^2 \frac{\boldsymbol{K}\left(\frac{1}{k(h)}\right) - \boldsymbol{E}\left(\frac{1}{k(h)}\right)}{\boldsymbol{K}\left(\frac{1}{k(h)}\right)} + \frac{4\pi^2 k(h)^2}{\boldsymbol{K}\left(\frac{1}{k(h)}\right)^2} \sum_{m=1}^{\infty} \frac{mq\left(\frac{1}{k(h)}\right)^m}{1 - q\left(\frac{1}{k(h)}\right)^{2m}} \cos(m\theta).$$

Formula (900.05) of Byrd and Friedman, *Handbook of Elliptic Integrals for Engineers and Scientists*, 1971.

Stability criterion

Theorem

Let $\eta = G(h_0)$ be a stationary state and G smooth and decreasing. Then the stability condition holds true if and only if

$$1 + \int_{\mathbb{R}\times\mathbb{T}} G'(h_0(x,v))\cos^2(x)\mathrm{d}x\mathrm{d}v - \int_J G'(h_0(a))C_0(a)^2\mathrm{d}a > 0.$$

Condition for classical orbital stability in energy space.

Barré-Yamaguchi 2015

Lemou-Luz-Méhats 2017.

Theorem

$\alpha > 0$, $\beta > 0$ such that $\alpha^2\beta < \frac{2}{\pi}$, then there exists $M_0 > 0$ satisfying such that

$$\eta(x,v) = \alpha e^{-\beta\left(\frac{v^2}{2} - M_0\cos(x)\right)},$$

is a stable stationary states.

Conclusions

- Next step : nonlinear equation for Vlasov-HMF.
- Dimension 1 in space crucial . Generically the same scenario for Vlasov-Poisson or Euler 2D.
- There exists a nonlinear symplectic structure for

$$\partial_t f(t,x,v) + \{f, \frac{\delta \mathcal{H}}{\delta f}[f]\}(t,x,v) = 0$$

Can we do normal form with this structure ?