On principal eigenvalues for time-periodic operators

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Talk Outline

1. $\mathcal{R}_0$
2. Frequency
3. Large drift
4. Small diffusion

SIS reaction-diffusion model

[Allen-Bolker-L.-Nevai[1]]

$$\begin{cases} S_t = d_S \Delta S - \beta(x)\dfrac{SI}{S+I} + \gamma(x)I, & x \in \Omega,\ t > 0, \\ I_t = d_I \Delta I + \beta(x)\dfrac{SI}{S+I} - \gamma(x)I, & x \in \Omega,\ t > 0, \\ \nabla S \cdot \mathbf{n} = \nabla I \cdot \mathbf{n} = 0 & x \in \partial\Omega,\ t > 0, \\ \int_\Omega [S(x,0) + I(x,0)]\mathrm{d}x = N. \end{cases} \tag{1}$$

- $S(x,t), I(x,t)$: density of susceptible and infected individuals
- d_S, d_I: diffusion rates of susceptible and infected
- $\beta(x), \gamma(x)$: disease transmission and recovery rates
- $N > 0$: total number of individuals at $t = 0$

[1] DCDS 21 (2008) 1-20

Basic reproduction number

- Let $0 < \lambda_1 < \lambda_2 < ...$ denote the eigenvalues of

$$\begin{cases} -d_I \Delta\varphi + \gamma(x)\varphi = \lambda\beta(x)\varphi \quad \text{in } \Omega, \\ \nabla\varphi \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

- Next generation approach (Diekmann-Heesterbeek; Van Den Driessche-Watmough; Wang-Zhao):

$$\mathcal{R}_0 = \frac{1}{\lambda_1}$$

Theorem 1

(i) *If* $\mathcal{R}_0 \leq 1$, *then the disease will die out;*

(ii) *If* $\mathcal{R}_0 > 1$, *the disease will persist.*

An unusual prediction

Lemma 2 (Allen et al. 2008 DCDS)

If β/γ is non-constant, then $\mathcal{R}_0$ is strictly decreasing in d_I.

- Variational characterization of $\mathcal{R}_0$:

$$\mathcal{R}_0 = \sup_{\varphi \in H^1(\Omega), \varphi \neq 0} \frac{\int_\Omega \beta \varphi^2}{d_I \int_\Omega |\nabla \varphi|^2 + \int_\Omega \gamma \varphi^2}$$

- If β/γ is a constant, then $\mathcal{R}_0 = \beta/\gamma$.

- Increasing d_I will increase access to medical resources?

Time periodic SIS model

[R. Peng & X.-Q. Zhao[2]]

$$\begin{cases} \partial_\tau \tilde{S} - d_S \Delta \tilde{S} = -\frac{\beta(x,\omega\tau)\tilde{S}\tilde{I}}{\tilde{S}+\tilde{I}} + \gamma(x,\omega\tau)\tilde{I} & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_\tau \tilde{I} - d_I \Delta \tilde{I} = \frac{\beta(x,\omega\tau)\tilde{S}\tilde{I}}{\tilde{S}+\tilde{I}} - \gamma(x,\omega\tau)\tilde{I} & \text{in } \Omega \times \mathbb{R}_+, \\ \nabla \tilde{S} \cdot \mathbf{n} = \nabla \tilde{I} \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times \mathbb{R}_+. \end{cases}$$

- Set $t = \omega\tau$, $S(x,t) = \tilde{S}(x, t/\omega)$, and $I(x,t) = \tilde{I}(x, t/\omega)$

[S. Liu & L. 2021]

$$\begin{cases} \omega\partial_t S - d_S \Delta S = -\frac{\beta(x,t)SI}{S+I} + \gamma(x,t)I & \text{in } \Omega \times \mathbb{R}_+, \\ \omega\partial_t I - d_I \Delta I = \frac{\beta(x,t)SI}{S+I} - \gamma(x,t)I & \text{in } \Omega \times \mathbb{R}_+, \\ \nabla S \cdot \mathbf{n} = \nabla I \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times \mathbb{R}_+. \end{cases} \tag{2}$$

[2] Nonlinearity 25 (2012) 1451-1471

Basic reproduction number

- Characterization of $\mathcal{R}_0 = \mathcal{R}_0(d_I, \omega)$:

$$\begin{cases} \omega\partial_t\varphi = d_I\Delta\varphi - \gamma\varphi + \frac{1}{\mathcal{R}_0}\beta\varphi & \text{in } \ \Omega \times [0,1], \\ \nabla\varphi \cdot \mathbf{n} = 0 & \text{on } \ \partial\Omega \times [0,1], \\ \varphi(x,0) = \varphi(x,1) & \text{in } \ \Omega. \end{cases}$$

- Peng & Zhao 2012: If $\mathcal{R}_0 < 1$, the unique disease free periodic solution (DFP) of (2) is globally stable; If $\mathcal{R}_0 > 1$, (2) admits at least one positive periodic solution.

