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Free and projective generalized multinormed spaces

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Introduction: L–spaces, L–contractive operators and projectivity

A multinormed space is a comparatively recent mathematical structure which is, roughly speaking, one of versions of a quantized normed space. These spaces found important applications to the theory of Banach lattices and to the geometric theory of normed spaces. To define them, one takes a linear space E and introduces, for every $n \in \mathbb{N}$, a norm on the column of length n with coordinates in E, that is on $E^n = E \otimes \mathbb{C}^n$. If these norms are connected by certain axioms with participation of a given $p; 1 \leq p \leq \infty$, we say that a p-multinormed space is given. I do not give the exact definition because I shall introduce a more general notion that will be the main subject of my talk.

First of all, we choose and fix (so far arbitrary) normed space $\mathbf{L}$, which we shall call *base space*. Our principal examples are the space $L_p(X)$ of functions on a measure space X and its dense subspace $L_p^0(X)$, consisting of simple functions, that is of linear combinations of characteristic functions of measurable subsets of X. We prepare our first definition.

Let E be a linear space. Consider the pure algebraic tensor product $\mathbf{L} \otimes E$. Note that it is a left module over the algebra $\mathcal{B}(\mathbf{L})$ of bounded operators on $\mathbf{L}$ with the outer multiplication " $\cdot$ ", well defined on elementary tensors by $a \cdot (\xi \otimes x) := a(\xi) \otimes x; a \in \mathcal{B}(\mathbf{L}), \xi \in \mathbf{L}, x \in E$.

Definition 1. Suppose that some norm on $\mathbf{L} \otimes E$ is given. This norm is called $\mathbf{L}$-*norm on* E, if for all $a \in \mathcal{B}(\mathbf{L})$ and $u \in \mathbf{L} \otimes E$ we have the inequality $\|a \cdot u\| \leq \|a\|\|u\|$, The space E, endowed by an $\mathbf{L}$-norm, is called $\mathbf{L}$-*space*.

An important example of an $\mathbf{L}$-norm will be considered later.

(I spoke about “one of versions of the quantization” because of the following reason. If, instead of a norm on $\mathbf{L} \otimes E$, we would introduce a proper, in a sense, norm on $A \otimes E$, where A is a “good” operator algebra, consisting, for example, of compact operators, we would get a so-called quantum, or an abstract operator space.)

L-spaces are generalizations of aforementioned p-multinormed spaces. Indeed, let X be the set $\mathbb{N}$ of natural numbers with the counting measure (the measure of a point is 1). Then $L_p^0(X)$ coincides with ℓ_p^0, the space of sequences, ending with zeroes, with the norm inherited from ℓ_p. It is easy to verify that the introduction of a ℓ_p^0-norm on E is equivalent to the introduction of a family of norms, for every n, on the column space of length n with coordinates from E that satisfies the axioms of a p-multinormed space. (Here I ask you to believe me.)

We proceed to the second main definition.

If an operator $\varphi : G \to E$ between linear spaces is given, we shall use, for the operator $\mathbf{1_L} \otimes \varphi := \mathbf{L} \otimes G \to \mathbf{L} \otimes E$, well defined on elementary tensors by $\xi \otimes x \mapsto \xi \otimes \varphi(x)$, the brief notation φ_∞. The latter is called the *amplification of* φ. Obviously, this is a morphism of left $\mathcal{B}(\mathbf{L})$-modules.

Imitating the definition of a completely bounded operator between quantum spaces, we give

Definition 2. An operator $\varphi : G \to E$ between **L**–spaces is called **L**–*bounded*, if its amplification φ_∞ is bounded, and it is called **L**–*contractive*, if φ_∞ is a contractive (= not increasing norms of vectors) operator.

The set of **L**-contractive operators between **L**-spaces G and E is denoted by $\mathcal{CC}(G, E)$; it is this set which is of most interest for us.

