Globally Convergent Coderivative-Based Generalized Newton Methods in Nonsmooth Optimization

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Abstract. This paper proposes and justifies two new globally convergent Newton-type methods to solve unconstrained and constrained problems of nonsmooth optimization by using tools of variational analysis and generalized differentiation. Both methods are coderivative-based and employ generalized Hessians (coderivatives of subgradient mappings) associated with objective functions, which are either of class $\mathcal{C}^{1,1}$, or are represented in the form of convex composite optimization, where one of the terms may be extended-real-valued. The proposed globally convergent algorithms are of two types. The first one extends the damped Newton method and requires positive-definiteness of the generalized Hessians for its well-posedness and efficient performance, while the other algorithm is of the Levenberg-Marquardt type being well-defined when the generalized Hessians are merely positive-semidefinite. The obtained convergence rates for both methods are at least linear, but becomes superlinear under the so-called semismooth* property of subgradient mappings. Problems of convex composite optimization are investigated with and without the strong convexity assumption on of smooth parts of objective functions by implementing the machinery of forward-backward envelopes. Numerical experiments are conducted for a basic class of Lasso problems by providing performance comparisons of the new algorithms with some other first-order and second-order methods that are highly recognized in nonsmooth optimization.

Key words. Nonsmooth optimization, variational analysis, generalized Newton methods, global convergence, linear and superlinear convergence rates, convex composite optimization, Lasso problems

Mathematics Subject Classification (2000) 90C31, 49J52, 49J53

1 Introduction

It has been well recognized that the classical Newton method furnishes a highly efficient algorithm to solve unconstrained optimization problems of the type

$$\text{minimize } \varphi(x) \text{ subject to } x \in \mathbb{R}^n \tag{1.1}$$

with $\mathcal{C}^2$-smooth objective functions φ, provided that the Hessian matrix $\nabla^2\varphi(\bar{x})$ is positive-definite at the reference solution $\bar{x}$ and the starting point x^0 is chosen sufficiently close to $\bar{x}$. In this case, the Newton iterations exhibit the local convergence with a quadratic rate; see, e.g., [2, 22, 64].

To solve the optimization problem (1.1) globally, various line search algorithms are implemented by using iterative procedures of the form

$$x^{k+1} := x^k + \tau_k d^k \quad \text{for all } k \in \mathbb{N} := \{1, 2, \ldots\} \tag{1.2}$$

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with a step size $\tau_k \geq 0$ and a search direction $d^k \neq 0$. For Newton-type methods, the search directions are chosen by solving the linear equations

$$-\nabla\varphi(x^k) = H_k d^k, \tag{1.3}$$

where $H_k := \nabla^2\varphi(x^k)$ in the classical case, while H_k is an appropriate approximation of the Hessian for various *quasi-Newton methods*. An efficient way to choose H_k is provided by the *BFGS* (*Broyden-Fletcher-Goldfarb-Shanno*) *method*; see [14, 22, 37] for more details on this and related algorithms. If $H_k = \nabla^2\varphi(x^k)$ is positive-definite, algorithm (1.2) with the *backtracking line search* is called the *damped Newton method* [2, 7] to distinguish it from the *pure Newton method*, which uses a fixed step size. When the Hessian $\nabla^2\varphi(x^k)$ is merely positive-semidefinite, H_k in (1.3) is often taken as the regularized Hessian $H_k := \nabla^2\varphi(x^k) + \mu_k I$ with the sequence $\{\mu_k\}$ being chosen as $\mu_k := c\|\nabla\varphi(x^k)\|$ for some constant $c > 0$. The corresponding algorithm is called the *Levenberg-Marquardt method*. We refer the reader to [13, 41, 76] for many interesting results in this direction.

Among the most popular Newton-type methods to solve problems of nonsmooth optimization (1.1) with objective functions of class $\mathcal{C}^{1,1}$ (i.e., continuously differentiable with Lipschitzian gradients) is the *semismooth Newton method*. The literature on this method and its modifications is enormous; the reader is referred to, e.g., [22, 37, 39, 66] and the bibliographies therein for various developments and historical remarks. In fact, most of the known results address solving the equations $f(x) = 0$ with Lipschitzian vector functions f, as well as their generalized versions, to which optimization problems are reduced via stationary conditions (observe that this is not the case of our paper). The main idea behind the semismooth Newton method is the usage of Clarke's *generalized Jacobian* of Lipschitzian mappings. In this way, local convergence results, together with some globalization procedures, were obtained for this method under the *nonsingularity* of generalized Jacobians. The reader is referred to, e.g., [34, 75] for infinite-dimensional versions of the semismooth Newton method with applications to optimization and control problems governed by partial differential equations. Other versions of Newton-type methods to solve nonsmooth equations, generalized equations, optimization and variational problems can be found in [5, 16, 18, 22, 35, 37, 39, 58, 67] among other publications. We are not in a position here to review numerous contributions to Newtonian methods that are not directly related to our paper; see more commentaries below concerning publications related to our results.

In this paper, we develop two *globally convergent* generalized Newton algorithms to solve optimization problems (1.1) starting with the case of $\mathcal{C}^{1,1}$ objective functions (i.e., those being second-order nonsmooth) and then considering problems of *convex composite optimization* with objectives represented as sums of two convex functions such that one of them is smooth, while the other one may be extended-real-valued, which allows us to include problems of constrained optimization. The developed algorithms constitute the coderivative-based *generalized damped Newton method* (GDNM) and the *generalized Levenberg-Marquardt method* (GLMM) for the classes of problems under consideration.

Roughly speaking, the major feature of both generalized Newton algorithms developed here is the replacement of the classical Hessian $\nabla^2\varphi$ of $\mathcal{C}^2$-smooth functions by the *generalized Hessian* (or *second-order subdifferential*) $\partial^2\varphi$ of extended-real-valued, lower semicontinuous ones. This construction was introduced by Mordukhovich [48] as the *coderivative* of the subgradient mapping, while enjoying nowadays comprehensive calculus rules and constructive calculations for important classes of functions that naturally appear in variational analysis, optimization, and optimal control; see Section 2 for more details, discussions, and references.

Coderivatives have been recently employed by Gfrerer and Outrata [28] to design a *pure* Newton-type algorithm of solving *generalized equations* and–very differently–by Mordukhovich and Sarabi [56] to find local minimizers of (1.1) that were assumed to be *tilt-stable* in the sense of Poliquin and Rockafellar [63]. The results of [56] were obtained first for $\mathcal{C}^{1,1}$ objectives and then were propagated to a general class of *prox-regular* functions by using Moreau envelopes. The *local* superlinear convergence of these Newtonian algorithms was established in [28, 56] under the *semismooth** assumption on the mapping in question, the property introduced in [28] as a less restrictive version of semismoothness. The paper by Khanh et al. [38] developed a coderivative-based algorithm of the pure Newton type to solve *subgradient inclusions* defined by prox-regular functions with justifying the local superlinear convergence of iterates under the semismooth* property of the corresponding subgradient mapping.

As mentioned, this paper designs and justifies *globally* convergent algorithms for unconstrained problems of $\mathcal{C}^{1,1}$ optimization and generally constrained problems of convex composite optimization. For the latter class, we employ the *forward-backward envelope* (FBE), the construction that has been rather recently introduced in variational analysis and optimization, while has been already proved to be very useful in the framework of composite optimization; see, e.g., [73] with the references therein.

A central assumption in our *GDNM algorithm* for problems of $\mathcal{C}^{1,1}$ *optimization* is the *positive-definiteness* of the generalized Hessian $\partial^2\varphi$, which is a direct extension of that for the classical Hessian $\nabla^2\varphi$ in the damped Newton method. This assumption alone ensures that the GDNM is *well-defined* and converges globally to a *tilt-stable minimizer* of (1.1) with at least some *linear rate*. The *Q-superlinear* convergence rate of GDNM is guaranteed under the semismooth* property of the gradient mapping $\nabla\varphi$ and some relationship between parameters of the algorithm and the problem data.

The proposed *GLMM algorithm* to solve problems of $\mathcal{C}^{1,1}$ optimization does not generally require the positive-definiteness of the generalized Hessian $\partial^2\varphi$: we construct it and verify its *well-posedness* and *global convergence* to stationary points of φ under merely *positive-semidefiniteness* of the generalized Hessian. To establish results on the linear and superlinear *convergence rates* of GLMM, the *metric regularity* of the gradient mappings is additionally imposed; the property that has been well understood, characterized, and broadly applied in variational analysis and optimization.

Considering further problems of *convex composite optimization* in the form

$$\text{minimize } \varphi(x) := f(x) + g(x) \;\text{ subject to }\; x \in \mathbb{R}^n, \tag{1.4}$$

where f is a convex smooth function and $g\colon \mathbb{R}^n \to \overline{\mathbb{R}} := (-\infty, \infty]$ is a lower semicontinuous (l.s.c.) extended-real-valued convex function, we reduce them to $\mathcal{C}^{1,1}$ optimization by using *FBE*. Employing *second-order calculus rules* allows us to express the generalized Hessian of FBE via the problem data and relate the metric regularity and tilt stability properties of φ from (1.4) to the corresponding ones for FBE. In this way, we establish constructive results on well-posedness, global convergence, and convergence rates for both GDNM and GLMM algorithms to solve (1.4) *with* and *without* the *strong convexity* assumption on f in a highly important case of quadratic functions in (1.4); see below.

The results established by using both GDNM and GLMM for problems of convex composite optimization with quadratic functions f in (1.4) are employed to solve a basis class of *Lasso problems*, which can be written in this form. Such problems were introduced by Tibshirani [74] motivated by applications to statistics, and since that they have been largely investigated and applied to practical models in machine learning, image processing, etc. Computing all the ingredients of both algorithms in Lasso terms, we conduct MATLAB numerical experiments by using random data sets. To compare the performance of GDNM and GLMM with other quite popular and efficient algorithms of nonsmooth optimization, we conduct parallel numerical experiments with the same Lasso data for the recent second-order *Semismooth Newton Augmented Lagrangian Methods* (SSNAL) developed in [42] and the two well-recognized first-order methods: the *Alternating Direction Method of Multipliers* (ADMM) taken from [6] and the *Fast Iterative Shrinkage-Thresholding Algorithm* (FISTA) developed in [3].

The subsequent parts of the paper are organized as follows. In Section 2 we briefly overview the tools of variational analysis and generalized differentiation used in our algorithmic developments. Section 3 describes and justifies the coderivative-based GDNM to solve problems of $\mathcal{C}^{1,1}$ optimization. In Section 4 we design the coderivative-based GLMM for the same class of problems with deriving well-posedness and global convergence results. Section 5 develops both GDNM and GLMM to solve problems of convex composite optimization. In Section 6 we conduct numerical experiments of employing GDNM and GLMM to solve the basic Lasso problem and then comparing the achieved numerical results with those obtained by using SSNAL, ADMM, and FISTA. The final Section 7 lists the main achievements of the paper and discusses some topics of our future research. For the reader's convenience, we place several technical lemmas in the Appendix.

2 Preliminaries from Variational Analysis

This section presents some preliminaries from variational analysis and generalized differentiation that are broadly employed in what follows. The reader can find more details in the monographs [50, 51, 70] from which we borrow the standard notation used below. Recall that $\mathbb{N} := \{1, 2, \ldots\}$.

Given a set-valued mapping (multifunction) $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ between finite-dimensional spaces, its (sequential Pailnlevé-Kuratowski) *outer limit* at $\bar{x}$ is defined by

$$\mathop{\text{Lim}\,\text{sup}}_{x\to\bar{x}} F(x) := \big\{ y \in \mathbb{R}^n \;\big|\; \exists \text{ sequences } \; x_k \to \bar{x},\; y_k \to y \;\text{ with }\; y_k \in F(x_k), \;\; k \in \mathbb{N} \big\}.$$

Using the notation $z \xrightarrow{\Omega} \bar{z}$ meaning that $z \to \bar{z}$ with $z \in \Omega$ for a given nonempty set $\Omega \subset \mathbb{R}^s$, the (Fréchet) *regular normal cone* to Ω at $\bar{z} \in \Omega$ is

$$\widehat{N}_\Omega(\bar{z}) := \Big\{ v \in \mathbb{R}^s \;\Big|\; \limsup_{z \xrightarrow{\Omega} \bar{z}} \frac{\langle v, z - \bar{z}\rangle}{\|z - \bar{z}\|} \le 0 \Big\},$$

while the (Mordukhovich) *limiting normal cone* to Ω at $\bar{z} \in \Omega$ is defined by

$$N_\Omega(\bar{z}) := \mathop{\text{Lim}\,\text{sup}}_{z \xrightarrow{\Omega} \bar{z}} \widehat{N}_\Omega(z) = \big\{ v \in \mathbb{R}^s \;\big|\; \exists\, z_k \xrightarrow{\Omega} \bar{z},\; v_k \to v \;\text{ as }\; k \to \infty \;\text{ with }\; v_k \in \widehat{N}_\Omega(z_k) \big\}. \tag{2.1}$$

The corresponding limiting *coderivative* of $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is defined (2.1) by

$$D^*F(\bar{x}, \bar{y})(v) := \big\{ u \in \mathbb{R}^n \;\big|\; (u, -v) \in N_{\operatorname{gph} F}(\bar{x}, \bar{y}) \big\}, \quad v \in \mathbb{R}^m, \tag{2.2}$$

where $\operatorname{gph} F := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in F(x)\}$, and where $\bar{y}$ is omitted in the coderivative notation if $F(\bar{x})$ is a singleton. If $F\colon \mathbb{R}^n \to \mathbb{R}^m$ is a single-valued mapping which is $\mathcal{C}^1$-smooth around $\bar{x}$, then

$$D^*F(\bar{x})(v) = \big\{ \nabla F(\bar{x})^* v \big\} \;\text{ for all }\; v \in \mathbb{R}^m$$

via the adjoint/transpose Jacobian matrix $\nabla F(\bar{x})^*$. The defined coderivative of general multifunctions satisfies comprehensive *calculus rules* based on *variational/extremal principles* of variational analysis. Among the most impressive and useful advantages of the coderivative (2.2) are complete characterizations in its terms the fundamental well-posedness properties (metric regularity, linear openness, and Lipschitzian behavior) of general multifunctions that were developed in [49] and were labeled in [70] as the *Mordukhovich criteria*. In this paper we employ these characterizations for the property of *metric regularity* of $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ meaning that there exist a number $\mu > 0$ and neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\operatorname{dist}\big(x; F^{-1}(y)\big) \le \mu \operatorname{dist}\big(y; F(x)\big) \;\text{ for all }\; (x, y) \in U \times V, \tag{2.3}$$

where $F^{-1}(y) := \{x \in \mathbb{R}^n \mid y \in F(x)\}$, and where 'dist' stands for the distance between a point and a set. If in addition F^{-1} has a single-valued localization around $(\bar{y}, \bar{x})$, i.e., there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ together with a single-valued mapping $\vartheta\colon V \to U$ such that $\operatorname{gph} F^{-1} \cap (V \times U) = \operatorname{gph} \vartheta$, then F is *strongly metrically regular* around $(\bar{x}, \bar{y})$ with modulus $\mu > 0$. The aforementioned coderivative characterization from [49, Theorem 3.6] tells us that, whenever F is closed-graph around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, its metric regularity around this point is equivalent to the implication

$$v \in \mathbb{R}^m,\; 0 \in D^*F(\bar{x}, \bar{y})(v) \Longrightarrow v = 0. \tag{2.4}$$

Moreover, the *exact regularity bound* of F at $(\bar{x}, \bar{y})$, i.e., the infimum of all $\mu > 0$ such that (2.3) holds for some neighborhoods U and V, is calculated by

$$\operatorname{reg} F(\bar{x}, \bar{y}) = \|D^*F(\bar{x}, \bar{y})^{-1}\| = \|D^*F^{-1}(\bar{y}, \bar{x})\| \tag{2.5}$$

via the norm of the coderivatives of F and F^{-1} as positive homogeneous multifunctions; see [50, 51, 70] for more discussions, different proofs, and infinite-dimensional extensions.

Another variational notion useful in what follows concerns a strong version of local monotonicity for set-valued operators. We say that $T\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *strongly locally monotone* with modulus $\kappa > 0$ around $(\bar{x}, \bar{y}) \in \operatorname{gph} T$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\langle x - u, v - w\rangle \ge \kappa \|x - u\|^2 \quad \text{for all } \; (x, v), (u, w) \in \operatorname{gph} T \cap (U \times V).$$

If in addition $\operatorname{gph} T \cap (U \times V) = \operatorname{gph} S \cap (U \times V)$ for any monotone operator $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ satisfying $\operatorname{gph} T \cap (U \times V) \subset \operatorname{gph} S$, then T is *strongly locally maximal monotone* with modulus $\kappa > 0$ around $(\bar{x}, \bar{y})$. We refer the reader to [53] and [51, Section 5.2] for coderivative characterizations of the latter property, which is significantly more relaxed than the strong metric regularity of T around $(\bar{x}, \bar{y})$.

Next we consider an extended-real-valued function $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ with the domain and epigraph

$$\operatorname{dom} \varphi := \big\{ x \in \mathbb{R}^n \,\big|\, \varphi(x) < \infty \big\}, \quad \operatorname{epi} \varphi := \big\{ (x, \alpha) \in \mathbb{R}^{n+1} \,\big|\, \alpha \geq \varphi(x) \big\}.$$

The (limiting) *subdifferential* of φ at $\bar{x} \in \operatorname{dom} \varphi$ is defined geometrically by

$$\partial \varphi(\bar{x}) := \big\{ v \in \mathbb{R}^n \,\big|\, (v, -1) \in N_{\operatorname{epi} \varphi}\big(\bar{x}, \varphi(\bar{x})\big) \big\} \tag{2.6}$$

via the limiting normal cone (2.1), while admitting various analytic representations and satisfying comprehensive calculus rules that can be found in[50, 51, 70]. Observe the useful scalarization formula

$$D^* F(\bar{x})(v) = \partial \langle v, F \rangle(\bar{x}) \;\text{ for all }\; v \in \mathbb{R}^m. \tag{2.7}$$

connecting the coderivative (2.2) of a locally Lipschitzian mapping $F\colon \mathbb{R}^n \to \mathbb{R}^m$ and the subdifferential (2.6) of the function $x \mapsto \langle v, F \rangle(x)$ whenever $v \in \mathbb{R}^m$.

