Universal dynamical approximation by Oberbeck-Boussinesque model

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1. PLAN

1. Oberbeck-Boussinesque model
2. Introduction. General approach.
3. Main Theorem
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2. Oberbeck- Boussinesque eqs

We consider the Oberbeck- Boussinesq approximation of the Navier-Stokes equations:

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nu \Delta \mathbf{v} - \nabla p + \kappa \mathbf{e}(1 + \gamma g)(u - u_0), \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0, \tag{2}$$

$$u_t + (\mathbf{v} \cdot \nabla) u = \Delta u + \eta, \tag{3}$$

where $\mathbf{v} = (v_1(x, y, t), v_2(x, y, t))^{tr}$, $u = u(x, y, t), p = p(x, y, t)$ are unknown functions defined on $\Omega \times \{t \geq 0\}$, the domain Ω is a rectangle, $\Omega = [0, \pi] \times [0, h] \subset \mathbf{R}^2$. Here $\mathbf{v}$ is the fluid velocity, where v_1 and v_2 are the normal and tangent velocity components, ν is the viscosity coefficient, p is the pressure, u is the temperature, $\eta(x, y)$ is a function describing a distributed heat source, $\mathbf{v} \cdot \nabla$ denotes the advection operator $v_1 \frac{\partial}{\partial x} + v_2 \frac{\partial}{\partial y}$.

3. OB model

The unit vector **e** is directed along the vertical y-axis: $\mathbf{e} = (0, 1)^{tr}$, κ is the coefficient of thermal expansion and a constant u_0 is the reference temperature. The term $\gamma g(x, y)$ is a space heterogeneous perturbation of the gravitational force, where $\gamma > 0$ is a small parameter, and $g(x, y)$ is a smooth function.

Picture of boundary conditions

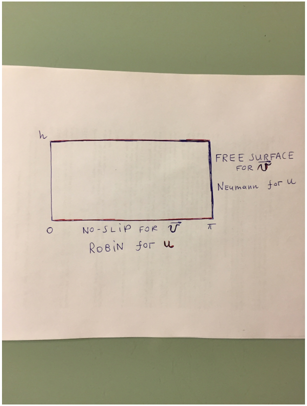

Figure: Boundary conditions and domain

4. Boundary conditions for fluid velocity

For the fluid velocity we set conditions of the free surface at the vertical boundaries $x = 0, \pi$:

$$v_1(x, y, t)\big|_{x=0,\pi} = 0, \quad \frac{\partial v_2(x, y, t)}{\partial x}\big|_{x=0,\pi} = 0 \tag{4}$$

and the no-slip condition at $y = 0, y = h$:

$$\mathbf{v}(x, y, t)\big|_{y=0,h} = \mathbf{0}. \tag{5}$$

5. Boundary conditions for temperature

For u

$$u_x(x, y, t)\big|_{x=0,\pi} = 0, \tag{6}$$

and

$$u_y(x, y, t)\big|_{y=0} = \beta u(x, 0, t), \quad u_y(x, y, t)\big|_{y=h} = \beta_1 u(x, h, t). \tag{7}$$

We complement (1)-(7) by initial conditions. So, we have a IBVP problem, which involves certain parameters.

6. IBVP parameters

Parameters $\mathcal{P}$ of our problem are coefficients $\nu, \beta, \beta_1, \gamma > 0, \kappa$ and functions $g(x, y), \eta(x, y)$.

7. Our Aim: Universal Dynamical Approximation

Our aim is to show that the family of semiflows $S^t_{\mathcal{P}}$ induced by our IBVP enjoys the property of Universal Dynamical Approximation (UDA).

This means that in a sense these semiflows can simulate any finite dimensional dynamics within any prescribed accuracy ϵ (in C^1-norm).

This property implies that semiflows $S^t_{\mathcal{P}}$ can generate all structurally stable dynamics (up to orbital topological equivalency).

Picture on structural stability

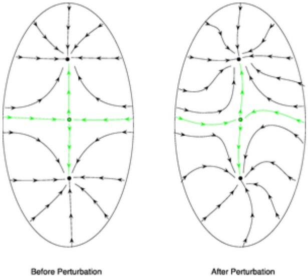

Figure: Structural stability

8. Universal Dynamical Approximation

Let us formulate UDA in more precise terms and further let us describe our plan to prove UDA. We will use some known results.

Remark: from the theory of dynamical system we use nothing but the definition of structural stability and basic theorems on existence of invariant manifolds. UDA property implies existence of hyperbolic chaos in OB.

Remark: The terminology UDA is proposed by Terence Tao.

9. Key Idea: Family of semiflows

Main idea is to consider families of semiflows instead of individual semiflows.