- If β and γ are independent of x, $\mathcal{R}_0$ is independent of ω, d_I:

$$\mathcal{R}_0 = \frac{\int_0^1 \beta(t)\,dt}{\int_0^1 \gamma(t)\,dt}$$

Dependence of $\mathcal{R}_0$ on d_I

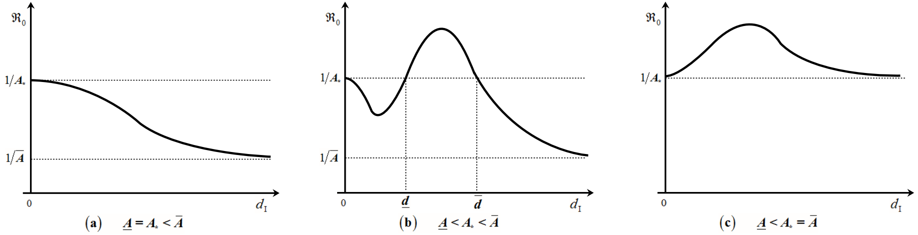

Figure: Graphs of $\mathcal{R}_0(d_I, \omega)$ as functions of d_I, with fixed ω.

- Including temporal variation in disease dynamics might be more realistic

A monotonicity result

[S. Liu, L, R. Peng & M. Zhou[3]]

$$\begin{cases} \omega\partial_t\varphi = d_I\Delta\varphi - \gamma\varphi + \lambda\varphi & \text{in } \Omega\times[0,1], \\ \nabla\varphi\cdot\mathbf{n} = 0 & \text{on } \partial\Omega\times[0,1], \\ \varphi(x,0) = \varphi(x,1) & \text{in } \Omega \end{cases} \tag{3}$$

Theorem 3

For fixed $d_I > 0$, the principal eigenvalue of (3) is increasing in ω.

- This answers a question of Hutson, Mischaikow, Polacik (JMB 2001)

[3] Proc. AMS, 47 (2019) 5291-5302

Monotonicity of $\mathcal{R}_0$

Theorem 4 (Liu & L 2021, to appear in JFA)

For each fixed d_I, $\mathcal{R}_0(\omega, d_I)$ is decreasing in ω. Furthermore, R_0 is either strictly increasing in ω or independent of ω.

- $\mathcal{R}_0$ is an increasing function of $T := 1/\omega$.
- In some scenarios, there is a critical period $T^* > 0$ such that $\mathcal{R}_0 < 1$ for $T < T^*$ and $\mathcal{R}_0 > 1$ for $T > T^*$.
- This might suggest why the outbreak of some diseases are less frequent than others, e.g. annually vs bi-annually

Level set of principal eigenvalue

- Principal eigenvalue $\lambda(d,\omega)$:

$$\begin{cases} \omega\partial_t\varphi - d\Delta\varphi + \gamma(x,t)\varphi = \lambda\beta(x,t)\varphi & \text{in } \ \Omega\times\mathbb{R} \\ \nabla\varphi\cdot\mathbf{n} = 0 & \text{on } \ \partial\Omega\times\mathbb{R} \\ \varphi(x,t) = \varphi(x,t+1) & \text{in } \ \Omega\times\mathbb{R} \end{cases} \tag{4}$$

- Three key quantities:

$$\begin{aligned} \overline{A} &:= \frac{\int_\Omega\int_0^1 \gamma \mathrm{d}x\mathrm{d}t}{\int_\Omega\int_0^1 \beta \mathrm{d}x\mathrm{d}t}, \\ A_* &:= \min_{x\in\bar{\Omega}} \frac{\int_0^1 \gamma \mathrm{d}t}{\int_0^1 \beta \mathrm{d}t}, \\ \underline{A} &:= \min_{x(\cdot)\in\mathcal{S}} \frac{\int_0^1 \gamma(x(t),t)\mathrm{d}t}{\int_0^1 \beta(x(t),t)\mathrm{d}t}, \end{aligned} \tag{5}$$

where $\mathcal{S} := \{x(\cdot)\in C(\mathbb{R};\bar{\Omega}) : \ x(t) = x(t+1)\}$; $\underline{A} \le A_* \le \overline{A}$

