Primitive recursive ordered fields and some applications

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Introduction

In our earlier work, computable ordered fields of reals were related to the field of computable reals and used to prove computability of some problems in algebra and analysis in the rigorous sense of computable analysis. In 2018, PTIME-computability of some problems on algebraic numbers established by Alaev-S (based on previous results from computer algebra) was applied to find upper complexity bounds for some problems in algebra and analysis. "Small" complexity classes (like PTIME or PSPACE) are often not closed under important constructions. E.g., the field $\mathbb{R}_{\mathrm{alg}}$ of algebraic reals is PTIME presentable but root-finding w.r.t. this presentation is only in EXPTIME. Thus, it seems reasonable to look at complexity classes in between PTIME and COMPUTABLE with better closure properties.

Introduction

Recently, there was a renewed interest in primitive recursive (PR) structures which are recognized as a principal model for an emerging new paradigm of computability — the so called online computability (Bazhenov, Downey, Kalimullin, Melnikov, Ng). PR-algorithms do not use an exhaustive search through a structure (usually written as unbounded WHILE...DO..., REPEAT...UNTIL..., or μ operator); thus, there is a possibility to count working time of the algorithm.

Although the upper complexity bounds for a PR-algorithm may be awfully large, this is a principal improvement compared with the general computability where estimation of complexity is impossible in principle. As stressed by Bazhenov et al, PR-presentability of a structure may often be improved even to PTIME-presentability. Thus, the importance of PR-presentability stems from the fact that it is in some respect close to feasible presentability but technically much easier, and has much better closure properties.

Introduction

In this work we establish PR-versions of some mentioned results and describe their applications in linear algebra and analysis. In particular, we find a PR-version (for the archimedean case) of the Ershov-Madison theorem on the computable real closure, relate the PR ordered fields of reals to the field of PR reals, give a sufficient condition for PR root-finding, propose (apparently, new) notions of PR-computability in analysis and apply them to obtain new results on computations of PDE-solutions. These results complement the results of Alaev-S about root-finding in the field $\mathbb{R}_{\mathrm{alg}}$ of algebraic reals and the results about the complexity of PDE-solutions. The class of PR real closed fields of reals is shown to be richer than the class of PTIME-presentable fields.

In programming terms, we identify important tasks which may be programmed without using the above-mentioned unbounded cycle operators.

Constructivisable structures

D e f i n i t i o n. A structure $\mathbb{B} = (B; \sigma)$ of a finite signature σ is called constructivizable iff there is a numbering β of B such that all signature predicates and functions, and also the equality predicate, are β-computable. Such a numbering β is called a constructivization of $\mathbb{B}$, and the pair $(\mathbb{B}, \beta)$ is called a constructive structure.

The "Russian" terminology used above was introduced by A.I. Mal'cev; the equivalent "American" notions for "constructivizable" and "constructive" are "computably presentable" and "computable", resp.

PR-constructivisable structures

PR-versions of the notions defined above are obtained by changing "computable" to "PR" in the definitions above. In particular, for numberings β and γ, β is *PR-reducible* to γ (in symbols $\beta \leq_{PR} \gamma$) iff $\beta = \gamma \circ f$ for some PR function f on $\mathbb{N}$, and β is *PR-equivalent* to γ (in symbols $\beta \equiv_{PR} \gamma$) iff $\beta \leq_{PR} \gamma$ and $\gamma \leq_{PR} \beta$. For $\nu : \mathbb{N} \to B$, a relation $P \subseteq B^n$ on B is ν-PR if the relation $P(\nu(k_1), \ldots, \nu(k_n))$ on $\mathbb{N}$ is PR. A function $f : B^n \to B$ is ν-PR if $f(\nu(k_1), \ldots, \nu(k_n)) = \nu g(k_1, \ldots, k_n)$ for some PR function $g : \mathbb{N}^n \to \mathbb{N}$. A structure $\mathbb{B} = (B; \sigma)$ is *PR-constructivizable* iff there is a numbering β of B such that all signature predicates and functions, and also the equality predicate, are β-PR. Such β is called a *PR-constructivization* of $\mathbb{B}$, and the pair $(\mathbb{B}, \beta)$ is called a *PR structure*.

PR functions

The PR functions are generated from the distinguished functions $o = \lambda n.0$, $s = \lambda n.n + 1$, and $I_i^n = \lambda x_1, \ldots, x_n.x_i$ by repeated applications of the operators of superposition S and primitive recursion R. Thus, any PR function is represented by a "correct" term in the partial algebra of functions over $\mathbb{N}$. Intuitively, any total function defined by an explicit definition using (not too complicated) recursion is PR; the unbounded μ-operator is of course forbidden but the bounded one is possible.