When we have an interesting concrete category, a typical problem is to describe its so-called projective objects, in the sense of this or that their reasonable definition. In algebra, "pure" and "Banach", these objects play the role of "homologically best". It was young Grothendieck, who was the first to investigate them in the frame-work of the classical analysis, namely in the theory of Banach spaces. I think it would be more understandable, if I shall at first give the respective definitions for "classical" Banach spaces, and then turn to their analogues for **L**-spaces.

Let me recall that an operator between two normed spaces is called *strictly coisometric* (or exact quotient mapping) if it maps the *closed* unit ball of the first space onto the *closed* unit ball of the second space, and it is called *coisometric* (or quotient mapping), if it maps the *open* unit ball of the first space onto the *open* unit ball of the second space. Students of the third year prove in their classes that the strict coisometry implies the coisometry, but the converse is, generally speaking, not true. And the most advanced of students are invited to prove that an operator is isometric (= preserves norms of vectors) $\Longleftrightarrow$ its adjoint operator is strictly coisometric $\Longleftrightarrow$ its adjoint operator is (just) coisometric.

Let P, G, E be linear spaces, $\tau : G \to E$, $\varphi : P \to E$ operators. We recall that an operator $\psi : P \to E$ is called a *lifting of* φ *across* τ, if it makes the diagram

$$\begin{array}{ccc} & & G \\ & \overset{\psi}{\nearrow} & \downarrow{\scriptstyle \tau} \\ P & \xrightarrow{\varphi} & E \end{array} \tag{1}$$

commutative.

In a similar way, one can define a lifting of a morphism in an arbitrary category.

In functional analysis there are two different approaches to the notion of the projectivity. The first one is, roughly speaking, general categorical, whereas the second one takes into account specific features of functional-analytic structures, based on the presence of norm.

Definition 3. A Banach space P is called *metrically projective*, if for every Banach spaces G, E, every strictly-coisometric operator $\tau : G \to E$ and every contractive operator $\varphi : P \to E$ there exists a contractive lifting of φ across τ. A Banach space P is called *extremely projective* (Banach geometers speak about "a space with the metric lifting property"), if for G, E as above, every coisometric operator $\tau : G \to E$ and for every $\varphi : P \to E$ with $\|\varphi\| < 1$ there exists a contractive lifting of φ across τ.

Grothendieck investigated extremely projective Banach spaces, and he proved that they are exactly $\ell_1(\Lambda)$, where Λ is an index set. Later the interest to the metric projectivity increased, and it was proved that this property characterizes the same $\ell_1(\Lambda)$. More of this, the same property, but considered in the context of all (not necessary complete) normed spaces, characterizes subspaces in $\ell_1(\Lambda)$, consisting of sequences, ending with zeroes, (with the same ℓ_1-norm).

In the "quantum" theory, by analogy with the classic one, its own twofold definition appears. We call an operator between **L**-spaces **L**-*strictly coisometric*, respectively, **L**-*coisometric*, if its amplification is strictly coisometric, respectively, coisometric.

Definition 4. An **L**–space P is called *metrically projective*, if for every **L**–spaces G, E, for every **L**-strictly-coisometric operator $\tau : G \to E$ and for every **L**-contractive operator $\varphi : P \to E$ there exists an **L**-contractive lifting of φ across τ.

An **L**–space P is called *extremely projective*, if for G, E as before, for every **L**-coisometric operator $\tau : G \to E$ and for every $\varphi : P \to E$ such that $\|\varphi_\infty\| < 1$, there exists an **L**-contractive lifting of φ across τ.

The main problem, considered in this talk, is as follows: which **L**–spaces are metrically, and which are extremely projective? Most of all, embarking from p-multinormed spaces, we are interested in the base spaces $L_p(X)$ and $L_p^0(X)$. Here is what is actually done:

1. For $\mathbf{L} := L_p^0(X)$ we have described all metrically projective $\mathbf{L}$-spaces.

2. For $\mathbf{L} = L_p(X)$ (as well as for the easier case $\mathbf{L} = L_p^0(X)$), we have described all extremely projective $\mathbf{L}$-spaces.