Asymptotics of principal eigenvalue

Theorem 5 (Liu & L 2021)

- *As* $d \to 0+$,

$$\lim_{\omega/\sqrt{d}\to 0} \lambda(d,\omega) = \underline{A}, \qquad \lim_{\omega/\sqrt{d}\to \infty} \lambda(d,\omega) = A_*$$

- *For* $\vartheta \in (0,\infty)$, $\lim_{d\to 0, \frac{\omega}{\sqrt{d}}\to\vartheta} \lambda(d,\omega) = A(\vartheta)$, *whereas*

$$\begin{cases} \vartheta\partial_t U + |\nabla U|^2 - \gamma = -A(\vartheta)\beta & \text{in } \Omega\times\mathbb{R}, \\ \nabla U\cdot\mathbf{n} = 0 & \text{on } \partial\Omega\times\mathbb{R}, \\ U(x,t) = U(x,t+1) & \text{in } \Omega\times\mathbb{R}. \end{cases} \tag{6}$$

Moreover, $A(\vartheta)$ *is continuous and non-decreasing in* ϑ, $A(\vartheta)\to\underline{A}$ *as* $\vartheta\to 0$, *and* $A(\vartheta)\to A_*$ *as* $\vartheta\to\infty$.

Level set of principal eigenvalue

- For each value $A \in (\underline{A}, \overline{A})$, define the level set

$$\Sigma_A := \{(d, \omega) : \omega, d > 0, \quad \lambda(d, \omega) = A\}.$$

- Σ_A has different topological structures for $A < A_*$ and $A > A_*$.

- One technical assumption:

(H) Set $c_* := \gamma - A_*\beta$. Assume $c_* \in C^{2,0}(\bar{\Omega} \times \mathbb{R})$ and there exist finite number of points $x_i \in \Omega$ $(1 \le i \le N)$ such that $\hat{c}_*(x_i) = 0$ and $\hat{c}_*(x) > 0$ for all $x \in \bar{\Omega} \setminus \{x_1, \ldots, x_N\}$, and each point x_i is a non-degenerate critical point.

Level set of principal eigenvalue

Theorem 6 (Liu & L, 2021)

If $\underline{A} = \overline{A}$, then $\lambda(d,\omega) \equiv \overline{A}$ for all $\omega, d > 0$; If $\underline{A} < \overline{A}$,

(i) *For $A \in (\underline{A}, A_*]$, there is $\omega_A(d) : (0, d_A] \to [0,\infty)$ s.t. $\lambda(d, \omega_A(d)) \equiv A$ in $(0, d_A]$, $\omega_A(d_A) = 0$, $\omega_A(d) > 0$ for $d \in (0, d_A)$, and if $A < \overline{A}$, $\omega_A(d)/\sqrt{d}$ is monotone decreasing.*

  (a) *for $A < A_*$, the limit $\lim_{d\to 0} \omega_A(d)/\sqrt{d}$ exists and is positive. In particular, $\omega_A(d) \to 0$ as $d \to 0$;*

  (b) *for $A = A^*$, if (H) holds, then $\omega_{A_*}(d) \to 0$ as $d \to 0$.*

(ii) *For $A \in (A_*, \overline{A})$, if $\gamma - \hat{\gamma} - A(\beta - \hat{\beta})$ is independent of x, then $\lambda(d_A, \omega) \equiv A$ for all $\omega > 0$; otherwise, there exist $\underline{d}_A \in (0, d_A)$ and continuous function $\omega_A(d) : (\underline{d}_A, d_A] \to [0,\infty)$ such that $\lambda(d, \omega_A(d)) \equiv A$ for $d \in (\underline{d}_A, d_A]$.*

  (a) *$\omega_A(d_A) = 0$, $\omega_A(d) > 0$ in $(0, d_A)$, $\omega_A(d) \to \infty$ as $d \searrow \underline{d}_A$;*