Consider an evolution equation in a Banach space **B** depending on parameter $\mathcal{P}$:

$$u_t = \mathbf{A}u + F(u, \mathcal{P}). \tag{8}$$

Assume that for some $\mathcal{P}$ that equation generates a global semiflow S^t. We obtain then a family $\mathcal{F}$ of global semiflows $S^t_{\mathcal{P}}$, where each semiflow depends on the parameter $\mathcal{P}$.

10. Special useful terminology: realisation of vector fields (RVF)

Suppose that for an integer $n > 0$ there is an appropriate value $\mathcal{P}_n$ of the parameter $\mathcal{P}$ such that the corresponding global semiflow $S^t_{\mathcal{P}_n}$ has an n- dimensional normally hyperbolic locally invariant manifold $\mathcal{M}_n$ embedded in our phase space $\mathbf{B}$ by a C^1-smooth map defined on a ball $B^n \subset \mathbb{R}^n$.

The restriction of semiflow $S^t_{\mathcal{P}_n}$ to $\mathcal{M}_n$ is defined then by a vector field Q on $\mathcal{M}^n$. Then we say that the family $S^t_{\mathcal{P}}$ realizes the vector field Q.

(That terminology is coined by Peter Polacik (USA) in 1991-1995. He also applied it for quasilinear parabolic PDE's).

11. Reduction to invariant manifold: picture

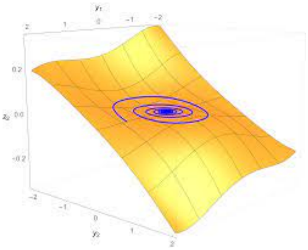

Figure: Realisation of a two dimensional field

12. Universal Dynamical Approximation (UDA)

The family $S^t_{\mathcal{P}}$ enjoys UDA if for each dimension n that family realizes a dense (in the norm $C^1(\mathbb{B}^n)$) set of vector fields Q on the ball $\mathbb{B}^n$.

Corollary 1: Theorem on Persistence of Hyperbolic Sets implies that some semiflows $S^t_{\mathcal{P}}$ exhibit a chaotic large time behaviour.

13. Main Theorem

Theorem

The family of the semiflows defined by our IBVP has the UDA property.

14. How to prove UDA

The approach is based on the classical center manifold technique and an idea that any n-dimensional dynamics can bifurcate from an equilibrium with n- zero eigenvalues if the number of parameters $\mathcal{P}$ is large enough. Such approach was used for finite dimensional systems by L. Shilnikov. This approach works for reaction-diffusion equations (Polacik-Dancer, 1995), quasilinear parabolic equations of second order, reaction-diffusion systems and some systems of ODE's with quadratic nonlinearities.

Let us describe our plan in more details.

15. Plan in more details

1. We choose parameters $\mathcal{P}$ in an appropriate way to provide existence of global semiflows in an appropriate phase space;
2. we consider a weakly nonlinear situation:

$$\mathbf{v} = \gamma \tilde{\mathbf{v}}, \ u = U(y) + \gamma \tilde{u},$$

   where $\tilde{\mathbf{v}}, \tilde{u}$ are new unknowns and $\gamma > 0$ is small enough;
3. We investigate the operator $\mathbf{L}$, which gives us a linearization of the problem at $(0, U)$; we show that this operator enjoys the bifurcation property described above: for each N there is a parameter value $\mathcal{P}_N$ such that the kernel of $\mathbf{L}$ consists of N eigenfunctions ;
4. We show existence of a sufficiently regular locally invariant manifold of dimension N;
5. We show that in such a way we can realize all quadratic vector fields; this implies UDA due to known results.

16. What is non-trivial ?

In this plan, *only the point* **3** *is really hard*.
Indeed, the first step can be done by an elegant technique developed by R. Temam (R. Temam, *Infinite dimensional dynamical systems in mechanics and physics*, Springer, New York etc. 1988).
Existence of invariant manifolds can be shown by standard methods.

17. Lotka-Volterra system

The last point are based on known results (the first one was probably by M. D. Korzuchin: Ph.D. thesis, Moscow (1969)). For example, the generalised Lotka-Volterra system

$$\frac{dx_i}{dt} = x_i\big(- r_i + \sum_{j=1}^{N} K_{ij} x_j\big), \tag{9}$$

exhibits UDA (Kozlov-Vakulenko, Nonlinearity, 2013), where parameters are N, r_i, K_{ij}. One can prove UDA for some other classes of quadratic systems (Vakulenko, Weber, Grigoriev, Studies in Applied Math. 2015)

18. Main difficulty

So, the key step is **3**: to find operator **L** with a large number of zero modes (eigenfunctions of **L** with zero eigenvalues). Idea of the construction follows the classical works. First let us remember the classical investigation of Lord Rayleigh on so-called Rayleigh-Bénard instability.