As a particular case of these results, we have described all metrically and all extremely projective p-multinormed spaces.

3. For $\mathbf{L} = L_p(X)$ we have found a broad class of metrically projective $\mathbf{L}$-spaces. (However, we do not know whether it embraces all metrically projective $\mathbf{L}$-spaces.)

Also there are results about general $\mathbf{L}$–spaces. We shall see that $\mathbf{L}$–spaces that are, in a sense, composed of certain sufficiently transparent "bricks" are projective. These bricks will be explicitly shown.

1 Functor $\odot$ and $\odot$-free L–spaces

To obtain the mentioned results, we use a mighty mean for the investigation of the projectivity provided by the theory of categories: freeness. What is it?

Let $\mathcal{K}$ be a (so far arbitrary) category, **Set**, as usually, the category of sets and maps, $\square : \mathcal{K} \to \mathbf{Set}$ some covariant functor, and M a set.

Definition 5. An object $\mathbf{F}^{\square}(M)$ in $\mathcal{K}$ is called a $\square$-*free object with the base* M, if, for every object $E \in \mathcal{K}$ there exists a bijection

$$\mathbf{I}_E : \mathbf{Set}(M, \square E) \to \mathcal{K}(\mathbf{F}^{\square}(M), E)$$

between the respective sets of morphisms, and these bijections are, as they say, *natural on the second argument.*

The latter means that for every morphism $\varphi : G \to E$ in $\mathcal{K}$ we have the commutative diagram

$$\begin{array}{ccc} \mathbf{Set}(M, \Box G) & \xrightarrow{\mathbf{I}_G} & \mathcal{CC}(\mathbf{F}^{\Box}(M), G) \\ {\scriptstyle (\Box\varphi)^*}\downarrow & & \downarrow{\scriptstyle \varphi^*} \\ \mathbf{Set}(M, \Box E) & \xrightarrow{\mathbf{I}_E} & \mathcal{CC}(\mathbf{F}^{\Box}(M), E).) \end{array} \tag{2}$$

where φ^* acts by composition as $\psi \mapsto \varphi \circ \psi$ and $(\Box\varphi)^*$ as $\rho \mapsto (\Box\varphi) \circ \rho$.

Henceforth, we shall write $\mathbf{F}(M)$ for $\mathbf{F}^{\Box}(M)$, if there is no confusion about the functor $\Box$.

But where is here the projectivity? The matter of fact is that there exists a general-categorical definition of the projectivity that also fits, as we shall soon see, for our concrete structures. Namely:

We shall call a morphism $\tau : G \to E$ in $\mathcal{K}$ $\Box$-admissible, if $\Box(\tau)$ is a surjective map (of sets).

Definition 6. An object $P \in \mathcal{K}$ is called $\Box$-*projective*, if for every $G, E \in \mathcal{K}$, $\Box$-admissible morphism $\tau : G \to E$ and every morphism $\varphi : P \to E$ there exists (in $\mathcal{K}$) a lifting of φ across τ.

Here is the gift of the theory of categories (read Mac Lane). I recall that an object $E \in \mathcal{K}$ is called a *retract* of an object $G \in \mathcal{K}$, if there exists in $\mathcal{K}$ a morphism from G into E, possessing a right inverse morphism. Here is a well known

Proposition 1. (i) For every set M, a $\square$-free object $\mathbf{F}^{\square}(M)$ (if it does exist), and every retract of this object are $\square$-projective.

(ii) If, in addition, every set is a base of a $\square$-free object, then (conversely) every $\square$–projective object is a retract of some $\square$-free object.

It is time to introduce our principal concrete category, denoted by $\mathbf{LNor_1}$. It objects are $\mathbf{L}$-spaces and its morphisms are $\mathbf{L}$-contractive operators. Let us also consider, for the better understanding, the "classical" prototype, so to say embryo, of this category, namely the category $\mathbf{Ban_1}$ with Banach spaces as its objects as contractive operators as its morphisms.