  (b) *$\omega_A(d)/\sqrt{d}$ is monotone decreasing in $(\underline{d}_A, d_A]$.*

Topology of level sets

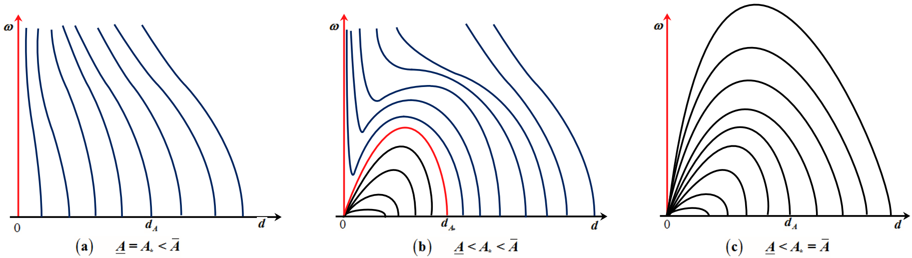

Figure: Illustrations of the level sets of principal eigenvalue $\lambda(d, \omega)$ in d-ω plane for three cases: $\underline{A} = A_* < \overline{A}$, $\underline{A} < A_* < \overline{A}$, and $\underline{A} < A_* = \overline{A}$

Dependence of $\mathcal{R}_0$ on d_I

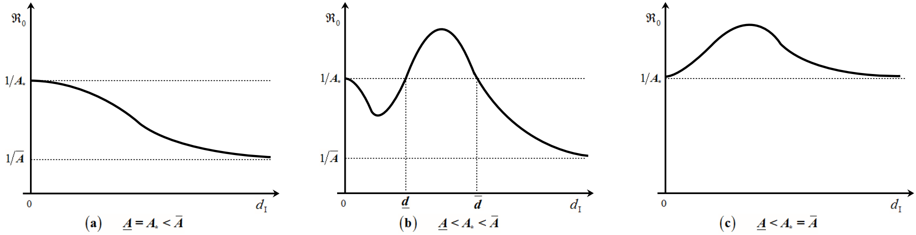

Figure: Graphs of $\mathcal{R}_0(d_I, \omega)$ as functions of d_I, with fixed ω. The generic case $\underline{A} < A_* < \overline{A}$ is shown in Fig. (b), where the graph of $\mathcal{R}_0$ intersects with the $1/A_*$-line at least twice for small frequency ω. For $\underline{A} < A_* = \overline{A}$, the graph of $\mathcal{R}_0$ is not monotonic either, as illustrated by Fig. (c).

- Including temporal variation in disease dynamics might be more realistic

Time-periodic parabolic operators

$$\begin{cases} \partial_t\varphi - D\Delta\varphi + \alpha\mathbf{v}\cdot\nabla\varphi + c\varphi = \lambda\varphi, & x\in\Omega, t\in[0,1], \\ b\varphi + (1-b)\nabla\varphi\cdot\mathbf{n} = 0 & x\in\partial\Omega, t\in[0,1], \\ \varphi(x,0) = \varphi(x,1), & x\in\Omega. \end{cases}$$

- $\mathbf{v}$: C^1 vector field, time-periodic with unit period
- c: continuous, time-periodic with unit period; $b \in [0,1]$
- Principal eigenvalue $\lambda = \lambda(D,\alpha)$: It is real and has the smallest real part among all eigenvalues (Hess, 1991)
- Q. How does principal eigenvalue depend on D, α?

Principal eigenvalue for elliptic operator

$$\begin{cases} -D\Delta\varphi + \alpha\mathbf{v}\cdot\nabla\varphi + c(x)\varphi = \lambda(\alpha)\varphi & \text{in } \Omega, \\ b\varphi + (1-b)\nabla\varphi\cdot\mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

- $\mathbf{v}(x)$: C^1 vector field; $c(x) \in C(\bar{\Omega})$; $b \in [0,1]$
- Principal eigenvalue $\lambda(D,\alpha)$: It is real, simple and has the smallest real part among all eigenvalues
- Q. How does principal eigenvalue depend on D, α?