19. Rayleigh temperature profile

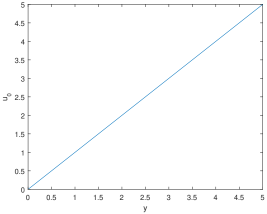

Figure: Classical Rayleigh-Bénard instability arises for a linear profile $U(y)$. The spectral problem for linearized operator $\mathbf{L}$ can be studied. Eigenfunctions of $\mathbf{L}$ have the form $\psi_{k,m} = \exp(ikx)\Psi_m(y)$, where ψ is the stream function (we can make the variable separation). Then one can show that there is a bifurcation at a single $k = k_c \neq 0$: $Re\ \lambda(k_c) = 0$.

20. Our main trick: a special temperature profile

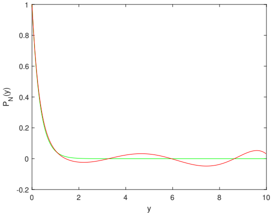

Figure: We choose the profile $U(y) = \exp(-by) + \mu P_N(y)$ as a sharply decreasing exponent $\exp(-by)$, where $b > 0$ is a large parameter, plus a small polynomial $P_N(y)$ of degree N. Then one can show that there is possible a bifurcation with zero modes $\psi(y)\exp(ik_j x)$ for a set of k_j, $j = 1, ..., N$.

21. Spectrum of linearized operator

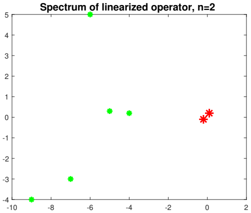

Figure: Location of spectrum for a weakly perturbed exponential temperature profile for $N = 2$. We have two eigenvalues close to origin (shown by red) and all remaining eigenvalues are in negative half plane far away from the imaginary axis (green). We can control location of red points and their number by a choice of polynomial $P_N(y)$.

22. Equation for k

For large $b > 0$ and bounded k we can construct an asymptotics of eigenfunctions, and find eq. for z. For an appropriate β this eq. takes the form:

$$(z+1)^2 = 4(1+az)^{-1} + Y_k(z), \tag{10}$$

where $z = \bar{k}(\lambda)/k$ is a new complex unknown, $\bar{k} = \sqrt{k^2+\lambda^2}$. Here Y_k is a small perturbation and $a = O(b^{-1})$.

22. Equation for k

The non-perturbed equation reduces to $(z+1)^2 = 4$ *and it does not involve the wave vector* k*!!*

It is a key point. The second point is that we can control the small perturbation Y_k by polynomial $P_N(y)$.

23. Some important details: parameter choice

Coefficient $\gamma > 0$ should be small to provide a weakly nonlinear regime at non-perturbed solution $\mathbf{v} = 0, u = U(y)$;
We use functions g, η to obtain a special temperature profile $U(y)$;
Coefficients β, β_1, ν control, together with $U(y)$, the spectrum of linear operator;
We reduce OB eqs. to a large system of diff. equations with quadratic nonlinearities;
Functions g, η help to control a matrix that defines that system of differential equations. So, we can realize all quadratic systems.
We have a finite set of conditions and functional parameters, which lie in an infinite dimensional space.

24. Other results

Consider a quasilinear parabolic equation of second order in bounded domains $D \subset \mathbb{R}^n$ with a smooth boundary. Results for quasilinear parabolic eqs. depends on dimension of Ω. If *dim* $\Omega = 1$, no chaos does not appear, only rest points or time periodic solutions (T. Zelenyak, after H. Matano, J. Mallet-Paret and others).

For *dim* $\Omega > 1$ the UDA property is shown by P. Poláčik, 1991-1995 (also K. P. Rybakovski, 1994). However, the invariant manifolds are unstable.

25. Reaction-diffusion systems (RDS)

A large class of RDS has UDA property, even with 2 reagents only.

$$u_t = d\Delta u + f(u, v) + \zeta, \tag{11}$$

$$v_t = D\Delta v + g(u, v) + \eta, \tag{12}$$

where $u = u(x, y, t)$ and $v = v(x, y, t)$ are unknown functions defined on $\Omega \times \{t \geq 0\}$, Ω is the strip $(-\infty, \infty) \times [0, 1] \subset \mathbf{R^2}$, $d, D > 0$ are diffusion coefficients, $\eta(x, y)$ and $\zeta(x, y)$ are smooth functions that can be interpreted as external sources independent of u, v. For semiflows defined by (12),(11) the following assertion is valid: either this semiflow $S^t(P)$ is monotone, or $S^t(P)$ can ϵ-realize all finite dimensional vector fields. Moreover, conditions to f, g have transparent chemical interpretation. They mean that the reagent v is neither an inhibitor nor an activator for u.

27. Publications

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30. Open Questions

1) Is it possible to prove UDA for reaction-diffusion systems with 2 components in 1D case ?

2) Euler equations in dimension 2 or 3. Or NS for viscous non-compressible fluid.

3) Kuramoto-Sivashinski equation.

31. Acknowledgements

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