Following [48], we define the *second-order subdifferential*, or *generalized Hessian*, $\partial^2 \varphi(\bar{x}, \bar{v})\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ at $\bar{x} \in \operatorname{dom} \varphi$ for $\bar{v} \in \partial \varphi(\bar{x})$ as the coderivative of the subgradient mapping

$$\partial^2 \varphi(\bar{x}, \bar{v})(u) := \big(D^* \partial \varphi\big)(\bar{x}, \bar{y})(u) \;\text{ for all }\; u \in \mathbb{R}^n. \tag{2.8}$$

If φ is $\mathcal{C}^2$-smooth around $\bar{x}$, then we have

$$\partial^2 \varphi(\bar{x})(u) = \big\{ \nabla^2 \varphi(\bar{x}) u \big\} \;\text{ for all }\; u \in \mathbb{R}^n, \tag{2.9}$$

while for φ of class $\mathcal{C}^{1,1}$ around $\bar{x}$, we get by the scalarization formula (2.7) that

$$\partial^2 \varphi(\bar{x})(u) = \partial \big\langle u, \nabla \varphi(\bar{x}) \big\rangle \;\text{ for all }\; u \in \mathbb{R}^n. \tag{2.10}$$

As follows from (2.10), calculus rules and computations of the second-order subdifferential for $\mathcal{C}^{1,1}$ functions reduce in fact to those for the first-order construction (2.6). We also have well-developed second-order calculus rules for (2.8) for rather general classes of extended-real-valued functions; see, e.g., [50, 51, 55] with many additional references. Furthermore, the second-order subdifferential has been computed and analyzed in terms of the given data for broad classes of structural functional systems appearing in numerous aspects of variational analysis, optimization, stability, and optimal control among other areas, with subsequent applications to optimality conditions, sensitivity analysis, numerical algorithms, stochastic programming, electricity markets, etc. The reader can find more information in, e.g., [9, 11, 15, 17, 31, 32, 54, 48, 50, 51, 55, 69, 77] along with other publications on such developments and related topics of second-order variational analysis. Some new results in this direction are presented in what follows.

In this paper, we use the fundamental notion of *tilt-stable local minimizers* and its second-order characterizations for the justification of the proposed Newton-type algorithms.

Definition 2.1 (tilt-stable local minimizers). *Given $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$, a point $\bar{x} \in \operatorname{dom} \varphi$ is a* TILT-STABLE LOCAL MINIMIZER *of φ if there exists a number $\gamma > 0$ such that the mapping*

$$M_\gamma\colon v \mapsto \operatorname{argmin}\big\{ \varphi(x) - \langle v, x \rangle \,\big|\, x \in \mathbb{B}_\gamma(\bar{x}) \big\}$$

is single-valued and Lipschitz continuous on some neighborhood of $0 \in \mathbb{R}^n$ with $M_\gamma(0) = \{\bar{x}\}$. By a MODULUS *of tilt stability of φ at $\bar{x}$ we understand a Lipschitz constant of M_γ around the origin.*

This notion was introduced by Poliquin and Rockafellar in [63] and characterized there via $\partial^2 \varphi$ for a broad class of prox-regular functions $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ that are overwhelmingly involved in second-order variational analysis. More recently, developing second-order subdifferential calculus and second-order growth conditions made it possible to establish complete characterizations of tilt-stable local minimizers for various classes of problems in constrained optimization including nonlinear programming, composite optimization, second-order cone programming, semidefinite programming, etc.; see, e.g., [10, 19, 20, 27, 51, 52, 55] among other publications on tilt stability in optimization.

Finally in this section, we recall the notions of convergence rates used for our algorithms.

Definition 2.2 (**rates of convergence**). *Let $\{x^k\} \subset \mathbb{R}^n$ be a sequence of vectors converging to $\bar{x}$ as $k \to \infty$ with $\bar{x} \neq x^k$ for all $k \in \mathbb{N}$. The convergence rate is said to be:*

(i) R-LINEAR *if we have*

$$0 < \limsup_{k\to\infty} \left(\|x^k - \bar{x}\|\right)^{1/k} < 1,$$

i.e., there exist $\mu \in (0,1)$, $c > 0$, and $k_0 \in \mathbb{N}$ such that

$$\|x^k - \bar{x}\| \le c\mu^k \quad \textit{for all} \;\; k \ge k_0.$$

(ii) Q-LINEAR *if we have*

$$\limsup_{k\to\infty} \frac{\|x^{k+1} - \bar{x}\|}{\|x^k - \bar{x}\|} < 1,$$

i.e., there exist $\mu \in (0,1)$ and $k_0 \in \mathbb{N}$ such that

$$\|x^{k+1} - \bar{x}\| \le \mu\|x^k - \bar{x}\| \quad \textit{for all} \;\; k \ge k_0.$$

(iii) Q-SUPERLINEAR *if we have*

$$\lim_{k\to\infty} \frac{\|x^{k+1} - \bar{x}\|}{\|x^k - \bar{x}\|} = 0.$$

3 Coderivative-Based Damped Newton Method in $\mathcal{C}^{1,1}$ Optimization

In this section, we concentrate on the unconstrained optimization problem (1.1), where the cost function $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is of class $\mathcal{C}^{1,1}$. A coderivative-based generalization of the pure Newton method to solve (1.1) *locally* was first suggested and investigated in [56] under the major assumption that a given point $\bar{x}$ is a *tilt-stable* local minimizer of (1.1). Then it was extended in [38] to solve directly the gradient system $\nabla\varphi(x) = 0$ under certain assumptions on a given solution $\bar{x}$ of the gradient equation ensuring the well-posedness and local superlinear convergence of the algorithm. One of the serious disadvantages of the pure Newton method and its generalizations is that the corresponding sequence of iterates may not converge if the starting point is not sufficiently close to the solution. This motivates us to design and justify a *globally* convergent *damped Newton* counterpart of the generalized pure Newton algorithms from [38, 56] with *backtracking line search* to solve (1.1). Here is the algorithm.

Algorithm 1 Globally coderivative-based damped Newton algorithm for $\mathcal{C}^{1,1}$ functions

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Input: $x^0 \in \mathbb{R}^n$, $\sigma \in \left(0, \frac{1}{2}\right)$, $\beta \in (0,1)$

1: **for** $k = 0, 1, \ldots$ **do**
2: If $\nabla\varphi(x^k) = 0$, stop; otherwise go to the next step
3: Choose $d^k \in \mathbb{R}^n$ such that $-\nabla\varphi(x^k) \in \partial^2\varphi(x^k)(d^k)$
4: Set $\tau_k = 1$.
5: **while** $\varphi(x^k + \tau_k d^k) > \varphi(x^k) + \sigma\tau_k\langle d^k, \nabla\varphi(x^k)\rangle$ **do**
6: set $\tau_k := \beta\tau_k$
7: **end while**
8: Set $x^{k+1} := x^k + \tau_k d^k$
9: **end for**

If φ is $\mathcal{C}^2$-smooth, Algorithm 1 reduces to the standard damped Newton method (as, e.g., in [2, 7]) due to (2.9). In the general case of $\varphi \in \mathcal{C}^{1,1}$, it follows from (2.2) that the direction d^k in (3) can be explicitly found from the inclusion

$$\big(-\nabla\varphi(x^k), -d^k\big) \in N\big((x^k, \nabla\varphi(x^k)); \operatorname{gph}\nabla\varphi\big).$$

Note also that, due to the scalarization formula (2.10), the Newton equation in Step 3 of Algorithm 1 can be equivalently written in the form

$$-\nabla\varphi(x^k) \in \partial\langle d^k, \nabla\varphi\rangle(x^k),$$

which merely requires the first-order subdifferential computation.

We start justifying Algorithm 1 with the verification of its *well-posedness*. The following proposition establishes the existence of *descent* Newton directions under the *positive-definiteness* of the generalized Hessian mapping $\partial^2\varphi(x)$.

Proposition 3.1 (existence of descent Newton directions). *Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be of class $\mathcal{C}^{1,1}$ on $\mathbb{R}^n$. Suppose that $\nabla\varphi(x) \neq 0$ and that the mapping $x \mapsto \partial^2\varphi(x)$ is positive-definite on $\mathbb{R}^n$, i.e.,*

$$\langle v, u\rangle > 0 \;\text{ for all }\; v \in \partial^2\varphi(x)(u) \;\text{ and }\; u \neq 0. \tag{3.1}$$

Then there exists a nonzero direction $d \in \mathbb{R}^n$ such that

$$-\nabla\varphi(x) \in \partial^2\varphi(x)(d). \tag{3.2}$$

Moreover, every such direction satisfies the inequality $\langle\nabla\varphi(x), d\rangle < 0$. Consequently, for each $\sigma \in (0,1)$ and $d \in \mathbb{R}^n$ satisfying (3.2) we find $\delta > 0$ such that

$$\varphi(x+\tau d) \le \varphi(x) + \sigma\tau\langle\nabla\varphi(x), d\rangle \;\text{ whenever }\; \tau \in (0,\delta). \tag{3.3}$$

Proof. By the positive-definiteness of $\partial^2\varphi$, it follows from [51, Theorem 5.16] that $\nabla\varphi$ is locally strongly maximal monotone around $(x, \nabla\varphi(x))$. Thus $\nabla\varphi$ is strongly metrically regular around $(x, \nabla\varphi(x))$ by [51, Theorem 5.13]. Using [38, Corollary 4.2] yields the existence of $d \in \mathbb{R}^n$ with $-\nabla\varphi(x) \in \partial^2\varphi(x)(d)$. To verify further that $d \neq 0$, suppose on the contrary that $d = 0$. Since $\nabla\varphi$ is locally Lipschitz around x, it follows from [50, Theorem 1.44] that

$$-\nabla\varphi(x) \in \partial^2\varphi(x)(0) = \big(D^*\nabla\varphi\big)(x)(0) = \{0\},$$

which contradicts the assumption that $\nabla\varphi(x) \neq 0$. Employing again the positive-definiteness of $\partial^2\varphi$ tells us that $\langle\nabla\varphi(x), d\rangle < 0$. Using finally [37, Lemmas 2.18 and 2.19], we arrive at (3.3) and thus complete the proof of the proposition. $\square$

The next theorem establishes the *global linear convergence* of Algorithm 1 to a *tilt-stable minimizer* of (1.1) under the positive-definiteness assumption on the generalized Hessian $\partial^2\varphi$.

Theorem 3.2 (global linear convergence of the coderivative-based damped Newton algorithm for $\mathcal{C}^{1,1}$ functions). *Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be of class $\mathcal{C}^{1,1}$, and let $x^0 \in \mathbb{R}^n$ be an arbitrary point such that the generalized Hessian $\partial^2\varphi(x)$ is positive-definite for all $x \in \Omega$, where*

$$\Omega := \big\{x \in \mathbb{R}^n \;\big|\; \varphi(x) \le \varphi(x^0)\big\}. \tag{3.4}$$

Then Algorithm 1 either stops after finitely many iterations, or produces a sequence $\{x^k\} \subset \Omega$ such that $\{\varphi(x^k)\}$ is monotonically decreasing. Moreover, if the iterative sequence $\{x^k\}$ has a limiting point $\bar{x}$ (in particular, when the level set Ω from (3.4) is bounded), then $\{x^k\}$ converges to $\bar{x}$, which is a tilt-stable local minimizer of φ. In this case, we have:

- **(i)** *The convergence rate of $\{\varphi(x^k)\}$ is at least Q-linear.*
- **(ii)** *The convergence rates of $\{x^k\}$ and $\{\|\nabla\varphi(x^k)\|\}$ are at least R-linear.*

Proof. Proposition 3.1 easily ensures by induction that Algorithm 1 either stops after finitely many iterations, or produces a sequence $\{x^k\} \subset \Omega$ such that $\varphi(x^{k+1}) < \varphi(x^k)$ for all $k \in \mathbb{N}$. Suppose next that $\{x^k\}$ has a limiting point $\bar{x}$. Since the set Ω is closed, we get $\bar{x} \in \Omega$, and hence have that $\partial^2\varphi(\bar{x})$ is positive-definite. Then [9, Proposition 4.6] gives us positive numbers κ and δ such that

$$\langle z, w\rangle \ge \kappa\|w\|^2 \quad \text{for all } \; z \in \partial^2\varphi(x)(w), \; x \in \mathbb{B}_\delta(\bar{x}), \; \text{ and } \; w \in \mathbb{R}^n. \tag{3.5}$$

Since φ is of class $\mathcal{C}^{1,1}$ around $\bar{x}$, we get without loss of generality that $\nabla\varphi$ is Lipschitz continuous on $\mathbb{B}_\delta(\bar{x})$ with some constant $\ell > 0$. By [50, Theorem 1.44] we have

$$\|z\| \le \ell\|w\| \quad \text{for all} \;\; z \in \partial^2\varphi(x)(w),\; x \in \mathbb{B}_\delta(\bar{x}), \;\; \text{and} \;\; w \in \mathbb{R}^n. \tag{3.6}$$

The rest of the proof is split into the following four claims.

Claim 1: *For any subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that $x^{k_j} \to \bar{x}$ as $j \to \infty$, the corresponding sequence $\{\tau_{k_j}\}$ in Algorithm 1 is bounded from below by a positive number γ, and we have*

$$\varphi(x^{k_j}) - \varphi(x^{k_j+1}) \ge \sigma\gamma\kappa\|d^{k_j}\|^2 \quad \textit{for all large } j \in \mathbb{N}. \tag{3.7}$$

Suppose on the contrary that $\{\tau_{k_j}\}_{j\in\mathbb{N}}$ is not bounded from below by a positive number. Combining this with $\tau_k \ge 0$ gives us a subsequence of $\{\tau_{k_j}\}$ that converges to 0. Assume without loss of generality that $\tau_{k_j} \to 0$ as $j \to \infty$. Since $-\nabla\varphi(x^{k_j}) \in \partial^2\varphi(x^{k_j})(d^{k_j})$ for all $j \in \mathbb{N}$, we deduce from (3.5) that

$$\langle -\nabla\varphi(x^{k_j}), d^{k_j}\rangle \ge \kappa\|d^{k_j}\|^2 \quad \text{for large} \;\; j \in \mathbb{N}. \tag{3.8}$$

The Cauchy-Schwarz inequality yields $\|\nabla\varphi(x^{k_j})\| \ge \kappa\|d^{k_j}\|$ for such j that verifies the boundedness of the sequence of directions $\{d^{k_j}\}$. Thus $x^{k_j} + \beta^{-1}\tau_{k_j}d^{k_j} \to \bar{x}$ as $j \to \infty$, and hence $x^{k_j} + \beta^{-1}\tau_{k_j}d^{k_j} \in \mathbb{B}_\delta(\bar{x})$ whenever j is sufficiently large. Furthermore, the exit condition of the backtracking line search in Step 5 of Algorithm 1 brings us to the strict inequality

$$\varphi(x^{k_j} + \beta^{-1}\tau_{k_j}d^{k_j}) > \varphi(x^{k_j}) + \sigma\beta^{-1}\tau_{k_j}\langle\nabla\varphi(x^{k_j}), d^{k_j}\rangle \tag{3.9}$$

for large j. Due to (3.8) and (3.9), we can apply Lemma 7.1 from the Appendix with

$$x := x^{k_j} \; y := x^{k_j} + \beta^{-1}\tau_{k_j}d^{k_j},\; c_1 := \sigma, \;\; \text{and} \;\; c_2 := \kappa\beta\tau_{k_j}^{-1}$$

for such j, which imply in turn that $\sigma > 1 - \frac{\ell}{2\beta\kappa}\tau_{k_j}$ for large j. Letting $j \to \infty$ gives us $\sigma \ge 1$, a contradiction due to the choice of $\sigma < 1$. Hence there exists $\gamma > 0$ such that $\tau_{k_j} \ge \gamma$ for all $j \in \mathbb{N}$. Moreover, using the estimate in (3.8) allows us to find $j_0 \in \mathbb{N}$ such that

$$\varphi(x^{k_j}) - \varphi(x^{k_j+1}) \ge \sigma\tau_{k_j}\langle -\nabla\varphi(x^{k_j}), d^{k_j}\rangle \ge \sigma\gamma\kappa\|d^{k_j}\|^2 \quad \text{for all} \;\; j \ge j_0, \tag{3.10}$$

which therefore justifies Claim 1.

Claim 2: *$\bar{x}$ is a tilt-stable local minimizer of φ.* To verify this, we only need to show that $\bar{x}$ is a stationary point of φ, by taking into account the positive-definiteness of $\partial^2\varphi(\bar{x})$ and the second-order characterization of tilt-stability from [63, Theorem 1.3]. Since $\bar{x}$ is a limiting point of $\{x^k\}$, there exists a subsequence $\{x^{k_j}\}_{j\in\mathbb{N}}$ of $\{x^k\}$ such that $x^{k_j} \to \bar{x}$ as $j \to \infty$. Due to Claim 1, we find $\gamma > 0$ such that (3.7) is satisfied. Since the sequence $\{\varphi(x^k)\}$ is nonincreasing and since $\varphi(\bar{x})$ is a limiting point of $\{\varphi(x^k)\}$, the sequence $\{\varphi(x^k)\}$ must converge to $\varphi(\bar{x})$ as $k \to \infty$. Letting $j \to \infty$ in the inequality (3.7), we have $\|d^{k_j}\| \to 0$ as $j \to \infty$. Moreover, the estimate (3.6) together with the inclusion $-\nabla\varphi(x^{k_j}) \in \partial^2\varphi(x^{k_j})(d^{k_j})$ implies that $\|\nabla\varphi(x^{k_j})\| \le \ell\|d^{k_j}\|$ for all large j. Passing to the limit as $j \to \infty$ in this inequality tells us that $\nabla\varphi(\bar{x}) = 0$, which readily justifies Claim 2.

Claim 3: *The iterative sequence $\{x^k\}$ is convergent.* To verify this claim, we use Ostrowski's condition from [22, Proposition 8.3.10]. Let us first show that no other limiting point of $\{x^k\}$ exists in $\mathbb{B}_\delta(\bar{x})$. Assuming the contrary, we find $\widetilde{x} \in \mathbb{B}_\delta(\bar{x})$ such that $\widetilde{x} \ne \bar{x}$ and that $\widetilde{x}$ is a limiting point of $\{x^k\}$. Arguing similarly to Claim 2 tells us that $\widetilde{x}$ is also a tilt-stable local minimizer of φ, which contradicts the strong convexity of φ on $\mathbb{B}_\delta(\bar{x})$. Supposing next that $\{x^{k_j}\}$ is an arbitrary subsequence of $\{x^k\}$ with $x^{k_j} \to \bar{x}$ as $j \to \infty$, we need to show that

$$\lim_{j\to\infty}\|x^{k_j+1} - x^{k_j}\| = 0. \tag{3.11}$$

Indeed, Claim 1 gives us $\gamma > 0$ such that (3.7) holds, which implies in turn that

$$\|x^{k_j+1} - x^{k_j}\|^2 = \tau_{k_j}^2\|d^{k_j}\|^2 \le \|d^{k_j}\|^2 \le \frac{1}{\sigma\gamma\kappa}\left(\varphi(x^{k_j}) - \varphi(x^{k_j+1})\right) \to 0$$

and hence verifies (3.11). Finally, it follows from [22, Proposition 8.3.10] that the sequence $\{x^k\}$ converges to $\bar{x}$ as $k \to \infty$, which therefore completes the proof of Claim 3.