Dirichlet boundary condition

$$\begin{cases} -\Delta\varphi + \alpha\mathbf{v}\cdot\nabla\varphi = \lambda(\alpha)\varphi \quad \text{in } \Omega, \\ \varphi = 0 \quad \text{on } \partial\Omega. \end{cases}$$

- Wentzell (1975):

$$\lim_{\alpha\to+\infty}\frac{\lambda(\alpha)}{\alpha^2} = \frac{1}{4}\lim_{T\to+\infty}\inf_X \frac{\int_0^T |X'(t)+\mathbf{v}(X(t))|^2\,dt}{T} < +\infty,$$

where $X \in C^1([0,T];\bar{\Omega})$.

- $\lambda(\alpha) = O(\alpha^2)$

Devinatz, Ellis, Friedman (1973-75)

- $\mathbf{v} \cdot \nabla\phi < 0$ in $\bar{\Omega}$ for some $\phi \in C^1(\bar{\Omega})$: $\exists c > 0$ s.t. for $\alpha \gg 1$,

$$\lambda(\alpha) \geq c\alpha^2;$$

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- $\mathbf{v} = -\nabla U(x)$ with $|\nabla U| > 0$ in $\bar{\Omega}$:

$$\lim_{\alpha\to\infty} \frac{\lambda(\alpha)}{\alpha^2} = \frac{1}{4}\min_{\bar{\Omega}} |\nabla U|^2;$$

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- $\mathbf{v}$ vanishes at some point in Ω: $\lambda(\alpha) = O(\alpha)$
- $\mathbf{v} \cdot \mathbf{n} < 0$ on $\partial\Omega$: $\lambda(\alpha) = O(\alpha e^{-c\alpha})$

Neumann boundary condition

$$\begin{cases} -\Delta\varphi + \alpha \mathbf{v} \cdot \nabla\varphi + c(x)\varphi = \lambda(\alpha)\varphi & \text{in } \Omega, \\ \nabla\varphi \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \\ \varphi > 0 & \text{in } \Omega. \end{cases}$$

- For any α,

$$\min_{\bar{\Omega}} c \le \lambda(\alpha) \le \max_{\bar{\Omega}} c.$$

- Question: $\lim_{\alpha\to\infty} \lambda(\alpha) = ?$

Divergent free drift

[Berestycki-Hamel-Nadirashvili, CMP 2005]

- Assume $div(\mathbf{v}) = 0$ in Ω and $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$, then

$$\lim_{\alpha \to \infty} \lambda(\alpha) = \inf_{\varphi \in \mathcal{I}} \frac{\int_\Omega (|\nabla \varphi|^2 + c(x)\varphi^2)}{\int_\Omega \varphi^2} < +\infty,$$

where

$$\mathcal{I} = \{\varphi \in H^1(\Omega) : \varphi \neq 0, \mathbf{v} \cdot \nabla \varphi = 0 \text{ a.e. in } \Omega\}$$

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  where

$$\mathcal{I} = \{\varphi \in H^1(\Omega) : \varphi \neq 0, \mathbf{v} \cdot \nabla \varphi = 0 \text{ a.e. in } \Omega\}$$

- If $\mathcal{I}$ consists of constants only,

$$\lim_{\alpha \to \infty} \lambda(\alpha) = \frac{1}{|\Omega|} \int_\Omega c$$

Potential flow

$$\begin{cases} -\Delta\varphi - \alpha\nabla m \cdot \nabla\varphi + c(x)\varphi = \lambda(\alpha)\varphi & \text{in } \Omega, \\ \nabla\varphi \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Potential flow

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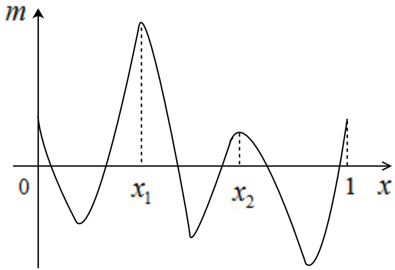
Theorem 1 (Chen-L, IUMJ 2008)

Suppose that $m \in C^2(\bar{\Omega})$ and all critical points of m are non-degenerate, $c \in C(\bar{\Omega})$. Then

$$\lim_{\alpha\to\infty}\lambda(\alpha) = \min_{x\in\mathcal{M}} c(x),$$

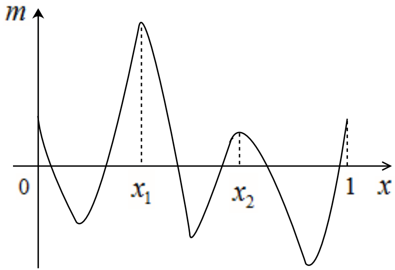
where $\mathcal{M}$ is the set of points of local maximum of m.