Consider the structure $(\mathcal{N}; +, \circ, J, s, q)$ where $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ is the set of unary functions on $\mathbb{N}$, $+$ and $\circ$ are binary operations on $\mathcal{N}$ defined by $(p + q)(n) = p(n) + q(n)$ and $(p \circ q)(n) = p(q(n))$, J is a unary operation on $\mathcal{N}$ defined by $J(p)(n) = p^n(0)$ where $p^0 = id_{\mathbb{N}}$ and $p^{n+1} = p \circ p^n$, s and q are distinguished elements defined by $s(n) = n + 1$ and $q(n) = n - [\sqrt{n}]^2$ where, for $x \in \mathbb{R}$, $[x]$ is the unique integer m with $m \leq x < m + 1$.

PR Ershov-Madison's theorem

By a classical theorem of Artin and Schreier, for any ordered field $\mathbb{A}$ there exists an algebraic ordered extension $\widehat{\mathbb{A}} \supseteq \mathbb{A}$ which is real closed.

Yu.L. Ershov and independently E.W. Madison proved a computable version of the Artin-Schreier theorem: if $\mathbb{A}$ is constructivizable then so is also $\widehat{\mathbb{A}}$.

We make search for a PR analogue of the Ershov-Madison theorem. Our proof below works only for *PR-Archimedean fields* which we define as the PR ordered subfields $(\mathbb{A}, \alpha)$ of $\mathbb{R}$ such that there is a PR function f with $\forall(\alpha(n) \leq f(n))$.

T h e o r e m. If $(\mathbb{A}, \alpha)$ is a PR-Archimedean subfield of $\mathbb{R}$ then so is also $(\widehat{\mathbb{A}}, \widehat{\alpha})$.

PR root finding

A computable field $(\mathbb{B}, \beta)$ *has computable root-finding* if, given a polynomial $p \in \mathbb{B}[x]$ of degree > 1, one can compute a (possibly, empty) list of all roots of p in $\mathbb{B}$. By Frölich-Shepherdson, $(\mathbb{B}, \beta)$ has computable root-finding iff it has computable splitting. As usual, the notion of PR root-finding is obtained by changing "computable" to "PR".

P r o p o s i t i o n. A PR field has PR root-finding iff it has PR splitting.

T h e o r e m. 1. If $\alpha \in \mathrm{pras}(\mathbb{R})$ then $(\widehat{\mathbb{A}}, \widehat{\alpha})$ and $(\overline{\mathbb{A}}, \overline{\alpha})$ have PR root-finding.
2. If $\alpha \in \mathrm{pras}(\mathbb{R})$ then $\widehat{\alpha} \in \mathrm{pras}(\mathbb{R})$.

PR fields vs PR reals

We search for a PR-analogue of the following fact: for any finite set F of computable reals there is a computable real closed ordered subfield $(\mathbb{B}, \beta)$ of the computable reals such that $F \subseteq B$. In particular, the union of all computable real closed fields of reals is the field $\mathbb{R}_{\rm c}$ of computable reals.

The PR-analogue of $\mathbb{R}_{\rm c}$ is the ordered field $\mathbb{R}_{\rm p}$ of PR reals. A real a is PR if $a = \lim_n q_n$ for a PR sequence $\{q_n\}$ of rational numbers which is fast Cauchy, i.e. $|q_n - q_{n+1}| < 2^{-n}$ for all n.
The PR-analogue of the results above is only partial:

P r o p o s i t i o n. Every PR ordered field of reals is a subset of $\mathbb{R}_{\rm p}$ but the union of such fields is a proper subset of $\mathbb{R}_{\rm p}$.

PR fields vs PR reals

Nevertheless, there is an easy criterion (a bit cumbersome formulation is omitted) of when a tuple of PR reals may be adjoined to a given PR-Archimedean field of reals. From this criterion we can deduce the following.

T h e o r e m. There is a PRAS-field of arbitrary transcendence degree. The ordered fields $\mathbb{Q}(e)$ and $\mathbb{Q}(\pi)$ are PRAS.

In contrast, we do currently do not know any example of a PTIME-presentable field of reals containing a transcendental number. There is a PRAS-field which is not PTIME-presentable.

As is well known, all eigenvalues of any symmetric real matrix are real. *Spectral decomposition* of such a matrix $A \in M_n(\mathbb{R})$ is a pair $((\lambda_1, \dots, \lambda_n), (\mathbf{v}_1, \dots, \mathbf{v}_n))$ where $\lambda_1 \leq \cdots \leq \lambda_n$ is the non-decreasing spectrum of A and $\mathbf{v}_1, \dots, \mathbf{v}_n$ is a corresponding orthonormal basis of eigenvectors, i.e. $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ for $i = 1, \dots, n$.