Claim 4: *The convergence rate of $\{\varphi(x^k)\}$ is at least Q-linear, while the convergence rates of $\{x^k\}$ and $\{\|\nabla\varphi(x^k)\|\}$ are at least R-linear.* Indeed, the strong convexity of φ on $\mathbb{B}_\delta(\bar{x})$ shows that

$$\langle \nabla\varphi(x) - \nabla\varphi(u), x - u\rangle \geq \kappa\|x-u\|^2 \quad \text{for all} \quad x,u \in \mathbb{B}_\delta(\bar{x}). \tag{3.12}$$

Since $x^k \to \bar{x}$ as $k \to \infty$, we have that $x^k \in U$ for all $k \in \mathbb{N}$ sufficiently large. Substituting $x := x^k$ and $u := \bar{x}$ into (3.12) and then using the Cauchy-Schwarz inequality together with the stationary condition $\nabla\varphi(\bar{x}) = 0$ give us the estimate

$$\|\nabla\varphi(x^k)\| \geq \kappa\|x^k - \bar{x}\| \tag{3.13}$$

for large k. The local Lipschitz continuity of $\nabla\varphi$ around $\bar{x}$ and the result of [37, Lemma A.11] ensure the existence of $\ell > 0$ such that

$$\varphi(x^k) - \varphi(\bar{x}) = |\varphi(x^k) - \varphi(\bar{x}) - \langle \nabla\varphi(\bar{x}), x^k - \bar{x}\rangle| \leq \frac{\ell}{2}\|x^k - \bar{x}\|^2 \quad \text{for large} \quad k. \tag{3.14}$$

Furthermore, estimate (3.6) together with the inclusion $-\nabla\varphi(x^k) \in \partial^2\varphi(x^k)(d^k)$ implies that

$$\|\nabla\varphi(x^k)\| \leq \ell\|d^k\| \quad \text{for large} \quad k. \tag{3.15}$$

Claim 1 tells us that the sequence $\{\tau_k\}$ is bounded from below by some constant $\gamma > 0$ such that

$$\varphi(x^k) - \varphi(x^{k+1}) \geq \sigma\gamma\kappa\|d^k\|^2 \quad \text{for large } k.$$

Combining the above inequality with (3.15) yields the estimates

$$\varphi(x^k) - \varphi(x^{k+1}) \geq \sigma\gamma\kappa\ell^{-2}\|\nabla\varphi(x^k)\|^2 \tag{3.16}$$

whenever k is sufficiently lsrge. Finally, using (3.13), (3.14), (3.16) and then applying Lemma 7.2 from the Appendix with the sequences

$$\alpha_k := \varphi(x^k) - \varphi(\bar{x}),\ \beta_k := \|\nabla\varphi(x^k)\|, \text{ and } \ \gamma_k := \|x^k - \bar{x}\|$$

as well as with the positive numbers $c_1 := \sigma\gamma\kappa\ell^{-2}$ and $c_2 := \kappa$, $c_3 = \ell/2$, we verify Claim 4 and thus completes the proof of the theorem. $\square$

Our next goal in this section is to establish the *Q-superlinear convergence* of Algorithm 1. Let us first recall some additional notions of variational analysis and generalized differentiation needed for these developments. A highly recognized concept used for Newton-type methods dealing with single-valued Lipschitz continuous mappings is known as semismoothness. A mapping $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is *semismooth* at $\bar{x}$ if it is locally Lipschitzian around this point and the limit

$$\lim_{\substack{A \in \text{co}\overline{\nabla} f(\bar{x}+tu') \\ u' \to u, t\downarrow 0}} Au' \tag{3.17}$$

exists for all $u \in \mathbb{R}^n$, where 'co' stands for the convex hull of a set, and where $\overline{\nabla} f$ is defined by

$$\overline{\nabla} f(x) := \big\{A \in \mathbb{R}^{m\times n} \big|\ \exists\, x_k \overset{\Omega_f}{\to} x \ \text{ such that } \ \nabla f(x_k) \to A\big\}, \quad x \in \mathbb{R}^n,$$

with $\Omega_f := \{x \in \mathbb{R}^n \mid f \ \text{ is differentiable at } \ x\}$; see [22, 37, 39, 66] for further discussions. Note that any semismooth mappings automatically admits the classical directional derivative at the reference point. Quite recently [28], the concept of semismoothness has been improved and extended to set-valued mappings. To formulate the latter notion, recall first the construction of the *directional limiting normal cone* to a set $\Omega \subset \mathbb{R}^s$ at $\bar{z} \in \Omega$ in the direction $d \in \mathbb{R}^s$ introduced in [29] by

$$N_\Omega(\bar{z}; d) := \big\{v \in \mathbb{R}^s \ \big|\ \exists\, t_k \downarrow 0,\ d_k \to d,\ v_k \to v \ \text{ with } \ v_k \in \widehat{N}_\Omega(\bar{z} + t_k d_k)\big\}. \tag{3.18}$$

It is obvious that (3.18) agrees with the limiting normal cone (2.1) for $d = 0$. The *directional limiting coderivative* of $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ at $(\bar{x}, \bar{y}) \in \text{gph}\, F$ in the direction $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$ is defined in [26] by

$$D^*F\big((\bar{x},\bar{y});(u,v)\big)(v^*) := \big\{u^* \in \mathbb{R}^n \ \big|\ (u^*, -v^*) \in N_{\text{gph}\, F}\big((\bar{x},\bar{y});(u,v)\big)\big\} \ \text{ for all } \ v^* \in \mathbb{R}^m. \tag{3.19}$$

Using (3.19), we come to the aforementioned property of set-valued mappings introduced in [28].

Definition 3.3 (**semismooth* property of set-valued mappings**). *A mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is* SEMISMOOTH* *at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ if whenever $(u,v) \in \mathbb{R}^n \times \mathbb{R}^m$ we have*

$$\langle u^*, u\rangle = \langle v^*, v\rangle \ \text{ for all } \ (v^*,u^*) \in \operatorname{gph} D^*F\big((\bar{x},\bar{y});(u,v)\big).$$

Among various properties of semismooth* mappings obtained in [28], recall that this property holds if the graph of $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is represented as a union of finitely many closed and convex sets, as well as for the normal cone mappings generated by convex polyhedral sets. Note also that the semismooth* property of single-valued locally Lipschitzian mappings $f\colon \mathbb{R}^n \to \mathbb{R}^m$ around $\bar{x}$ agrees with the semismooth property (3.17) at this point provided that f directionally differentiable at $\bar{x}$.

The next lemma presents an increment estimate for $\mathcal{C}^{1,1}$ functions with semismooth* gradients. The obtained estimates are of their own interest, while are instrumental to establish major superlinear convergence results in this and subsequent sections.

Lemma 3.4 (**increment estimate for functions with semismooth* derivatives**). *Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^1$-smooth function around $\bar{x} \in \mathbb{R}^n$ with $\nabla\varphi(\bar{x}) = 0$. Suppose that $\nabla\varphi$ is locally Lipschitzian around this point with modulus $\ell > 0$, and that $\nabla\varphi$ is semismooth* at this point. Assume that a sequence $\{x^k\}$ converges to $\bar{x}$ with $x^k \neq \bar{x}$ as $k \in \mathbb{N}$, and that a sequence $\{d^k\}$ is superlinearly convergent with respect to $\{x^k\}$, i.e., $\|x^k + d^k - \bar{x}\| = o(\|x^k - \bar{x}\|)$. Consider the following assertions:*

- **(i)** *$\nabla\varphi$ is directionally differentiable at $\bar{x}$, and there exists $\kappa > 0$ such that $\langle\nabla\varphi(x^k), d^k\rangle \le -\frac{1}{\kappa}\|d^k\|^2$ whenever $k \in \mathbb{N}$ is sufficiently large.*
- **(ii)** *There exists $\kappa > 0$ such that*

$$\varphi(x^k + d^k) - \varphi(x^k) \le \langle\nabla\varphi(x^k + d^k), d^k\rangle - \frac{1}{2\kappa}\|d^k\|^2 \tag{3.20}$$

whenever k is sufficiently large.

If either (i) *holds and $\sigma \in (0, 1/2)$, or* (ii) *holds and $\sigma \in (0, 1(2\ell\kappa))$, then we have the estimate*

$$\varphi(x^k + d^k) \le \varphi(x^k) + \sigma\langle\nabla\varphi(x^k), d^k\rangle \ \text{ for all large } \ k. \tag{3.21}$$

Proof. Suppose that (i) is satisfied. The directional differentiability and semismoothness* of $\nabla\varphi$ at $\bar{x}$ ensure by [28, Corollary 3.8] that this mapping is semismooth at $\bar{x}$. Then estimate (3.21) follows immediately from [22, Proposition 8.3.18].

Suppose now that (ii) is satisfied and let $\sigma \in (0, 1/(2\ell\kappa))$. Since the sequence $\{d^k\}$ is superlinearly convergent with respect to $\{x^k\}$, by using [22, Lemma 7.5.7] we get

$$\lim_{k\to\infty} \|x^k - \bar{x}\|/\|d^k\| = 1, \tag{3.22}$$

which also yields the limiting relationship

$$\|x^k + d^k - \bar{x}\| = o(\|d^k\|) \quad \text{as} \quad k \to \infty. \tag{3.23}$$

Then the assumed estimated (3.20) in (ii) leads us to the inequalities

$$\begin{aligned}
\varphi(x^k + d^k) - \varphi(x^k) - \sigma\langle\nabla\varphi(x^k), d^k\rangle \ &\le \ \langle\nabla\varphi(x^k + d^k), d^k\rangle - \frac{1}{2\kappa}\|d^k\|^2 - \sigma\langle\nabla\varphi(x^k), d^k\rangle \\
&\le \ \|\nabla\varphi(x^k + d^k)\| \cdot \|d^k\| - \frac{1}{2\kappa}\|d^k\|^2 + \sigma\|\nabla\varphi(x^k)\| \cdot \|d^k\| \\
&\le \ \ell\|x^k + d^k - \bar{x}\| \cdot \|d^k\| - \frac{1}{2\kappa}\|d^k\|^2 + \sigma\ell\|x^k - \bar{x}\| \cdot \|d^k\| \\
&\le \ \|d^k\|^2\left(\ell\frac{\|x^k + d^k - \bar{x}\|}{\|d^k\|} - \frac{1}{2\kappa} + \sigma\ell\frac{\|x^k - \bar{x}\|}{\|d^k\|}\right)
\end{aligned}$$

for all large $k \in \mathbb{N}$. Finally, it follows from $\sigma < 1/(2\ell\kappa)$, (3.22), and (3.23) that

$$\varphi(x^k + d^k) - \varphi(x^k) - \sigma\langle\nabla\varphi(x^k), d^k\rangle \le 0 \quad \text{when} \quad k \ \text{ is sufficiently large.}$$

This verifies (3.21) and completes the proof of the lemma. $\square$

Now we are ready to justify the Q-superlinear rate of convergence of iterates in Algorithm 1 under some additional assumptions and relationships between parameters of the problem and the algorithm.

Theorem 3.5 (superlinear convergence of the coderivative-based damped Newton algorithm in $\mathcal{C}^{1,1}$ optimization). *In the setting of Theorem 3.2 ensuring the convergence of $\{x^k\}$ to a tilt-stable minimizer of φ $\bar{x}$ as $k \to \infty$, suppose that $\nabla\varphi$ is locally Lipschitzian around $\bar{x}$ with some constant $\ell > 0$ being also semismooth* at this point. Then the rate of the convergence of $\{x^k\}$ is at least Q-superlinear if either one of the following two conditions is satisfied:*

(i) *$\nabla\varphi$ is directionally differentiable at $\bar{x}$.*

(ii) *$\sigma \in (0, 1(2\ell\kappa))$, where $\kappa > 0$ is a modulus of tilt stability of $\bar{x}$.*

Moreover, in both cases (i) *and* (ii) *the sequence $\{\varphi(x^k)\}$ converges Q-superlinearly to $\varphi(\bar{x})$, and the sequence $\{\nabla\varphi(x^k)\}$ converges Q-superlinearly to 0 as $k \to \infty$.*

Proof. Fixing a title-stable minimizer $\bar{x}$ with modulus $\kappa > 0$ from the assertions of Theorem 3.2, we split the proof of this theorem into the three claims.

Claim 1: *The sequence of directions $\{d^k\}$ is superlinearly convergent with respect to $\{x^k\}$.* Indeed, by the characterization of tilt-stable minimizers via the combined second-order subdifferential taken from [52, Theorem 3.5] and [9, Proposition 4.6], we find a positive number δ such that the inequality

$$\langle z, w\rangle \geq \frac{1}{\kappa}\|w\|^2 \quad \text{for all} \;\; z \in \partial^2\varphi(x)(w),\; x \in \mathbb{B}_\delta(\bar{x}), \;\; \text{and} \;\; w \in \mathbb{R}^n \tag{3.24}$$

is satisfied. Employing the subadditivity property of coderivatives from [38, Lemma 5.6] gives us

$$\partial^2\varphi(x^k)(d^k) \subset \partial^2\varphi(x^k)(x^k + d^k - \bar{x}) + \partial^2\varphi(x^k)(-x^k + \bar{x}).$$

Since $-\nabla\varphi(x^k) \in \partial^2\varphi(x^k)(d^k)$, for all $k \in \mathbb{N}$ there exists $v^k \in \partial^2\varphi(x^k)(-x^k + \bar{x})$ such that

$$-\nabla\varphi(x^k) - v^k \in \partial^2\varphi(x^k)(x^k + d^k - \bar{x}).$$

Using further (3.24) and the Cauchy-Schwarz inequality, we get

$$\|x^k + d^k - \bar{x}\| \leq \kappa\|\nabla\varphi(x^k) + v^k\| \quad \text{for sufficiently large} \;\; k \in \mathbb{N}. \tag{3.25}$$

The semismoothness* of $\nabla\varphi$ at $\bar{x}$ together with $\nabla\varphi(\bar{x}) = 0$ implies by [38, Lemma 5.5] that

$$\|\nabla\varphi(x^k) + v^k\| = \|\nabla\varphi(x^k) - \nabla\varphi(\bar{x}) + v^k\| = o(\|x^k - \bar{x}\|). \tag{3.26}$$

Then it follows from (3.25) and (3.26) that $\|x^k + d^k - \bar{x}\| = o(\|x^k - \bar{x}\|)$, which justifies the claim.

Claim 2: *We have $\tau_k = 1$ for all $k \in \mathbb{N}$ sufficiently large provided that either condition* (i)*, or condition* (ii) *of this theorem is satisfied.* To verify this claim, let us show that (3.21) holds for large k under the imposed assumptions. Suppose first that (i) is satisfied. Due to the inclusion $-\nabla\varphi(x^k) \in \partial^2\varphi(x^k)(d^k)$ and estimate (3.24), we have $\langle\nabla\varphi(x^k), d^k\rangle \leq -\frac{1}{\kappa}\|d^k\|^2$ if k is large enough. Then the fulfillment of (3.21) in case (i) follows directly from Lemma 3.4. In case (ii), we know from Claim 1 that $\{d^k\}$ converges to 0 and that $x^k + d^k \to \bar{x}$ as $k \to \infty$. Employing the uniform second-order growth condition for tilt-stable minimizers from [52, Theorem 3.2] gives us a neighborhood U of $\bar{x}$ such that

$$\varphi(x) \geq \varphi(u) + \langle\nabla\varphi(u), x - u\rangle + \frac{1}{2\kappa}\|x - u\|^2 \quad \text{for all} \;\; x, u \in U,$$

and thus verifies (3.20). Using Lemma 3.4 brings us to (3.21), which is claimed.

Claim 3: *The conclusions on the Q-superlinear convergence in the theorem hold in both cases* (i) *and* (ii). We see from Claim 2 that $\tau_k = 1$ for all k sufficiently large, and thus Algorithm 1 eventually becomes the generalized pure Newton algorithm from [38, Algorithm 5.3]. Hence the claimed Q-superlinear convergence results follow from [38, Theorems 5.7 and 5.12]. $\square$

4 Coderivative-Based Levenberg-Marquardt Method

Observe that the positive-definiteness of the generalized Hessian $\partial^2\varphi(x)$ in Algorithm 1 cannot be replaced by the less demanding positive-semidefiniteness of $\partial^2\varphi(x)$ to ensure the existence of descent Newton direction for Algorithm 1 as in Proposition 3.1. Indeed, consider the simplest linear function $\varphi(x) := x$ on $\mathbb{R}$. Then we obviously have that $\varphi''(x) \geq 0$ for all $x \in \mathbb{R}$, while there are no Newton directions $d \in \mathbb{R}$ satisfying the backtracking line search condition (3.3). This means that Algorithm 1 cannot be even constructed without the positive-definiteness of $\partial^2\varphi(x)$. Here we propose the following globally convergent coderivative-based *generalized Levenberg-Marquardt algorithm* to solve problems of $\mathcal{C}^{1,1}$ optimization that is *well-posed* and exhibits the *convergence* of its subsequences to *stationary points* of φ under merely the *positive-semidefiniteness* of the generalized Hessian. Linear and superlinear *convergence rates* are achieved under some additional assumptions.

Algorithm 2 Globally coderivative-based Levenberg-Marquardt algorithm for $\mathcal{C}^{1,1}$ functions

Input: $x^0 \in \mathbb{R}^n$, $c > 0$, $\sigma \in \left(0, \frac{1}{2}\right)$, $\beta \in (0,1)$

for $k = 0, 1, \ldots$ **do**
  If $\nabla\varphi(x^k) = 0$, stop; otherwise let $\mu_k := c\|\nabla\varphi(x^k)\|$ and go to next step
  Choose $d^k \in \mathbb{R}^n$ such that $-\nabla\varphi(x^k) \in \partial^2\varphi(x^k)(d^k) + \mu_k d^k$
  Set $\tau_k = 1$
  while $\varphi(x^k + \tau_k d^k) > \varphi(x^k) + \sigma\tau_k\langle\nabla\varphi(x^k), d^k\rangle$ **do**
    set $\tau_k := \beta\tau_k$
  end while
  Set $x^{k+1} := x^k + \tau_k d^k$
end for

Our first major result in this section establishes the well-posedness and global convergence of iterates generated by Algorithm 2 to stationary points of φ under only the positive-semidefiniteness assumption on the generalized Hessian $\partial^2\varphi(x)$.