An illustration of Theorem 1



$$\boxed{\lim_{\alpha\to\infty} \lambda(\alpha) = \min\{c(0),\ c(x_1),\ c(x_2),\ c(1)\}.}$$

An illustration of Theorem 1



$$\boxed{\lim_{\alpha\to\infty} \lambda(\alpha) = \min\left\{c(0),\ c(x_1),\ c(x_2),\ c(1)\right\}.}$$

- Concentration of mass: As $\alpha \to \infty$, principal eigenfunction $\varphi(\alpha, \cdot)$ concentrates on some local maximum of m

Time-periodic parabolic operator

$$\begin{cases} \varphi_t - \varphi_{xx} - \alpha m_x(x,t)\varphi_x + c(x,t)\varphi = \lambda(\alpha)\varphi & \text{in } (0,1)\times[0,1], \\ \varphi_x(0,t) = \varphi_x(1,t) = 0 & \text{on } [0,1], \\ \varphi(x,t) = \varphi(x,t+1) & \text{on } (0,1). \end{cases}$$

- What is the limit of $\lambda(\alpha)$ when $\alpha \to \infty$?

Time-periodic parabolic operator

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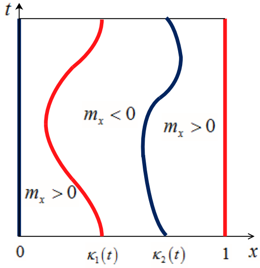
- What is the limit of $\lambda(\alpha)$ when $\alpha \to \infty$?
- The limit relies on spatially critical points of m:

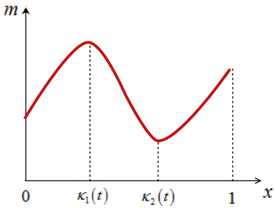
$$\{(x,t) \in [0,1]\times[0,1] : m_x(x,t) = 0\}$$

Example

Assumption on m: $\exists \kappa_1(t), \kappa_2(t) \in (0,1)$ such that

$$\begin{cases} m_x > 0, & x \in (0, \kappa_1(t)) \cup (\kappa_2(t), 1), \ t \in [0,1], \\ m_x = 0, & x \in \{\kappa_1(t), \kappa_2(t)\}, \ t \in [0,1], \\ m_x < 0, & x \in (\kappa_1(t), \kappa_2(t)), \ t \in [0,1]. \end{cases}$$





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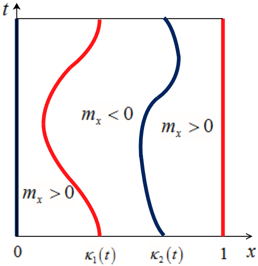

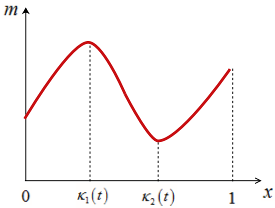

$$\boxed{\lim_{\alpha\to\infty} \lambda(\alpha) = \min\left\{\int_0^1 c(\kappa_1(s), s)\,\mathrm{d}s, \ \int_0^1 c(1, s)\,\mathrm{d}s\right\}.}$$

Asymptotic with large advection

Theorem 2 (Liu-L-Peng-Zhou, SIMA 2021)

Suppose that all spatially critical points of m are non-degenerate. Let $\{(\kappa_i(t), t) : t \in [0,1], 1 \le i \le N\}$ *be the set of points of local maximum of m. Then*

$$\lim_{\alpha\to\infty} \lambda(\alpha) = \min_{1\le i\le N} \left\{ \int_0^1 c\left(\kappa_i(s), s\right) \mathrm{d}s \right\}.$$

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Suppose that all spatially critical points of m are non-degenerate. Let $\{(\kappa_i(t), t) : t \in [0, 1], 1 \leq i \leq N\}$ *be the set of points of local maximum of m. Then*

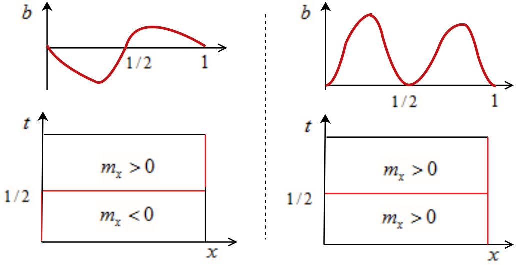
$$\lim_{\alpha\to\infty} \lambda(\alpha) = \min_{1\leq i\leq N} \left\{ \int_0^1 c\,(\kappa_i(s), s)\,\mathrm{d}s \right\}.$$