P r o p o s i t i o n. Let $\alpha \in \mathrm{pras}(\mathbb{R})$. Given n and a symmetric matrix $A \in M_n(\widehat{\mathbb{A}})$, one can primitive recursively find a spectral decomposition of A uniformly on n.

P r o p o s i t i o n. Let $\alpha \in \mathrm{pras}(\mathbb{R})$. Given n and a matrix $A \in M_n(\overline{\mathbb{A}})$, one can primitive recursively and uniformly on n find a Jordan normal form $J \in M_n(\overline{\mathbb{A}})$ for A and a non-degenerate matrix $C \in M_n(\overline{\mathbb{A}})$ with $A = C^{-1}JC$.

PR-computability of PDEs

We consider the initial-value problem

$$\begin{cases} A\frac{\partial \mathbf{u}}{\partial t} + \sum\limits_{i=1}^{m} B_i \frac{\partial \mathbf{u}}{\partial x_i} = f(t,x), \ \ t \geq 0, \\ \mathbf{u}|_{t=0} = \varphi(x_1,\ldots,x_m). \end{cases} \tag{1}$$

Here $A = A^* > 0$ and $B_i = B_i^*$ are constant symmetric $n \times n$-matrices, $t \geq 0$, $x = (x_1,\ldots,x_m) \in Q = [0,1]^m$, $\varphi : Q \to \mathbb{R}^n$ and $\mathbf{u} : [0,+\infty) \times Q \rightharpoonup \mathbb{R}^n$ is a partial function acting on the domain H of existence and uniqueness of the Cauchy problem (1). The solution $\mathbf{u}$ depends continuously on $\varphi, f, A, B_1,\ldots,B_m$.

PR-computability of PDEs

Symmetric hyperbolic systems are used to describe a wide variety of physical processes like those considered in the theories of elasticity, acoustics, electromagnetism etc., see e.g. [Friedrichs 1954, Godunov 1971,76, Landau, Lifschitz 1986 etc.].

They were first considered in 1954 by K.O. Friedrichs. He proved the existence theorem based on **finite difference approximations**, in contrast with the Schauder-Cauchy-Kovalevskaya method based on approximations by analytic functions and a careful study of infinite series. The methods of Friedrichs are used to construct different stable difference schemes, in particular the Godunov scheme we used in our works.

The notion of a hyperbolic system (applicable also to broader classes of systems) is due to I.G. Petrovski.

Questions

Questions: Is the solution **u** computable

I. from given initial conditions φ and right-hand part f (with fixed computable coefficients),
II. from φ, f **and** coefficients A, B_i

and in which sense?

III. If yes, what is the complexity of computations?

Results on computability in PDEs

I. For fixed computable matrices, the solution operator $(\varphi, f) \mapsto \mathbf{u}$ of (1), (2) is computable provided that the first and second partial derivatives of φ, f are uniformly bounded.

II. 1) The operator $(A, B_1, \ldots, B_m) \mapsto H$ is computable;
2) The solution operator $(\varphi, f, A, B_1, \ldots, B_m, n_A, n_1, \ldots, n_m) \mapsto \mathbf{u}$ of (1), (2) is computable under some additional spectral conditions on A, B_i.
Here n_A is the cardinality of spectrum of A (i.e. the number of different eigenvalues);
n_i are the cardinalities of spectra of the matrix pencils $\lambda A - B_i$.
Eigenvectors are in general not computable!
3) The solution operator $(\varphi, f, A, B_1, \ldots, B_m) \mapsto \mathbf{u}$ of (1) is computable when the coefficients of A, B_i run through an arbitrary computable real closed subfield of $\mathbb{R}$.

PR-computability on $\mathcal{N}$

For any $n \geq 0$, any term $t = t(v_1, \ldots, v_n)$ of the Robinson algebra determines the n-ary operator $\mathbf{t}$ on the Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ by setting $\mathbf{t}(g_1, \ldots, g_n)$ to be the value of t for $v_i = g_i$. Such operators are called PR. We give an example.

Let **Ca** be the set of sequences $q \in \mathbb{Q}$ such that $\varkappa \circ q \in \mathbb{Q}^{\mathbb{N}}$ is fast Cauchy, where $\varkappa$ is a bijective PR-constructivization of $\mathbb{Q}$. Let $\tilde{q}(n) = q(n)$ if $\forall i < n(|\varkappa(q_i) - \varkappa(q_{i+1})| < 2^{-i})$ and $\tilde{q}(n) = q(i_0)$ otherwise, where $i_0 = \mu i < n(|\varkappa(q_i) - \varkappa(q_{i+1})| \geq 2^{-i})$. Note $\tilde{q} \in \mathbf{Ca}$ for $q \in \mathcal{N}$, and $\tilde{q} = q$ for $q \in \mathbf{Ca}$, in particular $\mathbf{Ca} = \{\tilde{q} \mid q \in \mathcal{N}\}$.