Theorem 4.1 (well-posedness and convergence of the coderivative-based Levenberg-Marquardt algorithm). *Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ be of class $\mathcal{C}^{1,1}$ around a reference point $x \in \mathbb{R}^n$ such that $\nabla\varphi(x) \neq 0$ and that $\partial^2\varphi(x)$ is positive-semidefinite, i.e.,*

$$\langle z, u\rangle \geq 0 \ \ \textit{for all} \ \ z \in \partial^2\varphi(x)(u) \ \ \textit{and} \ \ u \in \mathbb{R}^n. \tag{4.1}$$

Then the following assertions hold:

(i) *For any $\varepsilon > 0$, there exists a nonzero direction $d \in \mathbb{R}^n$ such that*

$$-\nabla\varphi(x) \in \partial^2\varphi(x)(d) + \varepsilon d. \tag{4.2}$$

Moreover, every such direction satisfies the inequality $\langle\nabla\varphi(x), d\rangle < 0$. Consequently, for each $\sigma \in (0,1)$ and $d \in \mathbb{R}^n$ satisfying (3.2) we have $\delta > 0$ such that

$$\varphi(x + \tau d) \leq \varphi(x) + \sigma\tau\langle\nabla\varphi(x), d\rangle \ \ \textit{whenever} \ \ \tau \in (0, \delta). \tag{4.3}$$

(ii) *Picking any starting point $x^0 \in \mathbb{R}^n$ such that $\partial^2\varphi(x)$ is positive-semidefinite on the level set Ω from (3.4), we have that Algorithm 2 either stops after finitely many iterations, or produces a sequence of iterates $\{x^k\} \subset \Omega$ such that the sequence of values $\{\varphi(x^k)\}$ is monotonically decreasing. Moreover, all the limiting points of $\{x^k\}$ satisfy the stationary condition.*

Proof. To justify (i), fix x satisfying the assumptions therein and consider the function

$$\varphi_\varepsilon(\cdot) = \varphi(\cdot) + \frac{\varepsilon}{2}\|\cdot\|^2 \ \ \text{for any} \ \ \varepsilon > 0 \ \ \text{on} \ \ \mathbb{R}^n.$$

It follows from the second-order subdifferential sum rule in [50, Proposition 1.121] that

$$\partial^2\varphi_\varepsilon(x)(w) = \partial^2\varphi(x)(w) + \varepsilon w \quad \text{for all} \ \ w \in \mathbb{R}^n. \tag{4.4}$$

Thus $z-\varepsilon w \in \partial^2\varphi(x)(w)$ whenever $z \in \partial^2\varphi_\varepsilon(x)(w)$. Due to the positive-semidefiniteness of $\partial^2\varphi(x)(w)$, we get $\langle z, w\rangle \geq \varepsilon\|w\|^2$, which implies that $\partial^2\varphi_\varepsilon(x)$ is positive-definite. It follows from [51, Theorem 5.16] that $\nabla\varphi_\varepsilon$ is locally strongly maximally monotone around $(x, \nabla\varphi_\varepsilon(x))$. Hence the gradient mapping $\nabla\varphi_\varepsilon$ is strongly metrically regular around this point due to [51, Theorem 5.13] telling us that the inverse mapping $\nabla\varphi_\varepsilon^{-1} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ admits a single-valued localization $\vartheta : V \to U$ around $(\nabla\varphi_\varepsilon(x), x)$, which is locally Lipschitzian around the point $\nabla\varphi_\varepsilon(x)$. Combining this with the scalarization formula (2.7) yields the representations

$$D^*\nabla\varphi_\varepsilon^{-1}\big(\nabla\varphi_\varepsilon(x), x\big)\big(\nabla\varphi(x)\big) = D^*\vartheta\big(\nabla\varphi_\varepsilon(x)\big)\big(\nabla\varphi(x)\big) = \partial\big\langle\nabla\varphi(x), \vartheta\big\rangle\big(\nabla\varphi_\varepsilon(x)\big). \tag{4.5}$$

Since ϑ is locally Lipschitzian, we deduce from [50, Theorem 1.22] that $\partial\langle\nabla\varphi_\varepsilon(x), \vartheta\rangle(\nabla\varphi_\varepsilon(x)) \neq \emptyset$. Thus it follows from (2.7) that

$$D^*\nabla\varphi_\varepsilon^{-1}\big(\nabla\varphi_\varepsilon(x), x\big)\big(\nabla\varphi(x)\big) \neq \emptyset. \tag{4.6}$$

Picking any $-d \in D^*\nabla\varphi_\varepsilon^{-1}(\nabla\varphi_\varepsilon(x), x)(\nabla\varphi(x))$ and easily representing the coderivative of the inverse mapping $\nabla\varphi_\varepsilon^{-1}$ via that of $\nabla\varphi_\varepsilon$, we deduce from (4.6) the inclusion

$$-\nabla\varphi(x) \in D^*\nabla\varphi_\varepsilon(x)(d) = \partial^2\varphi_\varepsilon(x)(d).$$

Due to (4.4), it follows from the above that $-\nabla\varphi(x) \in \partial^2\varphi(x)(d) + \varepsilon d$.

To verify (i), it remains to show that $d \neq 0$. Supposing the contrary and using the local Lipschitz continuity of $\nabla\varphi$, we obtain from [50, Theorem 1.44] that

$$-\nabla\varphi(x) \in \partial^2\varphi(x)(0) = \big(D^*\nabla\varphi\big)(x)(0) = \{0\},$$

which contradicts the imposed assumption $\nabla\varphi(x) \neq 0$. The positive-definiteness of $\partial^2\varphi_\varepsilon(x)$ yields $\langle\nabla\varphi(x), d\rangle < 0$ that ensures in turn the fulfillment of (3.3) due to [37, Lemmas 2.18 and 2.19].

Next we proceed with the proof of (ii). It follows from (i) while arguing by induction that Algorithm 2 either stops after finitely many iterations, or produces a sequence of iterates $\{x^k\} \subset \Omega$ such that $\varphi(x^{k+1}) < \varphi(x^k)$ for all $k \in \mathbb{N}$. Let us first show that the sequence $\{d^k\}$ is bounded. Indeed, the construction of the directional sequence $\{d^k\}$ in Algorithm 2 gives us the inclusions

$$-\nabla\varphi(x^k) - \mu_k d^k \in \partial^2\varphi(x^k)(d^k) \quad \text{for all} \quad k \in \mathbb{N}, \tag{4.7}$$

and thus we get that $\langle -\nabla\varphi(x^k) - \mu_k d^k, d^k\rangle \geq 0$, i.e.,

$$\langle\nabla\varphi(x^k), -d^k\rangle \geq \mu_k\|d^k\|^2, \quad k \in \mathbb{N}. \tag{4.8}$$

Employing the Cauchy-Schwarz inequality and replacing μ_k with $c\|\nabla\varphi(x^k)\|$ lead us to

$$c\|\nabla\varphi(x^k)\| \cdot \|d^k\| = \mu_k\|d^k\| \leq \|\nabla\varphi(x^k)\|$$

and readily implies that $\|d^k\| \leq 1/c$ for all k.

Fix now a limiting point $\bar{x}$ of the sequence of iterates $\{x^k\}$ and find a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that $x^{k_j} \to \bar{x}$ as $j \to \infty$. Since the sequence $\{\varphi(x^k)\}$ is nonincreasing and $\varphi(\bar{x})$ is a limiting point of $\{\varphi(x^k)\}$, this sequence converges to $\varphi(\bar{x})$ as $k \to \infty$. Moreover, we have

$$\varphi(x^{k+1}) - \varphi(x^k) \leq \sigma\tau_k\langle\nabla\varphi(x^k), d^k\rangle < 0 \quad \text{for all} \quad k \in \mathbb{N},$$

which yields the equality

$$\lim_{k\to\infty} \tau_k\langle\nabla\varphi(x^k), d^k\rangle = 0. \tag{4.9}$$

Using the boundedness of the sequence $\{d^k\}$, we find a subsequence $\{d^{k_j}\}$ converging to some $\bar{d} \in \mathbb{R}^n$. Let us verify the condition

$$\big\langle\nabla\varphi(\bar{x}), \bar{d}\big\rangle = 0. \tag{4.10}$$

Indeed, if $\limsup_{j\to\infty} \tau_{k_j} > 0$, then (4.10) follows immediately from (4.9). Otherwise, we have $\lim_{j\to\infty} \tau_{k_j} = 0$, and the exit condition of the backtracking line search in Step 5 of Algorithm 2 brings us to

$$\varphi(x^{k_j} + \tau'_{k_j} d^{k_j}) > \varphi(x^{k_j}) + \sigma\tau'_{k_j}\langle\nabla\varphi(x^{k_j}), d^{k_j}\rangle \tag{4.11}$$

for all $j \in \mathbb{N}$, where $\tau'_{k_j} := \tau_{k_j}/\beta$. Dividing now both sides of (4.11) by τ'_{k_j} and letting $j \to \infty$ imply by the smoothness of φ that

$$\langle\nabla\varphi(\bar{x}), \bar{d}\rangle = \lim_{j\to\infty} \frac{\varphi(x^{k_j} + \tau'_{k_j} d^{k_j}) - \varphi(x^{k_j})}{\tau'_{k_j}} \geq \sigma\langle\nabla\varphi(\bar{x}), \bar{d}\rangle.$$

Hence this tells us that $\langle\nabla\varphi(\bar{x}), \bar{d}\rangle \geq 0$ by $\sigma < 1$. Furthermore, letting $j \to \infty$ in $\langle\nabla\varphi(x^{k_j}), d^{k_j}\rangle \leq 0$, we get $\langle\nabla\varphi(\bar{x}), \bar{d}\rangle \leq 0$ and thus obtain (4.10). Combining (4.8) and (4.10) verifies that $\mu_{k_j}\|d^{k_j}\|^2 \to 0$ as $j \to \infty$. By the definition of $\{\mu_k\}$ and the convergence $x^{k_j} \to \bar{x}$, we have $\mu_{k_j} = c\|\nabla\varphi(x^{k_j})\| \to c\|\nabla\varphi(\bar{x})\|$ as $j \to \infty$, which ensures therefore that

$$c\,\|\nabla\varphi(\bar{x})\| \cdot \|\bar{d}\|^2 = \lim_{j\to\infty} \mu_{k_j} \left\|d^{k_j}\right\|^2 = 0. \tag{4.12}$$

Since φ is of class $\mathcal{C}^{1,1}$ around $\bar{x}$, it follows from [50, Theorem 1.44] and (4.7) that there exists $\ell > 0$ such that $\|\nabla\varphi(x^{k_j}) + \mu_{k_j} d^{k_j}\| \leq \ell\|d^{k_j}\|$ for all j sufficiently large. This yields

$$\left\|\nabla\varphi(x^{k_j})\right\|^2 + 2\mu_{k_j}\left\langle\nabla\varphi(x^{k_j}), d^{k_j}\right\rangle + \mu_{k_j}^2\left\|d^{k_j}\right\|^2 \leq \ell^2\left\|d^{k_j}\right\|^2$$

for such j. Letting $j \to \infty$ in the above inequality, we arrive at $\|\nabla\varphi(\bar{x})\|^2 \leq \ell^2\left\|\bar{d}\right\|^2$ due to (4.10) and the second equality in (4.12). Using the obtained estimate together with the first part of (4.12) gives us $\nabla\varphi(\bar{x}) = 0$ and thus completes the proof of the theorem. □

The next theorem establishes the linear and superlinear *convergence rates* of iterates in Algorithm 2 to *tilt-stable minimizers* under the *metric regularity assumption* on $\nabla\varphi$ at the solution point $\bar{x}$. Note that the latter property is constructively characterized by (2.4) and (2.10) as

$$\{u \in \mathbb{R}^n \mid 0 \in \partial\langle u, \nabla\varphi\rangle(\bar{x})\} = \{0\}.$$

Theorem 4.2 (linear and superlinear global convergence of coderivative-based Levenberg–Marquardt algorithm). *In the setting of Theorem 4.1, let $\bar{x}$ be a limiting point of $\{x^k\}$ such that $\nabla\varphi$ is metrically regular around this point. Then $\bar{x}$ is a tilt-stable local minimizer of φ and Algorithm 2 converges to $\bar{x}$ with the convergence rates as follows:*

- **(i)** *The sequence of values $\{\varphi(x^k)\}$ converges to $\varphi(\bar{x})$ at least Q-linearly.*
- **(ii)** *The sequences $\{x^k\}$ and $\{\nabla\varphi(x^k)\}$ converge at least R-linearly to $\bar{x}$ and 0, respectively.*
- **(iii)** *The convergence rates of $\{x^k\}$, $\{\varphi(x^k)\}$, and $\{\nabla\varphi(x^k)\}$ are at least Q-superlinear if $\nabla\varphi$ is semismooth* at $\bar{x}$ and either one of the following two conditions holds:*
  - **(a)** *$\nabla\varphi$ is directionally differentiable at $\bar{x}$.*
  - **(b)** *$\sigma \in (0, 1/(2\ell\kappa))$, where $\kappa > 0$ and $\ell > 0$ are moduli of metric regularity and Lipschitz continuity of $\nabla\varphi$ around $\bar{x}$, respectively.*

Proof. We split the proof into the seven major claims of their own interest.

Claim 1: *$\bar{x}$ is a tilt-stable local minimizer of φ.* Due to Theorem 4.1, $\bar{x}$ is a stationary point of φ and $\bar{x} \in \Omega$, which implies that $\partial^2\varphi(\bar{x})$ is positive-semidefinite. This property and the assumed metric regularity of $\nabla\varphi$ around $\bar{x}$ with modulus κ allow us to conclude by using [20, Theorem 4.13] that $\bar{x}$ is a tilt-stable local minimizer of φ with the same modulus κ.

Claim 2: *For any subsequence $\{x^{k_j}\}$ of $\{x^k\}$ with $x^{k_j} \to \bar{x}$ as $j \to \infty$, the corresponding sequence $\{\tau_{k_j}\}$ in Algorithm 2 is bounded from below by a positive number $\gamma > 0$, and we have*

$$\varphi(x^{k_j}) - \varphi(x^{k_j+1}) \geq \frac{\sigma\gamma}{\kappa}\|d^{k_j}\|^2 \quad \textit{for all large } j \in \mathbb{N}. \tag{4.13}$$

Indeed, supposing on the contrary that $\{\tau_{k_j}\}$ is not bounded from below by a positive number and combining this with $\tau_k \geq 0$ give us a subsequence of $\{\tau_{k_j}\}$ that converges to 0. Let $\tau_{k_j} \to 0$ as $j \to \infty$ without loss of generality. Then using the characterization of tilt-stable minimizers via the combined second-order subdifferential taken from [52, Theorem 3.5] and [9, Proposition 4.6], we find $\delta > 0$ with

$$\langle z, w\rangle \geq \frac{1}{\kappa}\|w\|^2 \quad \text{for all} \;\; z \in \partial^2\varphi(x)(w), \; x \in \mathbb{B}_\delta(\bar{x}), \;\; \text{and} \;\; w \in \mathbb{R}^n. \tag{4.14}$$

Since $-\nabla\varphi(x^{k_j}) - \mu_{k_j} d^{k_j} \in \partial^2\varphi(x^{k_j})(d^{k_j})$ for all $j \in \mathbb{N}$, it follows from (4.14) that

$$\langle -\nabla\varphi(x^{k_j}), d^{k_j}\rangle \geq \left(\mu_{k_j} + \frac{1}{\kappa}\right)\|d^{k_j}\|^2 \geq \frac{1}{\kappa}\|d^{k_j}\|^2 \quad \text{for all } j \text{ sufficiently large.} \tag{4.15}$$

Then Claim 1 of Theorem 4.1 tells us that the sequence $\{d^k\}$ is bounded. Hence $x^{k_j} + \beta^{-1}\tau_{k_j} d^{k_j} \to \bar{x}$ as $j \to \infty$, and therefore $x^{k_j} + \beta^{-1}\tau_{k_j} d^{k_j} \in \mathbb{B}_\delta(\bar{x})$ for all large j. The exit condition of the backtracking line search in Step 5 of Algorithm 2 yields

$$\varphi(x^{k_j} + \beta^{-1}\tau_{k_j} d^{k_j}) > \varphi(x^{k_j}) + \sigma\beta^{-1}\tau_{k_j}\langle\nabla\varphi(x^{k_j}), d^{k_j}\rangle \tag{4.16}$$

for large j. Due to (4.15) and (4.16), we can apply Lemma 7.1 from the Appendix for such j with $x := x^{k_j}$, $y := x^{k_j} + \beta^{-1}\tau_{k_j} d^{k_j}$, $c_1 := \sigma$, and $c_2 := \kappa^{-1}\beta\tau_{k_j}^{-1}$, which implies that $\sigma > 1 - \frac{\ell\kappa}{2\beta}\tau_{k_j}$. Letting $j \to \infty$ gives us $\sigma \geq 1$, a contradiction due to $\sigma < 1$. This verifies the existence of $\gamma > 0$ such that $\tau_{k_j} \geq \gamma$ for all $j \in \mathbb{N}$. Using finally the estimate in (4.15), we find $j_0 \in \mathbb{N}$ such that

$$\varphi(x^{k_j}) - \varphi(x^{k_j+1}) \geq \sigma\tau_{k_j}\langle -\nabla\varphi(x^{k_j}), d^{k_j}\rangle \geq \frac{\sigma\gamma}{\kappa}\|d^{k_j}\|^2 \quad \text{for all} \;\; j \geq j_0, \tag{4.17}$$

which therefore justifies Claim 2.

Claim 3: *The iterative sequence $\{x^k\}$ is convergent.* To verify this claim, we are based on Ostrowski's condition from [22, Proposition 8.3.10]. Let us first show that there is no other limiting point of $\{x^k\}$ in $\mathbb{B}_\delta(\bar{x})$. On the contrary, suppose that there exists $\widetilde{x} \in \mathbb{B}_\delta(\bar{x})$ such that $\widetilde{x} \neq \bar{x}$ and $\widetilde{x}$ is a limiting point of $\{x^k\}$. It follows from Theorem 4.1 that $\widetilde{x}$ is a stationary point of φ, which contradicts the strong convexity of φ on $\mathbb{B}_\delta(\bar{x})$. Supposing next that $\{x^{k_j}\}$ is an arbitrary subsequence of $\{x^k\}$ such that $x^{k_j} \to \bar{x}$ as $j \to \infty$, we wish to show that

$$\lim_{j\to\infty}\|x^{k_j+1} - x^{k_j}\| = 0. \tag{4.18}$$

Indeed, find by Claim 2 such $\gamma > 0$ that (4.13) holds, which implies that

$$\|x^{k_j+1} - x^{k_j}\|^2 = \tau_{k_j}^2\|d^{k_j}\|^2 \leq \|d^{k_j}\|^2 \leq \frac{\kappa}{\sigma\gamma}\left(\varphi(x^{k_j}) - \varphi(x^{k_j+1})\right) \to 0$$

as $j \to \infty$ and thus verifies (4.18). Employing now [22, Proposition 8.3.10] ensures the convergence of $\{x^k\}$ to $\bar{x}$ as $k \to \infty$ and therefore completes the proof of this claim.