If E is a normed space, we shall denote by $B(E)$ its closed unit ball.

Let us consider the functor $\bigcirc : \mathbf{Ban_1} \to \mathbf{Set}$, taking a Banach space E to the set $B(E)$, and taking a contractive operator to its restriction to the respective unit balls. Then (as an easy exercise) one can see that $\bigcirc$-*admissible morphisms are exactly strictly coisometric operators.* It is easy to show that $\bigcirc$-free spaces are $\ell_1(\Lambda)$. From this it is possible to deduce (but after some work) what was said before: metrically, as well as extremely projective Banach spaces are the same $\ell_1(\Lambda)$. Therefore in the classical situation the projectivity coincides with the freeness (not a frequent phenomenon).

But what about our principal category $\mathbf{LNor_1}$? Here we need a more complicated functor. For a given $\mathbf{L}$ let us consider a family (to begin with, arbitrary) of its contractively complemented subspaces $L^\nu : \nu \in \Lambda$, where Λ is an index set. "Contractively complemented" means that there exists a projection from $\mathbf{L}$ onto L^ν of norm 1.

Now consider, for every **L**-space E, the set $\odot(E) := \mathsf{X}\{B(L^\nu \otimes E); \nu \in \Lambda\}$, that is the Cartesian product of the closed unit balls in $L^\nu \otimes E$ (with the norm, inherited from $\mathbf{L} \otimes E$). Thus, its elements can be thought as 'rows' $(..., v_\nu, ...)_{\nu\in\Lambda}$, with $v_\nu \in B(L^\nu \otimes E)$. Let us introduce the functor

$$\odot : \mathbf{LNor_1} \to \mathbf{Set},$$

taking E to $\odot(E)$ and an **L**-contractive operator $\varphi : G \to E$ to the map $\odot(\varphi) : \odot(G) \to \odot(E), (..., v_\nu, ...) \mapsto (..., \varphi_\infty(v_\nu), ...)$. (Such a map is obviously well defined.)

Besides, we shall call the base space $\mathbf{L}$ *properly presented* relative to a family $L^\nu : \nu \in \Lambda$ of its subspaces if, for every finite collection of vectors in $\mathbf{L}$ there exists $\nu \in \Lambda$ such that L^ν contains all these vectors. Our principal example is the family of all finite-dimensional subspaces in $L^0_p(X)$. (It is well known that they are contractively complemented; the respective projections of norm 1 are called conditional expectations, and they are of an independent interest.)

Proposition 2. Let $\mathbf{L}$ be an arbitrary base space, and $L^\nu : \nu \in \Lambda$ an arbitrary family of its contractively complemented subspaces. Then for the respective functor $\odot : \mathbf{LNor_1} \to \mathbf{Set}$ every $\mathbf{L}$-strictly coisometric operator is $\odot$-admissible. If, in addition, $\mathbf{L}$ is properly presented relative to such a family, then (conversely) every $\odot$-admissible operator is $\mathbf{L}$-strictly coisometric.

From what is said, taking into account the aforementioned general-categorical applications of the freeness to the projectivity (Proposition 1), we see that the following question appears: in which situations the functor $\odot$ admits free objects? Here is the answer:

Theorem 1. *Let* $\odot : \mathbf{LNor_1} \to \mathbf{Set}$ *be the functor, corresponding to an arbitrary* $\mathbf{L}$ *and arbitrary family* $L^\nu; \nu \in \Lambda$ *of its contractively complemented subspaces. Then the following assertions are equivalent:*

(*i*) *for every set* M *there exists an* $\odot$*-free* $\mathbf{L}$*-space with the base* M

(*ii*) *at least for one* M *there exists an* $\odot$*-free* $\mathbf{L}$*-space with the base* M

(*iii*) *all subspaces* L^ν *are finite-dimensional.*

So, it is to be finite-dimensional what matters. In particular, we see that, for our principal example $\mathbf{L} := L_p^0(X)$ and the family of all its finite-dimensional subspaces, every set is the base of a free $\mathbf{L}$-space.