- Peng-Zhao, CVPDE 2015: If $m_x > 0$, (resp. $m_x < 0$), then

$$\lim_{\alpha\to\infty} \lambda(\alpha) = \int_0^1 c\,(1, s)\,\mathrm{d}s \quad \left(\textit{resp.} \lim_{\alpha\to\infty} \lambda(\alpha) = \int_0^1 c\,(0, s)\,\mathrm{d}s \right).$$

An example: $m(x,t) = b(t)x$

- Assumption on b:



- Left figure: $\lim_{\alpha\to\infty} \lambda(\alpha) = \int_0^{\frac{1}{2}} c(0,s)\mathrm{d}s + \int_{\frac{1}{2}}^{1} c(1,s)\mathrm{d}s$;
- Right figure: $\lim_{\alpha\to\infty} \lambda(\alpha) = \int_0^{1} c(1,s)\mathrm{d}s$.

Small diffusion limit

$$\begin{cases} -D\Delta\varphi + c(x)\varphi = \lambda(D)\varphi & \text{in } \Omega, \\ \nabla\varphi\cdot\mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

- Concentration of mass:

$$\lim_{D\to 0}\lambda(D) = \min_{x\in\bar{\Omega}} c(x).$$

- Cooperative elliptic system: Dancer (2009); Lam-L (2016)
- Time-periodic cooperative system: Shen et al. (2017, 2018, 2019); Liang-Zhang-Zhao (2017, 2019); Bai-He (2020)

Small diffusion

$$\begin{cases} -D\Delta\varphi - \nabla m \cdot \nabla\varphi + c(x)\varphi = \lambda(D)\varphi & \text{in } \Omega, \\ \nabla\varphi \cdot \mathbf{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Theorem 3 (Chen-L, IUMJ 2012)

Assume $|\nabla m| \neq 0$ *on* $\partial\Omega$*, all critical points of* m *are non-degenerate.*

$$\lim_{D\to 0} \lambda(D) = \inf_{x\in\Sigma_1\cup\Sigma_2} \left\{ c(x) + \frac{1}{2}\sum_{i=1}^{n} \Big(|\kappa_i(x)| + \kappa_i(x) \Big) \right\},$$

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where $\kappa_i(x)$ *are eigenvalues of* $D^2 m(x)$*, and*

$$\Sigma_1 = \{x \in \Omega : |\nabla m| = 0\},$$
$$\Sigma_2 = \{x \in \partial\Omega : |\nabla m| = \nabla m \cdot \mathbf{n} > 0\}$$

Time-periodic parabolic operators

$$\begin{cases} \varphi_t - D\varphi_{xx} - m_x(x,t)\varphi_x + c(x,t)\varphi = \lambda(D)\varphi & \text{in } (0,1)\times[0,1], \\ \varphi_x(0,t) = \varphi_x(1,t) = 0 & \text{on } [0,1], \\ \varphi(x,0) = \varphi(x,1) & \text{on } (0,1). \end{cases}$$

- Set $p_+(x,t) := \max\{p(x,t), 0\}$, $\hat{p}(x) = \int_0^1 p(x,t)\,dt$.
- Redefine $\hat{m}_{xx}(0)$ and $\hat{m}_{xx}(1)$ as

$$\hat{m}_{xx}(0) = \begin{cases} -\infty & \text{if } \hat{m}_x(0+) < 0, \\ +\infty & \text{if } \hat{m}_x(0+) > 0, \end{cases}$$

$$\hat{m}_{xx}(1) = \begin{cases} +\infty & \text{if } \hat{m}_x(1-) < 0, \\ -\infty & \text{if } \hat{m}_x(1-) > 0. \end{cases}$$

Small diffusion

$$\begin{cases} \frac{dP}{dt} = -m_x\left(P(t), t\right), \quad t > 0, \\ P(t) = P(t+1). \end{cases} \tag{ODE}$$

Theorem 4 (Liu-L-Peng-Zhou, TAMS 2021)