Claim 4: *The convergence rate of $\{\varphi(x^k)\}$ is at least Q-linear, while the convergence rates of $\{x^k\}$ and $\{\|\nabla\varphi(x^k)\|\}$ are at least R-linear.* Indeed, the strong convexity of φ on $\mathbb{B}_\delta(\bar{x})$ implies that

$$\varphi(x) \geq \varphi(u) + \langle\nabla\varphi(u), x - u\rangle + \frac{1}{2\kappa}\|x - u\|^2 \;\; \text{and} \;\; \langle\nabla\varphi(x) - \nabla\varphi(u), x - u\rangle \geq \frac{1}{\kappa}\|x - u\|^2 \tag{4.19}$$

for all $x, u \in \mathbb{B}_\delta(\bar{x})$. By the convergence $x^k \to \bar{x}$ we have that $x^k \in U$ for all k sufficiently large, which we assumed from now on. Substituting $x := x^k$ and $u := \bar{x}$ into (4.19) and then using the Cauchy-Schwarz inequality together with $\nabla\varphi(\bar{x}) = 0$ yield the estimates

$$\varphi(x^k) \geq \varphi(\bar{x}) + \frac{1}{2\kappa}\|x^k - \bar{x}\|^2 \;\; \text{and} \tag{4.20}$$

$$\|\nabla\varphi(x^k)\| \geq \frac{1}{\kappa}\|x^k - \bar{x}\|. \tag{4.21}$$

The local Lipschitz continuity of $\nabla\varphi$ around $\bar{x}$ and the result of [37, Lemma A.11] ensure the existence of a positive number ℓ such that

$$\varphi(x^k) - \varphi(\bar{x}) = |\varphi(x^k) - \varphi(\bar{x}) - \langle\nabla\varphi(\bar{x}), x^k - \bar{x}\rangle| \leq \frac{\ell}{2}\|x^k - \bar{x}\|^2. \tag{4.22}$$

Moreover, since $-\nabla\varphi(x^k) - \mu_k d^k \in \partial^2\varphi(x^k)(d^k)$, by using [50, Theorem 1.44] we have

$$\|\nabla\varphi(x^k) + \mu_k d^k\| \leq \ell\|d^k\|. \tag{4.23}$$

It follows from the convergence $x^k \to \bar{x}$ and $\nabla\varphi(\bar{x}) = 0$ that $\mu_k = c\|\nabla\varphi(x^k)\| \to 0$ as $k \to \infty$, which implies that $\mu_k \leq \ell$. Combining the latter with (4.23) gives us the estimates

$$\|\nabla\varphi(x^k)\| \leq \|\nabla\varphi(x^k) + \mu_k d^k\| + \mu_k\|d^k\| \leq 2\ell\|d^k\|. \tag{4.24}$$

By Claim 2 we have that $\{\tau_k\}$ is bounded from below by some constant $\gamma > 0$ and that

$$\varphi(x^k) - \varphi(x^{k+1}) \geq \frac{\sigma\gamma}{\kappa}\|d^k\|^2,$$

which together with (4.24) yields the inequality

$$\varphi(x^k) - \varphi(x^{k+1}) \geq \frac{\sigma\gamma}{4\kappa\ell^2}\|\nabla\varphi(x^k)\|^2. \tag{4.25}$$

Combining finally (4.21), (4.22), and (4.25) and then applying Lemma 7.2 with the sequences $\alpha_k := \varphi(x^k) - \varphi(\bar{x})$, $\beta_k := \|\nabla\varphi(x^k)\|$, $\gamma_k := \|x^k - \bar{x}\|$ and positive numbers $c_1 := (\sigma\gamma)/(4\kappa\ell^2)$, $c_2 := 1/\kappa$, and $c_3 := \ell/2$, we verify all the conclusions of this claim.

Claim 5: *The sequence $\{d^k\}$ superlinearly converges with respect to $\{x^k\}$ provided that $\nabla\varphi$ is semismooth* at $\bar{x}$.* Indeed, the subadditivity property of coderivatives taken from [38, Lemma 5.6] yields

$$\partial^2\varphi(x^k)(d^k) \subset \partial^2\varphi(x^k)(x^k + d^k - \bar{x}) + \partial^2\varphi(x^k)(-x^k + \bar{x}).$$

Since $-\nabla\varphi(x^k) - \mu_k d^k \in \partial^2\varphi(x^k)(d^k)$ for all $k \in \mathbb{N}$, there exists $v^k \in \partial^2\varphi(x^k)(-x^k + \bar{x})$ such that

$$-\nabla\varphi(x^k) - \mu_k d^k - v^k \in \partial^2\varphi(x^k)(x^k + d^k - \bar{x}).$$

It follows from (4.14) and the Cauchy-Schwarz inequality that

$$\|x^k + d^k - \bar{x}\| \leq \kappa\|\nabla\varphi(x^k) + v^k + \mu_k d^k\| \leq \kappa\left(\|\nabla\varphi(x^k) - \nabla\varphi(\bar{x}) + v^k\| + \mu_k\|d^k\|\right). \tag{4.26}$$

The inclusion $-\nabla\varphi(x^k) - \mu_k d^k \in \partial^2\varphi(x^k)(d^k)$ and the estimate (4.14) tell us that

$$\langle\nabla\varphi(x^k), -d^k\rangle \geq \left(\kappa^{-1} + \mu_k\right)\|d^k\|^2 \geq \kappa^{-1}\|d^k\|^2. \tag{4.27}$$

By employing the Cauchy-Schwarz inequality again, we have that

$$\|d^k\| \leq \kappa\|\nabla\varphi(x^k)\| = \kappa\|\nabla\varphi(x^k) - \nabla\varphi(\bar{x})\| \leq \kappa\ell\|x^k - \bar{x}\|. \tag{4.28}$$

Furthermore, it follows from the Lipschitz continuity of $\nabla\varphi$ on $\mathbb{B}_\delta(\bar{x})$ and from $\nabla\varphi(\bar{x}) = 0$ that

$$\mu_k = c\|\nabla\varphi(x^k)\| = c\|\nabla\varphi(x^k) - \nabla\varphi(\bar{x})\| \leq \ell\|x^k - \bar{x}\|. \tag{4.29}$$

Using now the semismooth* property of the gradient mapping $\nabla\varphi$ at $\bar{x}$ together with the stationary condition $\nabla\varphi(\bar{x}) = 0$ implies by [38, Lemma 5.5] that

$$\|\nabla\varphi(x^k) + v^k\| = \|\nabla\varphi(x^k) - \nabla\varphi(\bar{x}) + v^k\| = o(\|x^k - \bar{x}\|). \tag{4.30}$$

Combining (4.26), (4.28), (4.29), and (4.30) gives us $\|x^k + d^k - \bar{x}\| = o(\|x^k - \bar{x}\|)$ as $k \to \infty$, which verifies the claimed superlinear convergence of $\{d^k\}$.

Claim 6: *We have $\tau_k = 1$ for all large $k \in \mathbb{N}$ provided that $\nabla\varphi$ is semismooth* at $\bar{x}$ and that either condition* (a)*, or condition* (b) *of the theorem holds.* To proceed, it suffices to verify the estimate in (3.21) under both conditions (a) and (b). It (a) is satisfied, then this estimate and the assertion of the claim follows directly (4.27) and Lemma 3.4. Assuming now the condition in (b) and using Claim 5, we easily see that the sequence $\{d^k\}$ converges to 0 and that $x^k + d^k \to \bar{x}$ as $k \to \infty$. Employing (4.19), we get (3.20). Then (3.21) follows from Lemma 3.4, and thus this claim is verified.

Claim 7: *The conclusions on the Q-superlinear convergence rates in* (iii) *hold in both cases* (a) *and* (b) *of the theorem.* To verify this, we get from Claim 6 that $\tau_k = 1$ for all k sufficiently large. It follows from $\|x^k + d^k - \bar{x}\| = o(\|x^k - \bar{x}\|)$ that

$$\|x^{k+1} - \bar{x}\| = \|x^k + \tau_k d^k - \bar{x}\| = \|x^k + d^k - \bar{x}\| = o(\|x^k - \bar{x}\|) \quad \text{as} \quad k \to \infty,$$

which justifies the Q-superlinear convergence rate for $\{x^k\}$. The Q-superlinear convergence rate of $\{\varphi(x^k)\}$ follows immediately from (4.20) and (4.22) while the Q-superlinear convergence rate for $\{\nabla\varphi(x^k)\}$ is a consequence of (4.21) and (4.29). This completes the proof of the theorem. $\square$

Remark 4.3 (convergence rates may not be superlinear without metric regularity). It is known that in the $\mathcal{C}^2$-smooth case, the Levenberg-Marquardt algorithm may not achieve the Q-superlinear convergence rate without the nonsingularity of the Hessian matrix at the reference point. A simple example is provided by the function $\varphi(x) := x^4$ on $\mathbb{R}$ with the reference point $\bar{x} := 0$. Observe that this function satisfies all the assumptions of Theorem 3.5(iii) but the metric regularity of φ' around $\bar{x}$, which therefore demonstrates that the latter assumption is essential for the Q-superlinear convergence of the iterates in Algorithm 2.

5 Coderivative-Based Newton Methods in Composite Optimization

In this section we consider a broad and highly important class of optimization problems given by

$$\text{minimize} \;\; \varphi(x) := f(x) + g(x), \quad x \in \mathbb{R}^n, \tag{5.1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ is a convex and smooth function, while the regularizer $g\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ is a convex and extended-real-valued one. This class is known as problems of *convex composite optimization.*

Problems written in format (5.1) frequently arise in many applied areas such as machine learning, compressed sensing, image processing, etc. Since g is generally extended-real-valued, the unconstrained format (1.4) encompasses problems of *constrained optimization.* If, in particular, g is the indicator function of a closed and convex set, then (1.4) becomes a constrained optimization problems studied, e.g., in the book [59] with numerous applications.

One of the most well-recognized algorithms to solve (1.4) is the forward-backward splitting (FBS), or proximal splitting, method [12, 44]. Since this method is of first order, its rate of convergence is at most linear. Another approach to solve (1.4) is to use second-order methods such as proximal Newton methods, proximal quasi-Newton methods, etc.; see, e.g., [4, 40, 57]. Although the latter approach has several benefits over first-order methods (as rapid convergence and high accuracy), a severe limitation of these methods is the cost of solving subproblems.

To develop here new globally convergent Newton methods to solve convex composite optimization problems of type (5.1), we first recall the classical notions of convex and variational analysis; see, e.g., [70]. Given an extended-real-valued, proper, l.s.c. function $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ and a positive number γ, the *Moreau envelope* $e_\gamma\varphi$ and the *proximal mapping* $\operatorname{Prox}_{\gamma\varphi}$ are defined by, respectively,

$$e_\gamma\varphi(x) := \inf_{y \in \mathbb{R}^n} \left\{ \varphi(y) + \frac{1}{2\gamma}\|y - x\|^2 \right\}, \tag{5.2}$$

$$\operatorname{Prox}_{\gamma\varphi}(x) := \operatorname*{argmin}_{y \in \mathbb{R}^n} \left\{ \varphi(y) + \frac{1}{2\gamma}\|y - x\|^2 \right\}. \tag{5.3}$$

These notions have been well recognized in variational analysis and optimization as efficient tools of regularization and approximation of nonsmooth functions. More recently, the following extended notion, known now as the *forward-backward envelope*, has been introduced by Patrinos and Bemporad [61] for problems of convex composite optimization under the name of the *composite Moreau envelope*.

Definition 5.1 (forward-backward envelope). *Let $\varphi = f + g$ be as in (5.1), and let $\gamma > 0$. The* FORWARD-BACKWARD ENVELOPE *(FBE) of φ with parameter γ is*

$$\varphi_\gamma(x) := \inf_{y\in\mathbb{R}^n}\left\{f(x) + \langle\nabla f(x), y - x\rangle + g(y) + \frac{1}{2\gamma}\|y - x\|^2\right\}, \tag{5.4}$$

which is, by definition (5.2) of the Moreau envelope, is equivalent to

$$\varphi_\gamma(x) = f(x) - \frac{\gamma}{2}\|\nabla f(x)\|^2 + e_\gamma g\big(x - \gamma\nabla f(x)\big). \tag{5.5}$$

The FBE has already been used for developing some efficient algorithms to solve nonsmooth optimization problems; see, e.g., [61, 72, 73] with further references therein. The next results taken from [61, 72] list those properties of the forward-backward envelope for convex composite extended-real-valued functions that are needed to derive the main results of this section.

Proposition 5.2 (basic properties of FBE). *Let $\varphi = f + g$ be as in (5.1), and let $\gamma > 0$. Suppose that f is twice continuously differentiable over $\mathbb{R}^n$, and that ∇f is Lipschitz continuous on $\mathbb{R}^n$ with modulus $\ell > 0$. Then the following assertions hold:*

(i) *The FBE φ_γ of φ is continuous differentiable on $\mathbb{R}^n$ with its gradient calculated by*

$$\nabla\varphi_\gamma(x) = \gamma^{-1}\big(I - \gamma\nabla^2 f(x)\big)\big(x - \mathrm{Prox}_{\gamma g}(x - \gamma\nabla f(x))\big), \quad x \in \mathbb{R}^n. \tag{5.6}$$

Moreover, the set of optimal solutions to (5.1) agrees with the stationary points of φ_γ as

$$\mathrm{argmin}\varphi = \mathrm{zer}\nabla\varphi_\gamma := \big\{x \in \mathbb{R}^n \;\big|\; \nabla\varphi_\gamma(x) = 0\big\} \quad \textit{for all} \;\; \gamma \in (0, 1/\ell).$$

(ii) *Consider $f(x) := \frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle + \alpha$, where $A \in \mathbb{R}^{n\times n}$ is a positive-semidefinite symmetric matrix, $b \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$, and define the numbers*

$$L := 2\left(1 - \gamma\lambda_{\min(A)}\right)/\gamma \;\; \textit{and} \;\; K := \min\big\{(1 - \gamma\lambda_{\min(A)})\lambda_{\min(A)}, (1 - \gamma\lambda_{\max(A)})\lambda_{\max(A)}\big\}.$$

Then for all $\gamma \in (0, 1/\ell)$, the FBE φ_γ is convex and its gradient $\nabla\varphi_\gamma$ is globally Lipschitzian on $\mathbb{R}^n$ with modulus L. If A is positive-definite, then φ_γ is strongly convex with modulus K.

It follows from Proposition 5.2 that using the forward-backward envelope (5.4) makes it possible to pass from the nonsmooth and constrained problem (5.1) to the unconstrained one:

$$\text{minimize} \quad \varphi_\gamma(x) \quad \text{subject to} \;\; x \in \mathbb{R}^n \tag{5.7}$$

with a smooth cost function. Thanks to the explicit calculations of φ_γ in (5.5) and its gradient (5.6), we can extend Algorithm 1 and Algorithm 2 to cover problem (5.1) via passing to (5.7). The implementation of this procedure requires revealing appropriate assumptions on φ in (5.1), which ensure the fulfillment of those for φ_γ and thus allow us to apply the results of Sections 3, 4 to (5.7).

Note that (5.7) is not generally a problem of $\mathcal{C}^{1,1}$ optimization, since Proposition 5.2(i) does not ensure the Lipschitz continuity of $\nabla\varphi_\gamma$. The latter property is guaranteed by Proposition 5.2(ii) when f is a quadratic function and thus problem (5.1) is written as

$$\text{minimize} \;\; \varphi(x) := \frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle + \alpha + g(x), \quad x \in \mathbb{R}^n, \tag{5.8}$$

where $A \in \mathbb{R}^{n\times n}$ is a *positive-semidefinite* symmetric matrix, $b \in \mathbb{R}^n$, and $\alpha \in \mathbb{R}$. From now on, this is our standing framework for the rest of the section.

Let us highlight that problems of type (5.8) are important for their own sake, while they also arise frequently as *subproblems* for various efficient numerical algorithms including *sequential quadratic programming methods* (SQP) [5, 37], *augmented Lagrangian methods* [30, 33, 42, 65, 68, 69], *proximal Newton methods* [40, 57], etc. Observe furthermore that optimization problems of this type often appear in practical models related, e.g., to machine learning and statistics. In particular, *Lasso problems* considered in Section 6 can be written in form (5.8). Moreover, there are some other important classes of problems that are modeled as (5.8). They include problems in *support vector machine* [36], *convex clustering* [62, 71], *constrained quadratic optimization* [59], etc.

Now we start the procedure of designing and justifying globally convergent generalized Newton algorithms to solve the convex composite problem (5.8) by applying the corresponding results for the $\mathcal{C}^{1,1}$ optimization problem (5.7) obtained in Sections 3 and 4. The first step is the express the generalized Hessian of the FDE φ_γ from (5.7) in terns of the given data of (5.8). Recall that the sign '$\succ$' indicates the matrix positive-definiteness.

Proposition 5.3 **(calculating the generalized Hessian of FBE).** *Let $\varphi = f + g$ be as in (5.8), and let $\gamma > 0$ be such that $B := I - \gamma A \succ 0$. Then we have the calculation formula*

$$\bar{z} \in \partial^2\varphi_\gamma(\bar{x})(w) \iff B^{-1}\bar{z} - Aw \in \partial^2 g\left(\mathrm{Prox}_{\gamma g}(\bar{u}), \frac{1}{\gamma}\big(\bar{u} - \mathrm{Prox}_{\gamma g}(\bar{u})\big)\right)(w - \gamma B^{-1}\bar{z}) \tag{5.9}$$

for any $\bar{x} \in \mathbb{R}^n$, $w \in \mathbb{R}^n$, and $\bar{u} := \bar{x} - \gamma(A\bar{x} + b)$.

Proof. Fix x, w, and $\bar{u}$ as above and define the function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ by

$$h(x) := e_\gamma g\big(x - \gamma(Ax + b)\big) \quad \text{for all} \quad x \in \mathbb{R}^n.$$

It is clear that h is continuous differentiable with $\nabla h(\bar{x}) = (I - \gamma A)^*\nabla e_\gamma g(\bar{u}) = B\nabla e_\gamma g(\bar{u})$. It follows from the definition of FBE (5.5) and the second-order sum rule from [50, Proposition 1.121] that

$$\partial^2\varphi_\gamma(\bar{x})(w) = (A - \gamma A^*A)w + \partial^2 h(\bar{x})(w) = BAw + \partial^2 h(\bar{x})(w). \tag{5.10}$$

By using the second-order chain rule from [50, Theorem 1.127], we have

$$\partial^2 h(\bar{x})(w) = B\partial^2 e_\gamma g(\bar{u})(Bw). \tag{5.11}$$

Combining (5.10) and (5.11) gives us the relationship

$$\partial^2\varphi_\gamma(\bar{x})(w) = BAw + B\partial^2 e_\gamma g(\bar{u})(Bw),$$

which in turn yields the equivalencies

$$\bar{z} \in \partial^2\varphi_\gamma(\bar{x})(w) \iff \bar{z} - BAw \in B\partial^2 e_\gamma g(\bar{u})(Bw) \iff B^{-1}\bar{z} - Aw \in \partial^2 e_\gamma g(\bar{u})(Bw).$$

Employing finally [38, Lemma 6.4], we arrive at the inclusion

$$B^{-1}\bar{z} - Aw \in \partial^2 g\left(\mathrm{Prox}_{\gamma g}(\bar{u}), \frac{1}{\gamma}\big(\bar{u} - \mathrm{Prox}_{\gamma g}(\bar{u})\big)\right)(Bw - \gamma B^{-1}\bar{z} + \gamma Aw),$$

which completes the proof of the proposition due to $Bw + \gamma Aw = w$. $\square$

The next proposition shows that the metric regularity and tilt stability of the original objective φ in (5.8) is equivalent to the corresponding properties of its FBE φ_γ in (5.7). Moreover, we get a useful estimate of the inverse mapping of $\partial^2\varphi_\gamma$ in terms of the given data of (5.8).