How do they look, these $\odot$-free $\mathbf{L}$-spaces? To describe them, we need a notion of the so-called *minimal* $\mathbf{L}$-norm on a normed space, say E. We remember that we must introduce a norm on the space $\mathbf{L}\otimes E$. For an element u in the latter space we set $\|u\| := \sup\{|(f\otimes g)(u)|\}$, where the supremum is taken on all functionals from unit balls of dual spaces $\mathbf{L}^*$ and E^*. It is an $\mathbf{L}$-norm on E. (One could show that it is indeed the least of all reasonable, in a sense, $\mathbf{L}$-norms on E, but we do not need this.)

The spaces that we want to supply with the minimal $\mathbf{L}$-norm are of the form L^*, dual to a subspace L of our base space. In particular, when a family $L^\nu : \nu \in \Lambda$ of contractively complemented subspaces is given, we shall always suppose that every $(L^\nu)^*$ is endowed with the minimal $\mathbf{L}$-norm.

It is these $(L^\nu)^*$ that serve bricks for the building of our $\odot$-free $\mathbf{L}$-spaces. Namely, the latter spaces are the so-called coproducts of certain families of these bricks. The coproducts in $\mathbf{LNor_1}$ of a given family $(E_i)_{i\in\Delta}$ of $\mathbf{L}$-spaces can be explicitly defined. For this, we take the *algebraic* sum $\oplus_i E_i$ and make it an $\mathbf{L}$-space in the following way: we identify $\mathbf{L}\otimes(\oplus_i E_i)$ with $\oplus_i(\mathbf{L}\otimes E_i)$ and consider on the latter space the norm of the (non-completed) l_1-sum of its direct summands. (Otherwise, for an element $u\in\oplus_i(\mathbf{L}\otimes E_i); u=(...,u_i,...); u_i\in\mathbf{L}\otimes E_i$ we set $\|u\|_1:=\sum_i\|u_i\|$).) It is easy to verify that we obtain an $\mathbf{L}$-space, called the $\oplus_1$-*sum* of our spaces and denoted by $(\oplus_i E_i)_1$. This space, together with the corresponding family of the embedding of every E_i into $(\oplus_i E_i)_1$, is indeed the coproduct in $\mathbf{LNor_1}$ of the given family.

Theorem 2. *For every* $\mathbf{L}$ *and arbitrary family* $L^\nu; \nu \in \Lambda$ *of its contractively complemented finite-dimensional subspaces* $L^\nu; \nu \in \Lambda$ *we have the following:*

(i) $\mathbf{F}(\star) := (\oplus_\nu (L^\nu)^*)_1$ *is an* $\odot$*-free* $\mathbf{L}$*-space with a singleton* $\{\star\}$ *as its base.*

(ii) For any set M, $\mathbf{F}(M) := (\oplus_{t \in M} \mathbf{F}(t))_1$ *is a* $\odot$*-free* $\mathbf{L}$*-space with* M *as its base. Here,* $\mathbf{F}(t)$ *is* $\mathbf{F}(\star)$ *for* $t := \star$.

The proof, naturally, consists of a construction, for a respective M and an $\mathbf{L}$-space E, a desirable bijection between $\mathbf{Set}(M, \odot(E)$ and $\mathcal{CC}(\mathbf{F}(M), E)$, that is between the set of functions from M into $\mathsf{X}\{B(L^\nu \otimes E); \nu \in \Lambda\}$ and the set of all $\mathbf{L}$-contractive operators from $\mathbf{F}(M)$ into E. The following statement lies in the core of our construction.