Then $q \mapsto \tilde{q}$ is a unary PR operator on $\mathcal{N}$ which is a retraction.

PR-computability on metric spaces

We transfer primitive recursiveness on $\mathcal{N}$ to that on $\mathbb{R}$. Namely, we define $\gamma(q) = \lim_n \varkappa(\tilde{q}(n))$ and call this γ the *Cauchy representation* of $\mathbb{R}$.

A function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ is called PR if $f(\gamma(p_0), \ldots, \gamma(p_n)) = \gamma(g(p_0, \ldots, p_n))$ for some PR function $g : \mathcal{N}^{n+1} \to \mathcal{N}$.

More generally, we can straightforwardly define PR metric spaces and PR-computability of functions between such spaces using standard Cauchy representations (the only difference with the classical definition is that now the distance between points in the specified dense set is required to be uniformly PR).

T h e o r e m. Let $M, p \geq 2$ be integers. Then the solution operator $(A, B_1, \ldots, B_m, \varphi) \mapsto \mathbf{u}$ for (1) is a PR-computable function (uniformly on m, n) from $S_+ \times S^m \times C_s^{p+1}(Q, \mathbb{R}^n)$ to $C^p_{sL_2}(H, \mathbb{R}^n)$ where S and S^+ are respectively the sets of all symmetric and symmetric positively definite matrices from $M_n(\widehat{\mathbb{A}})$, $||\frac{\partial\varphi}{\partial x_i}||_s \leq M$ and $||\frac{\partial^2\varphi}{\partial x_i \partial x_j}||_s \leq M$ for $i, j = 1, 2, \ldots, m$.

T h e o r e m. Let $M, p \geq 2$ be integers and $A, B_1, \ldots, B_m \in M_n(\mathbb{R}_{\mathrm{p}})$ be fixed matrices satisfying the conditions in (1). Then the solution operator $\varphi \mapsto \mathbf{u}$ for (1) is a PR-computable function (uniformly on m, n) from $C_s^{p+1}(Q, \mathbb{R}^n)$ to $C^p_{sL_2}(H, \mathbb{R}^n)$, with the same constraints on φ as in the previous theorem.

T h e o r e m. Given integers $m, n, a \geq 1$, matrices $A, B_1 \ldots, B_m \in M_n(\widehat{\mathbb{A}})$, and rational functions $\varphi_1 \ldots, \varphi_n \in \widehat{\mathbb{A}}(x_1 \ldots, x_m)$, $f_1 \ldots, f_n \in \widehat{\mathbb{A}}(t, x_1 \ldots, x_m)$ as in (1), one can primitive recursively uniformly on m, n, a compute a rational $T > 0$ with $H \subseteq [0, T] \times Q$, a spatial rational grid step h dividing 1, a time grid step τ dividing T and an h, τ-grid function $v : G_N^\tau \to \widehat{\mathbb{A}}$ such that $||\mathbf{u} - \widetilde{\upsilon \mid_H}||_{sL_2} < a^{-1}$, where $\widetilde{\upsilon \mid_H}$ is the multilinear interpolation of the restriction of the grid function υ to H.

Conclusion

Our previous results stress interesting interaction between symbolic algorithms (which aim to find precise solutions), and approximate algorithms (which aim to find "good enough" approximations to precise solutions). The symbolic algorithms implemented e.g. in computer algebra systems correspond well to computations on discrete structures (with mathematical foundations in the classical computability and complexity theory). The approximate algorithms included into numerical mathematics packages correspond well to computations on continuous structures (with mathematical foundations in the field of computability and complexity in analysis).

Conclusion

We hope that our present results demonstrate that PR computations is a natural next step in the investigation of this interaction between symbolic and numeric computation: they provide a natural borderline between problems in algebra and analysis computable in principle and feasible problems.

Although PR functions were thoroughly investigated in computability theory and proof theory, their study in computable structure theory and computable analysis seems still in the very beginning.

Conclusion

Fast progress of computation theory in the last decades made informal descriptions of several algorithms in the standard texts in algebra and analysis insufficient and sometimes even incorrect. They typically remain correct when interpreted in countable discrete structures (like computable ordered field of reals), though a finer distinction between general computability and feasible computability is desirable.

Some popular algorithms of linear algebra interpreted in continuous structures (like the real or complex numbers) become even incorrect; it seems desirable to add corresponding comments (which refer to computable analysis approach) in new editions of such textbooks.

THANK YOU FOR YOUR ATTENTION!!

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