Proposition 5.4 **(metric regularity and tilt-stability of FBE).** *Let $\varphi = f + g$ be as in (5.8), and let $\gamma > 0$ be such that $B := I - \gamma A \succ 0$. Then for any $\bar{x} \in \mathbb{R}^n$ satisfying $0 \in \partial\varphi(\bar{x})$ we have the following assertions:*

(i) $\|\partial^2\varphi_\gamma(\bar{x})^{-1}\| \le \|\partial^2\varphi(\bar{x}, 0)^{-1}\| + \gamma\|B^{-1}\|$.

(ii) *$\partial\varphi$ is metrically regular around $(\bar{x}, 0)$ if and only if $\nabla\varphi_\gamma$ is metrically regular around $\bar{x}$.*

(iii) *$\bar{x}$ is a tilt-stable local minimizer of φ if and only if $\bar{x}$ is a tilt-stable local minimizer of φ_γ.*

Proof. It follows from Proposition 5.3 that

$$z \in \partial^2\varphi_\gamma(\bar{x})(w) \iff 0 \in Aw + \partial^2 g\left(\operatorname{Prox}_{\gamma g}(\bar{u}), \frac{1}{\gamma}(\bar{u} - \operatorname{Prox}_{\gamma g}(\bar{u}))\right)(w - \gamma B^{-1}\bar{z}) \tag{5.12}$$

with $\bar{u}$ defined therein. The convexity of φ ensures that $\bar{x}$ is an optimal solution to (5.8), and thus $\bar{x} - \operatorname{Prox}_{\gamma g}(\bar{u}) = 0$ by [1, Theorem 27.2]. Therefore, (5.12) is equivalent to

$$z \in Aw+\partial^2 g(\bar{x}, -A\bar{x}-b)(w-\gamma B^{-1}z) = A(w-\gamma B^{-1}z)+\partial^2 g(\bar{x}, -A\bar{x}-b)(w-\gamma B^{-1}z)+\gamma AB^{-1}z. \tag{5.13}$$

The second-order subdifferential sum rule from [50, Proposition 1.121] yields

$$\begin{aligned} A(w - \gamma B^{-1}z) + \partial^2 g(\bar{x}, -A\bar{x} - b)(w - \gamma B^{-1}z) &= \partial^2(f + g)(\bar{x}, 0)(w - \gamma B^{-1}z) \\ &= \partial^2\varphi(\bar{x}, 0)(w - \gamma B^{-1}z). \end{aligned} \tag{5.14}$$

Combining (5.12), (5.13), and (5.14) gives us the equivalence

$$z \in \partial^2\varphi_\gamma(\bar{x})(w) \iff z \in \partial^2\varphi(\bar{x}, 0)(w - \gamma B^{-1}z), \tag{5.15}$$

which verifies (i). It follows from the Mordukhovich criterion (2.4) and the equivalence (5.15) that $\partial\varphi$ is metrically regular around $(\bar{x}, 0)$ if and only if $\nabla\varphi_\gamma$ is metrically regular around $\bar{x}$, which justifies assertion (ii). Finally, [20, Proposition 4.5] tells us that a stationary point $\bar{x}$ is a tilt-stable local minimizer of an l.s.c. convex function if and only if its subgradient mapping is metrically regular around $(\bar{x}, 0)$. Using this observation together with (ii), we obtain (iii) and complete the proof. $\square$

Now we recall some other notions of variational analysis that are used to establish the superlinear convergence of both algorithms developed in this section to solve problem (5.8). These notions, introduced by Rockafellar, are taken from the book [70]. A set-valued mapping $S : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is *proto-differentiable* at $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ if for any $\bar{w} \in \mathbb{R}^n$, $\bar{z} \in \operatorname{Lim\,sup}_{t\downarrow 0,\, w\to\bar{w}}(S(\bar{x} + tw) - \bar{y})/t$, and $t_k \downarrow 0$ there exist $w_k \to \bar{w}$ and $z_k \to \bar{z}$ such that $z_k \in (S(\bar{x} + t_k w_k) - \bar{y})/t_k$ whenever $k \in \mathbb{N}$. Given $\varphi\colon \mathbb{R}^n \to \overline{\mathbb{R}}$ with $\bar{x} \in \operatorname{dom}\varphi$, consider the family of second-order finite differences

$$\Delta_\tau^2\varphi(\bar{x}, v)(u) := \frac{\varphi(\bar{x} + \tau u) - \varphi(\bar{x}) - \tau\langle v, u\rangle}{\frac{1}{2}\tau^2}$$

and define the *second subderivative* of φ at $\bar{x}$ for $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ by

$$d^2\varphi(\bar{x}, v)(w) := \liminf_{\substack{\tau\downarrow 0\\ u\to w}} \Delta_\tau^2\varphi(\bar{x}, v)(u).$$

Then φ is said to be *twice epi-differentiable* at $\bar{x}$ for v if for every $w \in \mathbb{R}^n$ and every choice of $\tau_k \downarrow 0$ there exists a sequence $w^k \to w$ such that

$$\frac{\varphi(\bar{x} + \tau_k w^k) - \varphi(\bar{x}) - \tau_k\langle v, w^k\rangle}{\frac{1}{2}\tau_k^2} \to d^2\varphi(\bar{x}, v)(w) \ \text{ as } \ k \to \infty.$$

Twice epi-differentiability has been recognized as an important concept of second-order variational analysis with numerous applications to optimization; see the aforementioned monograph by Rockafellar and Wets and the recent papers [45, 46, 47] developing a systematic approach to verify epi-differentiability via *parabolic regularity*, which is a major second-order property of sets and functions.

The next proposition expresses the properties of the FBE φ_γ in (5.7), which are needed for the superlinear convergence of our algorithms, in terms of the given data of (5.8).

Proposition 5.5 (semismoothness* and directional differentiability of FBE derivatives). *Let $\varphi = f + g$ be as in (5.8), and let $\gamma > 0$ be such that $B := I - \gamma A \succ 0$. Then for any $\bar{x} \in \mathbb{R}^n$ satisfying the stationary condition $0 \in \partial\varphi(\bar{x})$ the following assertions hold:*

(i) $\nabla\varphi_\gamma$ *is semismooth** *at* $\bar{x}$ *if* ∂g *is semismooth** *at* $(\bar{x},\bar{v})$*, where* $\bar{v} := -A\bar{x} - b$*.*

(ii) $\nabla\varphi_\gamma$ *is directionally differentiable at* $\bar{x}$ *if* g *is twice epi-differentiable at* $\bar{x}$ *for* $\bar{v}$*.*

Proof. Denote $h_\gamma(x) := \text{Prox}_{\gamma g}(x - \gamma(Ax+b))$ for all $x \in \mathbb{R}^n$ and get by Proposition 5.2 that

$$\nabla\varphi_\gamma(x) = \gamma^{-1}(I-\gamma A)\big(x - h_\gamma(x)\big) = Bx - Bh_\gamma(x), \quad x \in \mathbb{R}^n. \tag{5.16}$$

Since the stationary point $\bar{x}$ is an optimal solution to (5.8) due the convexity of φ, we have that $\bar{x} = h_\gamma(\bar{x})$ by [1, Theorem 27.2]. Since ∂g is semismooth* at $(\bar{x},\bar{v})$, we have that $\text{Prox}_{\gamma g}$ is semismooth* at $\bar{x} - \gamma(A\bar{x}+b)$ by using [23, Proposition 6]. It follows from Lemma 7.3 in the Appendix that h_γ is semismooth* at $\bar{x}$. Employing now (5.16) and [28, Proposition 3.6] tells us that the gradient mapping $\nabla\varphi_\gamma$ is semismooth* at $\bar{x}$. This verifies assertion (i).

To proceed with the proof of (ii), observe by [70, Theorem 13.40] that the twice epi-differentiability of g at $\bar{x}$ for $\bar{v}$ amounts to saying that the subgradient mapping ∂g is proto-differentiable of at $(\bar{x},\bar{v})$. Using [23, Corollary 8], we conclude that $\text{Prox}_{\gamma g}$ is directionally differentiable at $\bar{x} - \gamma(A\bar{x}+b)$, which yields in turn the directional differentiability of h_γ at $\bar{x}$. Thus the mapping $\nabla\varphi_\gamma$ is directionally differentiable at $\bar{x}$ due to (5.16). This verifies (ii) and completes the proof of the proposition. $\square$

Now we are ready to describe and then justify the proposed globally coderivative-based damped Newton method for solving the convex composite optimization problem (5.8).

Algorithm 3 Coderivative-based damped Newton algorithm for convex composite optimization

Input: $x^0 \in \mathbb{R}^n$, $\gamma > 0$ such that $B := I - \gamma A \succ 0$, $\sigma \in \left(0, \frac{1}{2}\right)$, $\beta \in (0,1)$, and φ_γ as in (5.4)

for $k = 0, 1, \ldots$ **do**
  If $\nabla\varphi_\gamma(x^k) = 0$, stop. Otherwise set $u^k := x^k - \gamma(Ax^k + b)$, $v^k := \text{Prox}_{\gamma g}(u^k)$
  Find $d^k \in \mathbb{R}^n$ st $-\frac{1}{\gamma}(x^k - v^k) - Ad^k \in \partial^2 g\left(v^k, \frac{1}{\gamma}(u^k - v^k)\right)(x^k - v^k + d^k)$
  Set $\tau_k = 1$
  while $\varphi_\gamma(x^k + \tau_k d^k) > \varphi_\gamma(x^k) + \sigma\tau_k\langle\nabla\varphi_\gamma(x^k), d^k\rangle$ **do**
    set $\tau_k := \beta\tau_k$
  end while
  Set $x^{k+1} := x^k + \tau_k d^k$
end for

Explicit expressions for the sequences $\{v^k\}$ and $\{d^k\}$ in Algorithm 3 depend on given structures of the regularizers g, which are efficiently specified in applied models of machine learning and statistics; see, e.g., Section 6. The next theorem provides explicit sufficient conditions to run Algorithm 3 for solving the class of convex composite optimization problems (5.8).

Theorem 5.6 (global convergence of coderivative-based damped Newton algorithm in convex composite optimization). *Consider problem* (5.8)*, where the matrix* A *is positive-definite. Then Algorithm 3 either stops after finitely many iterations, or produces a sequence* $\{x^k\}$ *such that it globally R-linearly converges to* $\bar{x}$*, which is the unique solution to* (5.8) *being a tilt-stable local minimizer of* φ *with modulus* $\kappa := 1/\lambda_{\min(A)}$*. Furthermore, the convergence rate of* $\{x^k\}$ *is at least Q-superlinear if the subgradient mapping* ∂g *is semismooth** *at* $(\bar{x},\bar{v})$*, where* $\bar{v} := -A\bar{x} - b$*, and if either one of two following conditions is satisfied:*

(i) $\sigma \in (0, 1/(2LK))$*, where* $L := 2\left(1 - \gamma\lambda_{\min(A)}\right)/\gamma$ *and* $K := \kappa + \gamma\|B^{-1}\|$*.*

(ii) g *is twice epi-differentiable at* $\bar{x}$ *for* $\bar{v}$*.*

Proof. We deduce from Propositions 5.2(ii) and 5.3 that solving the convex composite optimization problem (5.8) by Algorithm 3 reduces to solving the $\mathcal{C}^{1,1}$ optimization problem (5.7) by using Algorithm 1. The rest of the proof is split into the following two claims"

Claim 1: *Algorithm 3 either stops after finitely many iterations or produces a sequence sequence* $\{x^k\}$ *which globally R-linearly converges to the unique solution* $\bar{x}$ *of* (5.8)*, which is a tilt-stable local* *minimizer of* φ. Indeed, we get from Proposition 5.2(ii) that φ_γ is a strongly convex function, and its gradient is globally Lipschitz continuous with modulus L. It follows from[8, Theorem 5.1] that $\partial^2\varphi_\gamma(x)$ is positive-definite for all $x \in \mathbb{R}^n$. Then Theorem 3.2 tells us that Algorithm 3 either stops after finitely many iterations, or produces a sequence $\{x^k\}$ that globally R-linearly converges to $\bar{x}$, which is a stationary point of φ_γ. Employing again Proposition 5.2(ii) confirms that $\bar{x}$ is an optimal solution to (5.8). Furthermore, the strong convexity of φ with modulus $\lambda_{\min(A)}$ and Lemma 7.4 from the Appendix yield the uniqueness and tilt stability conclusions for $\bar{x}$.

Claim 2: *The Q-superlinear convergence of* $x^k \to \bar{x}$ *holds under the assumptions of the theorem.* The imposed semismooth* property of ∂g at $(\bar{x}, \bar{v})$ ensures the fulfillment of this property for $\nabla\varphi_\gamma$ at $\bar{x}$ by Lemma 5.5. Assume now that condition (i) of the theorem is satisfied. Then we get from Claim 1 that L is a Lipschitz constant of $\nabla\varphi_\gamma$ around $\bar{x}$. As follows from [20, Proposition 4.5], the modulus of tilt-stability of the l.s.c. convex function φ under consideration at $\bar{x}$ is the same as the modulus of metric regularity of φ around this point. Combining the latter with the statement of Proposition 5.4(i) and the precise calculation in (2.5) of the exact bound of metric regularity, we conclude that $\bar{x}$ is tilt-stable local minimizer of φ_γ with modulus K. Thus the claimed assertion on the superlinear convergence in this case follows directly from Theorem 3.5. Assuming finally by (ii) that g is twice epi-differentiable at $\bar{x}$ for $\bar{v}$, we deduce from Proposition 5.5(ii) that the mapping $\nabla\varphi_\gamma$ is directionally differentiable at $\bar{x}$. Thus the claimed superlinear convergence of $\{x^k\}$ follows in this case from the corresponding statement of Theorem 3.5. This completes the proof of the theorem. $\square$

Theorem 5.6 and the results of numerical experiments in Section 6 show that Algorithm 3, designed in terms of the computable data of (5.8), exhibits an excellent performance when A is positive-definite, i.e., in the strongly convex setting of (5.8). Otherwise, this algorithm is not even well-defined. To relax this positive-definiteness/strong convex assumption, we now propose and justify a new coderivative-based algorithm of the Levenberg-Marquardt type, which is well-defined and globally convergent to solutions of (5.8) for merely *positive-semidefinite* matrices A with linear and superlinear convergence rates under some additional assumptions that include the *metric regularity* of the subgradient mapping $\partial\varphi$. Observe that the latter assumption is *weaker* than the *strong convexity* of f in problems of composite optimization (5.1), including those with quadratic functions f as in (5.8). A simple class of functions φ illustrating this observation is given by $\varphi = f + |x|$. In particular, for $f \equiv 0$ we have

$$\partial^2\varphi(0,0) = \big\{ w \in \mathbb{R} \;\big|\; (w, -v) \in \mathbb{R} \times \{0\} \big\},$$

and thus $0 \in \partial^2\varphi(0,0)(v) \implies v = 0$, which tells us by (2.4) that $\partial\varphi$ is metrically regular around $(0,0)$.

Here is the aforementioned algorithm, which is more complicated than Algorithm 3, while being applied for problems (5.8) with positive-semidefinite matrices A. Note that the new algorithm *does not* require performing operations like computing inverse matrices that are expensive in large dimensions.

Algorithm 4 Coderivative-based Levenberg-Marquardt algorithm for convex composite optimization

Input: $x^0 \in \mathbb{R}^n$, $\gamma > 0$ such that $B := I - \gamma A \succ 0$, $\lambda > 0$, $\sigma \in \left(0, \frac{1}{2}\right)$, $\beta \in (0,1)$, and φ_γ as in (5.4)

for $k = 0, 1, \ldots$ **do**
  If $\nabla\varphi_\gamma(x^k) = 0$, stop. Otherwise set $u^k := x^k - \gamma(Ax^k + b), v^k := \mathrm{Prox}_{\gamma g}(u^k), \mu_k := \lambda\|\nabla\varphi_\gamma(x^k)\|$
  Find $z^k \in \mathbb{R}^n$ st $-\frac{1}{\gamma}(x^k - v^k) - (\mu_k I + AB)z^k \in \partial^2 g\left(v^k, \frac{1}{\gamma}(u^k - v^k)\right)\left(x^k - v^k + (B + \gamma\mu_k I)z^k\right)$
  Set $d^k = Bz^k$
  Set $\tau_k = 1$
  while $\varphi_\gamma(x^k + \tau_k d^k) > \varphi_\gamma(x^k) + \sigma\tau_k\langle\nabla\varphi_\gamma(x^k), d^k\rangle$ **do**
    set $\tau_k := \beta\tau_k$
  end while
  Set $x^{k+1} := x^k + \tau_k d^k$
end for

The next theorem fully describes the well-posedness and performance of Algorithm 4.

Theorem 5.7 (global convergence of coderivative-based Levenberg-Marquardt algorithm in convex composite optimization). *Consider problem (5.8) of convex composite optimization, where the matrix A is positive-semidefinite. Then we have the assertions:*

(i) *Algorithm 4 either stops after finitely many iterations, or produces a sequence $\{x^k\}$ for which all the limiting points of this sequence are optimal solutions to (5.8).*

(ii) *If in addition the subgradient mapping $\partial\varphi$ is metrically regular around $(\bar{x}, 0)$ with modulus $\kappa > 0$, where $\bar{x}$ is a limiting point of $\{x^k\}$, then the sequence $\{x^k\}$ globally R-linearly converges to $\bar{x}$, and $\bar{x}$ is a tilt-stable local minimizer of φ with modulus κ.*

(iii) *The rate of convergence of $\{x^k\}$ is at least Q-superlinear if the subgradient mapping ∂g is semismooth* at $(\bar{x}, \bar{v})$, where $\bar{v} := -A\bar{x} - b$, and if one of two following conditions holds:*

(a) *$\sigma \in (0, 1/(2LK))$, where $L := 2\left(1 - \gamma\lambda_{\min(A)}\right)/\gamma$ and $K := \kappa + \gamma\|B^{-1}\|$.*

(b) *g is twice epi-differentiable at $\bar{x}$ for $\bar{v}$.*

Proof. Using Propositions 5.2(ii) and 5.3, we can reduce Algorithm 4 for solving the convex composite optimization problem (5.8) to Algorithm 2 for solving the $\mathcal{C}^{1,1}$ optimization problem (5.7). Let us now proceed with the justification of this procedure by verifying each claim of the theorem.

Claim 1: *Assertion* (i) *holds.* Indeed, we have from Proposition 5.2(ii) that the gradient mapping $\nabla\varphi_\gamma$ is globally Lipschitz continuous with modulus L and φ_γ is a convex function; thus the generalized Hessian $\partial^2\varphi_\gamma(x)$ is positive-semidefinite for all $x \in \mathbb{R}^n$ by [8, Theorem 3.2]. Therefore, it follows from Theorem 4.1 that Algorithm 4 either stops after finitely many iterations, or produces a sequence $\{x^k\}$ whose limiting points are solutions to (5.7). This verifies assertion (i).

Claim 2: *Assertion* (ii) *holds.* Under the assumptions made in (ii), it follows from the established relationships between problems (5.8) and (5.7) and the application of Theorem 4.2 to the latter that the sequence $\{x^k\}$ globally R-linearly converges to $\bar{x}$ as $k \to \infty$. Then we get by [20, Proposition 4.5] that the tilt-stability of φ at $\bar{x}$ with modulus κ follows from the metric regularity of $\partial\varphi$ and the convexity of φ, which therefore verifies (ii).