Proposition 3. For every $\nu \in \Lambda$ there exists a well defined bijection

$$\mathbf{I}_E^\nu : B(L^\nu \otimes E) \to \mathcal{CC}((L^\nu)^*, E)$$

that takes an element $u \in B(L^\nu \otimes E)$ to the operator, acting as $g \in (L^\nu)^* \mapsto (g \otimes \mathbf{1}_E)(u) \in E$. Morecver, we have $\|u\| = \|\mathbf{I}_E^\nu(u))_\infty\|$.

(To understand this better, we recall that for $u = \xi \otimes x$ the vector $(g \otimes \mathbf{1}_E)(u)$ is, of course, $g(\xi)x$.)

As to the proof, I shall only say that we use the existence, in every finite-dimensional normed space, say L, of the so-called Auerbach basis, that is a normed basis $e_1, ..., e_n$ in L together with a normed basis $e_1^*, ..., e_n^*$ in L^* such that $e_k^*(e_l) = \delta_{kl}$. For $L := L^\nu$ this provides the element $w := \sum_k e_k \otimes e_k^* \in \mathbf{L} \otimes (L^\nu)^*$ which has norm 1 and happens to be very useful.

2 Applications to the projectivity

The connection between the freeness and the projectivity (expressed in Proposition 1), together with the knowledge of $\odot$-free **L**-spaces, enables us to characterize projective **L**-spaces.

We shall say that a given **L**-space P is *well composed*, if it is the coproduct in $\mathbf{LNor_1}$ of some family of spaces, dual to contractively complemented finite-dimensional subspaces of **L** (with possible repetitions). Otherwise, for some family $L^\nu; \nu \in \Lambda$ of such subspaces (where L^ν are permitted to coincide for different ν), our P is $(\oplus_\nu((L_\nu)^*)_1; \nu \in \Lambda$.

(We recall that every $(L_\nu)^*$ is endowed with the minimal **L**-norm.)

Of course, if all spaces L^ν are different and thus we can speak about the functor $\odot$, then all $\odot$-free **L**-spaces are well composed.

We already recalled what is a retract. As a more "tolerant" notion, an **L**-space E is called a *near-retract* of an **L**-space G, if there exists an **L**-contractive operator from G onto E such that for every $\varepsilon > 0$ it has a right inverse **L**-bounded operator $\rho : E \to G$ with $\|\rho_\infty\| < 1 + \varepsilon$.

Theorem 3. (*i*) *Every retract (in the category* $\mathbf{LNor}_1$*) of a well composed* **L***–space is metrically projective whereas every near-retract of a well composed* **L***–space is extremely projective. (In particular, of course, every well composed* **L***–space is metrically projective, and it is extremely projective.)*

(*ii*) *If, in addition,* **L** *is properly presented relative to the family* $L^\nu; \nu \in \Lambda$ *of all its contractively complemented finite-dimensional subspaces, then (conversely) every metrically projective* **L***–space is a retract, and every extremely projective* **L***–space is a near-retract of some well composed* **L***–space.*

Note that the part of this theorem concerning the metric projectivity can be deduced rather quickly from the general-categorical connection of the freeness and the projectivity. However the part concerning the extreme projectivity requires an additional “functional-analytic” argument.

Of course, p. (ii) concerns the spaces $L_p^0(X)$ (of simple functions) and, in particular, multinormed spaces. But what to do with $L_p(X)$, the completion of $L_p^0(X)$? Obviously, it is not properly presented. The question whether the part of p. (ii), concerning the metric projectivity, for $\mathbf{L} := L_p(X)$ is so far open.

But at the same time we have

Theorem 4. *Every extremely projective $L_p(X)$-space is a near-retract of some well composed $L_p(X)$-space.*

Why we can do for the extreme projectivity what we can not do for the metric projectivity? The deep reason of such a difference is as follows. An operator between $L_p^0(X)$-spaces, being considered (with the help of the extension by contunuity) as an operator between $L_p(X)$-spaces, preserves the property to be $\mathbf{L}$-coisomenric but, generally speaking, it can loose the property to be $\mathbf{L}$-*strictly* coisomenric.

THANK YOU

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