Claim 3: *Assertion* (iii) *holds.* We deduce from Proposition 5.5(i) that the semismoothness* at of g at $(\bar{x}, \bar{v})$ yields this property for $\nabla\varphi_\gamma$ at $\bar{x}$. Assuming first that condition (a) is satisfied, we deduce from Proposition 5.2(ii) that L is a Lipschitz constant of $\nabla\varphi_\gamma$ around $\bar{x}$. It follows from [20, Proposition 4.5] that the modulus of tilt-stability of the l.s.c. convex function φ at $\bar{x}$ is equal to the modulus of metric regularity of φ around this point. Combining the latter with Proposition 5.4(i) and the calculation formula for the exact regularity bound in (2.5) tells us that $\bar{x}$ is a tilt-stable local minimizer of φ_γ with modulus K. Hence assertion (iii) in case (a) follows from Theorem 4.2(a). Assuming now (b) implies by Proposition 5.5(ii) that $\nabla\varphi_\gamma$ is directionally differentiable at $\bar{x}$. Applying finally Theorem 4.2(b) to problem (5.7) ensures the Q-superlinearly convergence of sequence $x^k \to \bar{x}$ as $k \to \infty$ and therefore completes the proof of the theorem. $\square$

6 Solving Lasso Problems and Numerical Experiments

This section is devoted to specifying both Algorithms 3 and 4 for the case of the basic Lasso problem stated below, and then to conducting numerical experiments for this problem and comparing them with the performances of some major first-order and second-order algorithms.

The basic Lasso problem, known also as the *ℓ^1-regularized least square optimization problem*, was introduced by Tibshirani [74], and since that it has been largely investigated and applied to various issues in statistics, machine learning, image processing, etc. This problem is formulated as follows:

$$\text{minimize } \ \varphi(x) := \frac{1}{2}\|Ax - b\|_2^2 + \mu\|x\|_1 \quad \text{subject to } \ x \in \mathbb{R}^n, \tag{6.1}$$

where A is an $m \times n$ matrix, $\mu > 0$, and $b \in \mathbb{R}^m$, and where $\|\cdot\|_1$ and $\|\cdot\|_2$ stand for the standard p-norms on $\mathbb{R}^n$. It is easy to see that the Lasso problem (6.1) belongs to the class of convex composite optimization problems (5.8). Indeed, we can represent (6.1) as minimizing the nonsmooth convex function $\varphi(x) := f(x) + g(x)$, where

$$f(x) := \frac{1}{2}\langle \widetilde{A}x, x\rangle + \langle \widetilde{b}, x\rangle + \widetilde{\alpha} \quad \text{and} \quad g(x) := \mu\|x\|_1 \tag{6.2}$$

with $\widetilde{A} := A^*A$, $\widetilde{b} := -A^*b$, and $\widetilde{\alpha} := \frac{1}{2}\|b\|^2$, and where the matrix $\widetilde{A} = A^*A$ is symmetric and positive-semidefinite. Observe that the Lasso problem (6.1) always admits an optimal solution [74].

We start implementing Algorithms 3 and 4 to solve (6.1) with the explicit calculation of their involved ingredients (proximal and subgradient mappings, generalized Hessian for the regularizer g) given entirely via the problem data.

Proposition 6.1 (explicit calculations for the Lasso problem). *Let $g(x) = \mu\|x\|_1$ be the regularizer in the Lasso problem (6.1). Then we have the calculation formulas:*

$$\big(\mathrm{Prox}_{\gamma g}(x)\big)_i = \begin{cases} x_i - \mu\gamma & \text{if} \quad x_i > \mu\gamma, \\ 0 & \text{if} \quad -\mu\gamma \le x_i \le \mu\gamma, \\ x_i + \mu\gamma & \text{if} \quad x_i < -\mu\gamma. \end{cases} \tag{6.3}$$

$$\partial g(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; \begin{array}{c} v_j = \mathrm{sgn}(x_j),\ x_j \neq 0, \\ v_j \in [-\mu, \mu],\ x_j = 0 \end{array} \right\} \quad \text{whenever} \ \ x \in \mathbb{R}^n. \tag{6.4}$$

The generalized Hessian of g is calculated by

$$\partial^2 g(x,y)(v) = \left\{ w \in \mathbb{R}^n \;\middle|\; \Big(\frac{1}{\mu}w_i, -v_i\Big) \in G\Big(x_i, \frac{1}{\mu}y_i\Big),\ i = 1, \ldots, n \right\} \tag{6.5}$$

for each $(x,y) \in \mathrm{gph}\,\partial g$ and $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, where the mapping $G\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is defined by

$$G(t,p) := \begin{cases} \{0\} \times \mathbb{R} & \text{if} \quad t \neq 0,\ p \in \{-1, 1\}, \\ \mathbb{R} \times \{0\} & \text{if} \quad t = 0,\ p \in (-1, 1), \\ (\mathbb{R}_+ \times \mathbb{R}_-) \cup (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) & \text{if} \quad t = 0,\ p = -1, \\ (\mathbb{R}_- \times \mathbb{R}_+) \cup (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\}) & \text{if} \quad t = 0,\ p = 1, \\ \emptyset & \text{otherwise}. \end{cases} \tag{6.6}$$

Proof. The formula for the proximal mapping (6.3) follows from definition (5.3) and the form of $g(\cdot) = \|\cdot\|_1$. The calculations of ∂g and $\partial^2 g$ are taken from [38, Propositions 7.1 and 7.2], respectively. $\square$

Let us present the specifications of Algorithms 3 and 4 as well as Theorem 5.6 and 5.7 on their performances, respectively, for the Lasso problem (6.1).

Theorem 6.2 (solving Lasso). *Considering the Lasso problem (6.1), we have the following:*

(i) *Algorithm 3, with all its ingredients calculated in Proposition 6.1, either stops after finitely many iterations, or produces a sequence $\{x^k\}$ such that it globally Q-superlinearly converges to $\bar{x}$, which is the unique solution to (6.1) and a tilt-stable local minimizer of φ with modulus $\kappa := 1/\lambda_{\min(A^*A)}$, provided that the matrix A^*A is positive-definite.*

(ii) *Algorithm 4, with the positive-semidefinite matrix A^*A and the ingredients calculated in Proposition 6.1, either stops after finitely many iterations, or produces a sequence $\{x^k\}$ such that any limiting point $\bar{x}$ of it is a solution to (6.1). If in addition $\partial\varphi$ is metrically regular around $(\bar{x}, 0)$ with some modulus $\kappa > 0$, then the sequence $\{x^k\}$ globally Q-superlinearly converges to $\bar{x}$, which is a tilt-stable local minimizer of φ with modulus κ.*

Proof. Observe by (6.4) that the graph of the subgradient mapping ∂g is the union of finitely many closed convex sets, and hence ∂g is semismooth* at all the points in its graph; see [28]. Furthermore, g is a proper, convex, and piecewise linear-quadratic function on $\mathbb{R}^n$. Then it follows from [70, Proposition 13.9] that g is twice epi-differentiable on $\mathbb{R}^n$. Applying Theorem 5.6 and Theorem 5.7, we arrive at all the conclusions in (i) and (ii) of this theorem, respectively. $\square$

To run Algorithms 3 and 4, we need to explicitly determine the sequences $\{v^k\}$ and $\{d^k\}$ generated by these algorithms. The expressions for v^k follows directly from (6.3), which tells us that

$$\left(v^k\right)_i = \begin{cases} u_i - \mu\gamma & \text{if} \quad u_i > \mu\gamma, \\ 0 & \text{if} \quad -\mu\gamma \le u_i \le \mu\gamma, \\ u_i + \mu\gamma & \text{if} \quad u_i < -\mu\gamma. \end{cases}$$

Using further the formulas in (6.4)–(6.6), we express d^k in Algorithm 3 via the conditions

$$\begin{cases} \left(-\frac{1}{\gamma}(x^k - v^k) - A^*Ad^k\right)_i = 0 & \text{if} \quad \left(v^k\right)_i \ne 0, \\ \left(-x^k - v^k + d^k\right)_i = 0 & \text{if} \quad \left(v^k\right)_i = 0. \end{cases}$$

Thus d^k can be computed for each $k \in \mathbb{N}$ by solving the linear equation $X^k d = v^k - x^k$, where

$$(X^k)_i := \begin{cases} \gamma(A^*A)_i & \text{if} \quad (v^k)_i \ne 0, \\ I_i & \text{if} \quad (v^k)_i = 0. \end{cases} \tag{6.7}$$

Similarly, by employing the calculations of Proposition 6.1 in the framework of Algorithm 4 and by performing elementary transformations, we get the linear equation

$$(X^k + \gamma\mu_k I)d^k = B(v^k - x^k)$$

to find the direction d^k, where X^k is computed in (6.7).

Now we are ready to conduct numerical experiments for solving the Lasso problem (6.1) by using our globally convergent coderivative-based Generalized Damped Newton Method (GDNM) via Algorithm 3 and globally convergent coderivative-based Generalized Levenberg-Marquardt Method (GLMM) via Algorithm 4. The obtained calculations are compared with those obtained by implementing the following highly recognized first-order and second-order algorithms:

(i) The *Alternating Direction Methods of Multipliers*[1] (ADMM); see [6, 24, 25].

(ii) The *Fast Iterative Shrinkage-Thresholing Algorithm*[2] (FISTA) with the code presented in [3].

(iii) The *Semismooth Newton Augmented Lagrangian Method*[3] (SSNAL) recently developed in [42].

All the numerical experiments are conducted on a desktop with 10th Gen Intel(R) Core(TM) i5-10400 processor (6-Core, 12M Cache, 2.9GHz to 4.3GHz) and 16GB memory. All the codes are written in MATLAB 2016a.

We divide the numerical implementation into the two parts with different sizes of A and b from (6.1). In both parts, A is generated randomly with i.i.d. (identically and independent distributed) standard Gaussian entries, where $b = 0$ in the first part, and where b is generated randomly with values of components are from 0 to 1 in the second part. In some particular tests, we normalize each column of A so that A^*A is close to singular and mark them with symbol * for identification. In summary, A^*A is nonsingular in Tests 3, 4, 7, 8, and it is singular or close to singular in all the other tests. Each table contains 2 tests where $n > m$ and the matrix A^*A is nonsingular, 2 tests where $n > m$ and the matrix A^*A is singular, 2 tests where $n = m$ and the matrix A^*A is nonsingular, 2 tests where $n = m$ and the matrix A^*A is singular, and 2 test when $m > n$. To simplify the numerical implementations for solving the Lasso problem (6.1), we set $\mu := 10^{-3}$ as the tuning parameter for all the tests. If an algorithm cannot start iterating in the first step, it is marked by "Error" word; this concerns only some cases of GDNM when the matrix A^*A is not positive-definite. Both parts of our testing process are described in more detail as follows:

[1]https://web.stanford.edu/ boyd/papers/admm/lasso/lasso.html
[2]https://github.com/he9180/FISTA-lasso
[3]https://www.polyu.edu.hk/ama/profile/dfsun/

- In the first part, we set starting points as all-ones vectors. Since $\bar{x} := 0$ is the unique solution to (6.1), the value $\|x^k\|$ can be used to measure the accuracy of an approximate optimal solution x^k for (6.1). We stop the algorithms in our experiments when either the condition $\|x^k\| < 10^{-6}$ is satisfied, or they reach the maximum computation time of 10000 seconds. The results of numerical experiments in this part are displayed in Table 1. Since $b = 0$, the algorithm ADMM always terminates after the first step and returns to the zero vector as a solution; so we do not mention the results from ADMM in Table 1.

- In the second part, $x^0 := 0$ is the starting point for each algorithm, and the following *relative KKT residual* η_k in (6.8) suggested in [42] is used to measure the accuracy of an approximate optimal solution x^k for (6.1):

$$\eta_k := \frac{\|x^k - \mathrm{Prox}_{\mu\|\cdot\|_1}(x^k - A^*(Ax^k - b))\|}{1 + \|x^k\| + \|Ax^k - b\|}. \tag{6.8}$$

  We stop the algorithms in our experiments when either the condition $\eta_k < 10^{-6}$ is satisfied, or they reach the maximum computation time of 10000 seconds. The results of numerical experiments in this part are displayed in Table 2.

In Tables 1 and 2, 'TN' stands for the text number, 'iter' indicates the number of performed iterations, and 'CPI (central processing unit) time' stands for the time needed to achieve the prescribed accuracy of approximate solutions (the smaller the better).

As we can see from the results presented in Tables 1 and 2, our algorithms GDNM and GLMM are highly efficient when A^*A is *nonsingular*, where the Q-superlinear convergence is guaranteed by Theorem 6.2. They may behave even better than the other compared algorithms when $m \geq n$, which is the setting of various practically important models; see, e.g., [21] for Lasso applications to diabetes studies where m is much large than n, and [3] for $m = n$ with applications to image processing.

When the matrix A^*A is *singular* (or close to singular), our theoretical results do not guarantee the fast convergence of GDNM and GLMM, while the conducted numerical experiments show that GLMM performs better that GDNM and better than FISTA and ADMM in Table 2, while usually worse than SSNAL. A partial explanation for this is that SSNAL is actually a *hybrid* algorithm, which combines the first-order augmented Lagrangian method to solve *dual* subproblems, which are strongly convex and of lower dimensions, with the subsequent applications of the second-order semismooth Newton method. Such a combination exhibits a high efficiency in solving Lasso problems in the singular case.

Table 1: Solving (6.1) on random instances with $b = 0$

Problem size and TN			iter				CPU time			
TN	m	n	SSNAL	FISTA	GLMM	GDNM	SSNAL	FISTA	GLMM	GDNM
1	400	800	22	31522	85	Error	0.35	69.15	1.43	Error
2	4000	8000	24	33455	202	Error	39.11	10000	695.55	Error
3	2000	2000	23	12450	115	91	35.04	337.42	22.00	19.59
4	4000	4000	29	17291	140	127	405.78	1734.33	98.53	104.65
5*	2000	2000	13	201	85	88	6.65	5.67	13.62	17.10
6*	4000	4000	13	206	113	484	9.77	22.19	78.49	401.34
7	800	400	3	258	38	12	0.58	0.40	0.19	0.09
8	8000	4000	3	331	87	7	6.60	66.90	63.33	7.39
9*	800	400	9	60	38	11	0.21	0.15	0.18	0.08
10*	8000	4000	9	58	91	11	3.62	13.83	67.20	10.48

Table 2: Solving (6.1) on random instances

Problem size and TN			iter					CPU time				
TN	m	n	SSNAL	FISTA	ADMM	GLMM	GDNM	SSNAL	FISTA	ADMM	GLMM	GDNM
1	400	800	25	37742	22873	1813	Error	0.45	145.52	10.89	45.62	Error
2	4000	8000	153	19173	19173	2499	Error	847.87	10000	2359.36	10000	Error
3	2000	2000	43	239701	12785	59	12	78.38	8138.94	158.12	11.07	2.24
4	4000	4000	246	73374	5970	59	218	1253.45	10000	320.81	48.16	178.91
5*	2000	2000	22	3619	90501	394	292	18.11	123.38	1141.64	65.60	58.80
6*	4000	4000	24	3629	103868	520	555	231.40	462.53	5166.16	369.27	474.74
7	800	400	4	430	10	6	3	0.14	0.86	0.02	0.11	0.08
8	8000	4000	13	487	11	7	3	18.80	117.92	3.67	8.46	4.39
9*	800	400	11	245	426	31	7	0.18	0.53	0.12	0.23	0.11
10*	8000	4000	11	238	411	72	9	8.37	59.18	32.17	56.37	8.88

7 Conclusions and Further Research

In this paper we propose and develop two globally convergent generalized Newton methods to solve problems of $\mathcal{C}^{1,1}$ optimization and of convex composite optimization with extended-real-valued regularizers, which include nonsmooth problems of constrained optimization. The developed algorithms are far-going extensions of the classical damped Newton method and of the Levenberg-Marquardt algorithm with the replacement of the standard Hessian by its generalized version applied to nonsmooth (of the second order) functions. The later construction is coderivative generated, which gives the names of our generalized Newton methods. The obtained results demonstrate the efficiently of both algorithms, their global superlinear convergence under appropriate assumptions, and their applications to the solution of Lasso problems with conducting numerical experiments.

Our future research includes developing hybrid generalized Newton methods, which contain subproblems that can be efficiently solved by using first-order algorithms, and then combining them with the advanced second-order Newton-type techniques. In this way, we intend to establish the global superlinear convergence of iterates under relaxed assumptions that do not involve the positive-definiteness of the generalized Hessian in $\mathcal{C}^{1,1}$ optimization and variations of strong convexity for problems of convex composite optimization, which will go beyond those with quadratic smooth parts. The obtained results would allow us to develop new applications to Lasso problems as well as to other important classes of models in machine learning, statistic, and related disciplines.

Appendix: Some Technical Lemmas

This section contains four technical lemmas used in the text. The first lemma establishes a relationship between constants in smooth function estimates.

Lemma 7.1 (estimates for smooth functions). *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuous differentiable function on an open set containing the interval $[x, y]$ such that the gradient of φ is Lipschitz continuous on this interval with a constant $\ell > 0$. Suppose that there are constants $c_1 \in \mathbb{R}$ and $c_2 > 0$ for which*

$$\varphi(y) > \varphi(x) + c_1\langle\nabla\varphi(x), y - x\rangle \quad \text{and} \quad \langle\nabla\varphi(x), y - x\rangle \le -c_2\|y - x\|^2. \tag{7.1}$$

Then we have the relationship $c_1 > 1 - \ell/(2c_2)$.

Proof. It follows from [37, Lemma A.11] that

$$\varphi(y) \le \varphi(x) + \langle\nabla\varphi(x), y - x\rangle + \frac{\ell}{2}\|y - x\|^2. \tag{7.2}$$

Combining (7.2) with both estimate in (7.1) gives us

$$\left(1 - \frac{\ell}{2c_2}\right)\langle\nabla\varphi(x), y - x\rangle \ge \frac{\ell}{2}\|y - x\|^2 + \langle\nabla\varphi(x), y - x\rangle \ge \varphi(y) - \varphi(x) > c_1\langle\nabla\varphi(x), y - x\rangle,$$

which implies that $c_1 > 1 - \ell/(2c_2)$ as claimed. □

The next lemma provides conditions for the R-linear and Q-linear convergence of sequences.

Lemma 7.2 (estimates for convergence rates). *Let $\{\alpha_k\}, \{\beta_k\}$, and $\{\gamma_k\}$ be sequences of positive numbers. Assume that there exist positive numbers $c_i > 0$, $i = 1, 2, 3$, and $k_0 \in \mathbb{N}$ such that for all $k \ge k_0$ we have the estimates:*

(i) $\alpha_k - \alpha_{k+1} \ge c_1\beta_k^2$.

(ii) $\beta_k \ge c_2\gamma_k$.

(iii) $c_3\gamma_k^2 \ge \alpha_k$.

Then the sequence $\{\alpha_k\}$ Q-linearly converges to zero, and the sequences $\{\beta_k\}$ and $\{\gamma_k\}$ R-linearly converge to zero as $k \to \infty$.

Proof. Combining (i), (ii), and (iii) yields the inequalities

$$\alpha_k - \alpha_{k+1} \ge c_1\beta_k^2 \ge c_1 c_2^2 \gamma_k^2 \ge \frac{c_1 c_2^2}{c_3}\alpha_k \quad \text{for all} \;\; k \ge k_0,$$

which imply that $\alpha_{k+1} \le q\alpha_k$, where $q := 1 - (c_1c_2^2)/c_3 \in (0,1)$. This verifies that the sequence $\{\alpha_k\}$ Q-linearly converges to zero. Using the latter and the assumed condition (i) ensures that

$$\beta_k^2 \le \frac{1}{c_1}(\alpha_k - \alpha_{k+1}) \le \frac{1}{c_1}\alpha_k \le \frac{1}{c_1}q\alpha_{k-1} \le \ldots \le \frac{1}{c_1}q^k\alpha_0 \quad \text{for all} \;\; k \ge k_0,$$

which tells us that $\beta_k \le c\mu^k$, where $c := \sqrt{\alpha_0/c_1}$ and $\mu := \sqrt{q}$. This justifies the R-linear convergence of the sequence $\{\beta_k\}$ to zero. Furthermore, it easily follows from (ii) that the sequence $\{\gamma_k\}$ R-linearly converge to zero, and thus we are done with the proof. $\square$

Now we obtain a useful result on the semismooth* property of compositions.

Lemma 7.3 (semismooth* property of composition mappings). *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric nonsingular matrix, $b \in \mathbb{R}^n$, $\bar{x} \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and semismooth* at $\bar{y} := A\bar{x}+b$. Then the mapping $g : \mathbb{R}^n \to \mathbb{R}^n$ defined by $g(x) := f(Ax+b)$ is semismooth* at $\bar{x}$.*

Proof. Using the coderivative chain rule from [50, Theorem 1.66], we get

$$D^*g(x)(w) = A^*D^*f(Ax+b)(w) \quad \text{for all} \;\; w \in \mathbb{R}^n. \tag{7.3}$$

Denote $\mu := \sqrt{\max\{1, \|A\|^2\} \cdot \max\{1, \|A^{-1}\|^2\}} > 0$. Picking any $\varepsilon > 0$ and employing the semismooth* property of f at $\bar{y}$, we find $\delta > 0$ such that

$$|\langle x^*, y - \bar{y}\rangle - \langle y^*, f(y) - f(\bar{y})\rangle| \le \frac{\varepsilon}{\mu}\big\|(y - \bar{y}, f(y) - f(\bar{y}))\big\| \cdot \|(x^*, y^*)\| \tag{7.4}$$

for all $y \in \mathbb{B}_\delta(\bar{y})$ and all $(x^*, y^*) \in \operatorname{gph} D^*f(y)$. Denoting $r := \delta/\|A\| > 0$ gives us $y := Ax + b \in \mathbb{B}_\delta(\bar{y})$ whenever $x \in \mathbb{B}_r(\bar{x})$. Picking now $x \in \mathbb{B}_r(\bar{x})$ and $(z^*, w^*) \in \operatorname{gph} D^*g(x)$, we get $(A^{-1}z^*, w^*) \in \operatorname{gph} D^*f(Ax+b)$ due to (7.3). It follows from (7.4) that

$$\begin{aligned}
|\langle z^*, x - \bar{x}\rangle - \langle w^*, g(x) - g(\bar{x})\rangle| \;&=\; |\langle A^{-1}z^*, y - \bar{y}\rangle - \langle w^*, f(y) - f(\bar{y})\rangle| \\
&\le\; \frac{\varepsilon}{\mu}\|(y - \bar{y}, f(y) - f(\bar{y}))\| \cdot \big\|(A^{-1}z^*, w^*)\big\| \\
&=\; \frac{\varepsilon}{\mu}\big\|(Ax - A\bar{x}, g(x) - g(\bar{x}))\big\| \cdot \big\|(A^{-1}z^*, w^*)\big\| \\
&\le\; \varepsilon\|(x - \bar{x}, g(x) - g(\bar{x}))\| \cdot \|(z^*, w^*)\|,
\end{aligned}$$

which verifies the semismooth* property of g at $\bar{x}$. $\square$

The final lemma establishes tilt stability of strongly convex functions at stationary points.

Lemma 7.4 (strong convexity and tilt-stability). *Let $\varphi : \mathbb{R}^n \to \overline{\mathbb{R}}$ be an l.s.c. and strongly convex function with modulus $\kappa > 0$, and let $\bar{x} \in \operatorname{dom}\varphi$ such that $0 \in \partial\varphi(\bar{x})$. Then $\bar{x}$ is a tilt-stable local minimizer of φ with modulus κ^{-1}.*

Proof. By the second-order characterization of strongly convex functions [8, Theorem 5.1], we have

$$\langle z, w\rangle \ge \kappa\|w\|^2 \quad \text{for all} \;\; z \in \partial^2\varphi(x,y)(w), \;\; (x,y) \in \operatorname{gph}\partial\varphi, \;\; \text{and} \;\; w \in \mathbb{R}^n.$$

This implies in turn that $\bar{x}$ is a tilt-stable local minimizer with modulus κ^{-1} by second-order characterization of tilt stability taken from [52, Theorem 3.5]. $\square$

References

[1] Bauschke H.H, Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces, 2nd edition. Springer, New York (2017)

[2] Beck, A.: Introduction to Nonlinear Optimization: Theory, Algorithms, and Applications with MATLAB. SIAM, Philadelphia, PA (2014)

[3] Beck, A., Teboulle, M.: (2009) A fast iterative shrinkage-thresholding algorithm for linear inverse problems. SIAM J. Imaging Sci. 2, 183–202 (2009)

[4] Becker, S., Fadili, M.J.: A quasi-Newton proximal splitting method. Adv. Neural Inform. Process. Syst. 25, 2618–2626 (2012)

[5] Bonnans, J.F.: Local analysis of Newton-type methods for variational inequalities and nonlinear programming. Appl. Math. Optim. 29, 161–186 (1994)

[6] Boyd, S., Parikh, N,, Chu, E., Peleato, B., Eckstein, J.: Distributed optimization and statistical learning via the alternating direction method of multipliers. Found. Trends Mach. Learning, 3, 1–122 (2010)

[7] Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press, Cambridge, UK (2004)

[8] Chieu, N.H., Chuong, T.D., Yao, J.-C., Yen, N.D.: Characterizing convexity of a function by its Fréchet and limiting second-order subdifferentials. Set-Valued Var. Anal. 19, 75–96 (2011)

[9] Chieu, N.H., Lee, G.M., Yen, N.D.: Second-order subdifferentials and optimality conditions for $\mathcal{C}^1$-smooth optimization problems. Appl. Anal. Optim. 1, 461–476 (2017)

[10] Chieu, N.M., Hien, L.V., Nghia, T.T.A.: Characterization of tilt stability via subgradient graphical derivative with applications to nonlinear programming. SIAM J. Optim. 28, 2246–2273 (2018)

[11] Colombo, G., Henrion, R., Hoang, N.D., Mordukhovich, B.S.: Optimal control of the sweeping process over polyhedral controlled sets. J. Diff. Eqs. 260, 3397–3447 (2016)

[12] Combettes, P.L., Pesquet, J.-C.: Proximal splitting methods in signal processing. In: Bauschke, H.H. et al. (eds) Fixed-Point Algorithms for Inverse Problems in Science and Engineering, pp. 185–212. Springer, New York (2011)

[13] Dan, H., Yamashita, N., Fukushima, M.: Convergence properties of the inexact Levenberg-Marquardt method under local error bound conditions. Optim. Meth. Softw. 17, 605–626 (2002)

[14] Dennis, J.E., Moré, J.J.: Quasi-Newton methods, motivation and theory. SIAM Rev. 19, 46–89 (1977)

[15] Ding, C., Sun, D., Ye, J.J.: First-order optimality conditions for mathematical programs with semidefinite cone complementarity constraints. Math. Program. 147, 539–379 (2014)

[16] Dias, S., Smirnov, G.: On the Newton method for set-valued maps. Nonlinear Anal. TMA, 75, 1219–1230 (2012)

[17] Dontchev, A.L., Rockafellar, R.T.: Characterizations of strong regularity for variational inequalities over polyhedral convex sets. SIAM J. Optim. 6, 1087–1105 (1996)

[18] Dontchev, A.L., Rockafellar, R.T.: Implicit Functions and Solution Mappings: A View from Variational Analysis, 2nd edition. Springer, New York (2014)

[19] Drusvyatskiy, D., Lewis, A.S.: Tilt stability, uniform quadratic growth, and strong metric regularity of the subdifferential. SIAM J. Optim. 23, 256–267 (2013)

[20] Drusvyatskiy, D., Mordukhovich, B.S., Nghia, T.T.A.: Second-order growth, tilt stability, and metric regularity of the subdifferential. J. Convex Anal. 21, 1165–1192 (2014)

[21] Efron, B., Hastie, T., Johnstone, I., Tibshirani, R.: Least angle regression. Ann. Statist. 32, 407–499 (2004)

[22] Facchinei, F., Pang, J.-C.: Finite-Dimensional Variational Inequalities and Complementarity Problems, Vol. II. Springer, New York (2003)

[23] Friedlander, M.P., Goodwin, A., Hoheisel, T: From perspective maps to epigraphical projections. arXiv preprint arXiv:2102.06809 (2021)

[24] Gabay, D., Mercier, B.: A dual algorithm for the solution of nonlinear variational problems via finite element approximations. Comput. Math. Appl. 2, 17–40 (1976)

[25] Glowinski, R., Marroco, A.: Sur l'approximation, par éléments finis d'ordre un, et la résolution, par pénalisation-dualité, d'une classe de problémes de Dirichlet non linéares. Revue Francaise d'Automatique, Informatique et Recherche Operationelle 9, 41–76 (1975)

[26] Gfrerer, H.: On directional metric regularity, subregularity and optimality conditions for nonsmooth mathematical programs. Set-Valued Var. Anal. 21, 151–176 (2013)

[27] Gfrerer, H., Mordukhovich, B.S.: Complete characterization of tilt stability in nonlinear programming under weakest qualification conditions. SIAM J. Optim. 25, 2081–2119 (2015)

[28] Gfrerer, H., Outrata, J.V.: On a semismooth* Newton method for solving generalized equations. SIAM J. Optim. 31, 489–517 (2021)

[29] Ginchev, I., Mordukhovich, B.S.: On directionally dependent subdifferentials. C. R. Acad. Bulg. Sci. 64, 497–508 (2011)

[30] Hang, N.T.V,, Mordukhovich, B.S., Sarabi, M.E.: Augmented Lagrangian method for second-order conic programs under second-order sufficiency. J. Global Optim, to appear. arXiv:2005.04182 (2021)

[31] Henrion, R., Mordukhovich, B.S., Nam, N.M.: Second-order analysis of polyhedral systems in finite and infinite dimensions with applications to robust stability of variational inequalities. SIAM J. Optim. 20, 2199–2227 (2010)

[32] Henrion, R., Römisch, W.: On M-stationary points for a stochastic equilibrium problem under equilibrium constraints in electricity spot market modeling. Appl. Math. 52, 473–494 (2007)

[33] Hestenes, M.R.: Multiplier and gradient methods. J. Optim. Theory Appl. 4, 303–320 (1969)

[34] Hintermüller, M., Ito, K., Kunisch, K.: The primal-dual active set strategy as a semismooth Newton method, *SIAM J. Optim.* 13, 865–888 (2002)

[35] Hoheisel, T., Kanzow, C., Mordukhovich, B.S., Phan, H.M.: Generalized Newton's methods for nonsmooth equations based on graphical derivatives, Nonlinear Anal. TMA 75, 1324–1340 (2012); Erratum in Nonlinear Anal. TMA 86, 157–158 (2013)

[36] Hsieh, C.J., Chang, K.W., Lin, C.J.: A dual coordinate descent method for large-scale linear SVM. Proceedings 25th International Conference on Machine Learning, pp. 408–415. Helsinki, Finland (2008)

[37] Izmailov, A.F., Solodov, M.V.: Newton-Type Methods for Optimization and Variational Problems. Springer, New York (2014)

[38] Khanh, P.D., Mordukhovich, B.S., Phat, V.T.: A generalized Newton method for subgradient systems. arXiv:2009.10551 (2020)

[39] Klatte, D., Kummer, B.: Nonsmooth Equations in Optimization. Regularity, Calculus, Methods and Applications. Kluwer Academic Publishers, Dordrecht, The Netherlands (2002)

[40] Lee, J.D., Sun, Y., Saunders, M.A.: Proximal Newton-type methods for minimizing composite functions. SIAM J. Optim. 24, 1420–1443 (2014)

[41] Li, D. H., Fukushima, M., Qi, L., Yamashita, N.: Regularized Newton methods for convex minimization problems with singular solutions. Comput. Optim. Appl. 28, 131–147 (2004)

[42] Li, X., Sun, D., Toh, K.-C.: A highly efficient semismooth Newton augmented Lagrangian method for solving Lasso problems. SIAM J. Optim. 28, 433–458 (2018)

[43] Lichman, M.: UCI Machine Learning Repository. University of California, School of Information and Computer Science, Irvine, CA (2013)

[44] Lions, P.-L., Mercier, B.: Splitting algorithms for the sum of two nonlinear operators. SIAM J. Numer. Anal. 16, 964–979 (1979)

[45] Mohammadi, A., Mordukhovich, B.S., Sarabi, M.E.: Variational analysis of composite models with applications to continuous optimization. Math. Oper. Res., to appear. DOI: 10.1287/moor.2020.1074 (2020)

[46] Mohammadi, A., Mordukhovich, B.S., Sarabi, M.E.: Parabolic regularity in geometric variational analysis. Trans. Amer. Math. Soc. 374, 1711–1763 (2021)

[47] Mohammadi, A., Sarabi, M.E.: Twice epi-differentiability of extended-real-valued functions with applications in composite optimization. SIAM J. Optim. 30, 2379–2409 (2020)

[48] Mordukhovich, B.S.: Sensitivity analysis in nonsmooth optimization. In: Field, D.A., Komkov, V.(eds) Theoretical Aspects of Industrial Design, pp. 32–46. SIAM Proc. Appl. Math. 58. Philadelphia, PA (1992)

[49] Mordukhovich, B.S.: Complete characterizations of openness, metric regularity, and Lipschitzian properties of multifunctions. Trans. Amer. Math. Soc. 340, 1–35 (1993)

[50] Mordukhovich, B.S.: Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications. Springer, Berlin (2006)

[51] Mordukhovich, B.S.: Variational Analysis and Applications. Springer, Cham, Switzerland (2018)

[52] Mordukhovich, B.S., Nghia, T.T.A.: Second-order characterizations of tilt stability with applications to nonlinear programming. Math. Program. 149, 83–104 (2015)

[53] Mordukhovich, B.S., Nghia, T.T.A.: Local monotonicity and full stability of parametric variational systems. SIAM J. Optim. 26, 1032–1059 (2016)

[54] Mordukhovich, B.S., Outrata, J.V.: On second-order subdifferentials and their applications. SIAM J. Optim. 12, 139–169 (2001)

[55] Mordukhovich, B.S., Rockafellar, R.T.: Second-order subdifferential calculus with applications to tilt stability in optimization. SIAM J. Optim. 22, 953–986 (2012)

[56] Mordukhovich, B.S., Sarabi, M.E.: Generalized Newton algorithms for tilt-stable minimizers in nonsmooth optimization. SIAM J. Optim. 31, 1184–1214 (2021)

[57] Mordukhovich, B.S., Yuan, X., Zheng, S., Zhang. J.: A globally convergent proximal Newton-type method in nonsmooth convex optimization. arXiv:2011.08166 (2020)

[58] Nesterov, Yu.: Lectures on Convex Optimization, 2nd edition. Springer, Cham, Switzerland (2018)

[59] Nocedal, J., Wright, S.: Numerical Optimization. Springer, New York (2006)

[60] Outrata, J.V., Sun, D.: On the coderivative of the projection operator onto the second-order cone. Set-Valued Anal. 16, 999–1014 (2008)

[61] Patrinos, P., Bemporad, A.: Proximal Newton methods for convex composite optimization. In: IEEE Conference on Decision and Control, 2358–2363 (2013)

[62] Pelckmans, K., De Brabanter, J., De Moor, B., Suykens, J.A.K.: Convex clustering shrinkage. In: PASCAL Workshop on Statistics and Optimization of Clustering, pp. 1–6. London, UK (2005)

[63] Poliquin, R.A., Rockafellar, R.T.: Tilt stability of a local minimum. SIAM J. Optim. 8, 287–299 (1998)

[64] Polyak, B.T.: Introduction to Optimization. Optimization Software, New York (1987)

[65] Powell, M.J.D.: A method for nonlinear constraints in minimization problems. In: Fletcher, R. (ed) Optimization, pp. 283–298. Academic Press, New York (1969)

[66] Qi, L., Sun, J.: A nonsmooth version of Newton's method. Math. Program. 58, 353–367 (1993)

[67] Robinson, S.M.: Newton's method for a class of nonsmooth functions. Set-Valued Anal. 2, 291–305 (1994)

[68] Rockafellar, R.T.: Augmented Lagrangian multiplier functions and duality in nonconvex programming. SIAM J. Control 12, 268–285 (1974)

[69] Rockafellar, R.T.: Augmented Lagrangians and hidden convexity in sufficient conditions for local optimality. Math. Program., to appear. http://sites.math.washington.edu/∼rtr/papers/rtr256-HiddenConvexity.pdf (2021)

[70] Rockafellar, R.T., Wets R.J-B.: Variational Analysis. Springer, Berlin (1998)

[71] She, Y.: Sparse regression with exact clustering. Elect. J. Stat. 4, 1055–1096 (2010)

[72] Stella, L., Themelis, A., Patrinos, P.: Forward–backward quasi-Newton methods for nonsmooth optimization problems. Comput. Optim. Appl. 67, 443–487 (2017)

[73] Stella, L., Themelis, A., Patrinos, P.: Forward-backward envelope for the sum of two nonconvex functions: further properties and nonmonotone linesearch algorithms. SIAM J. Optim. 28, 2274–2303 (2018)

[74] Tibshirani, R.: Regression shrinkage and selection via the Lasso. J. R. Stat. Soc. 58, 267–288 (1996)

[75] Ulbrich, M.: Semismooth Newton Methods for Variational Inequalities and Constrained Optimization Problems in Function Spaces. SIAM, Philadelphia, PA (2011)

[76] Yamashita, N., Fukushima, M.: On the rate of convergence of the Levenberg-Marquardt method. In: G. Alefeld, X. Chen (eds.) Topics in Numerical Analysis. 15, 239–249. Springer Vienna, Vienna (2001)

[77] Yao, J.-C., Yen, N.D.: Coderivative calculation related to a parametric affine variational inequality. Part 1: Basic calculation. Acta Math. Vietnam. 34, 157–172 (2009)