Assume $m_x(0,t) \neq 0$ *and* $m_x(1,t) \neq 0$ *for* $t \in [0,1]$*.*

(i) *If* (ODE) *has finite number of periodic solutions* $\{P_i(t)\}_{i=1}^N$*, satisfying* $0 = P_0 < P_1(t) < \ldots < P_N(t) < P_{N+1} = 1$*, and* $m_{xx}\left(P_i(t), t\right) \neq 0$ *for* $1 \leq i \leq N$ *and* $t \in [0,1]$*, then*

$$\lim_{D \to 0} \lambda(D) = \min_{0 \leq i \leq N+1} \left\{ \int_0^1 \Big[c\left(P_i(s), s\right) + [m_{xx}]_+ \left(P_i(s), s\right) \Big] \mathrm{d}s \right\}.$$

Small diffusion limit

$$\begin{cases} \frac{dP}{dt} = -m_x\left(P(t), t\right), \\ P(t) = P(t+1). \end{cases} \tag{ODE}$$

Theorem 5 (Liu-L-Peng-Zhou)

Assume $m_x(0,t) \neq 0$ and $m_x(1,t) \neq 0$ for $t \in [0,1]$.

(ii) *If* (ODE) *has no periodic solutions, then*

$$\lim_{D\to 0} \lambda(D) = \min\left\{\hat{c}(0) + [\hat{m}_{xx}]_+(0),\ \hat{c}(1) + [\hat{m}_{xx}]_+(1)\right\}.$$

Special case: $m = \alpha b(t)x$

$$\begin{cases} \varphi_t - D\varphi_{xx} - \alpha b(t)\varphi_x + c(x,t)\varphi = \lambda(D)\varphi & \text{in } (0,1)\times[0,1], \\ \varphi_x(0,t) = \varphi_x(1,t) = 0 & \text{on } [0,1], \\ \varphi(x,0) = \varphi(x,1) & \text{on } [0,1]. \end{cases}$$

Theorem 6 (Liu-L-Peng-Zhou)

(i) *If $\hat{b} \neq 0$, then for all $\alpha > 0$,*

$$\lim_{D\to 0} \lambda(D) = \begin{cases} \hat{c}(1) & \text{for } \hat{b} > 0, \\ \hat{c}(0) & \text{for } \hat{b} < 0. \end{cases}$$

Special case: $m = \alpha b(t)x$

Theorem 7 (Liu-L-Peng-Zhou)

(ii) *If* $\hat{b} = 0$, *set* $P(t) = -\int_0^t b(s)\mathrm{d}s$, $\overline{P} = \max P$, $\underline{P} = \min P$. *Then*

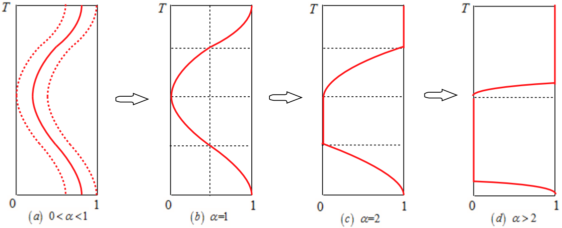
$$\lim_{D\to 0} \lambda(D) = \begin{cases} \min\limits_{y\in[-\alpha\underline{P},1-\alpha\overline{P}]} \int_0^1 c(\alpha P(s)+y,s)\mathrm{d}s & 0<\alpha\leq \frac{1}{\overline{P}-\underline{P}}, \\ \int_0^1 c(\tilde{P}_\alpha(s),s)\mathrm{d}s, & \alpha > \frac{1}{\overline{P}-\underline{P}}, \end{cases}$$

where $\tilde{P}_\alpha$ *is the unique periodic solution of* $\dot{\tilde{P}}(t) = -\alpha F\left(\tilde{P}(t),t\right)$ *with*

$$F(x,t) = \begin{cases} b(t) & 0<x<1,\ t\in[0,1], \\ \min\{b(t),0\}, & x=0,\ t\in[0,1], \\ \max\{b(t),0\}, & x=1,\ t\in[0,1]. \end{cases} \tag{7}$$

An explicit example

- $b(t) = -\pi \sin(2\pi t)$, $P(t) = \frac{1}{2}\cos(2\pi t) - \frac{1}{2}$, $\overline{P} = 0$, $\underline{P} = -1$



(a,b) $\alpha \le 1$: $\exists y_\alpha \in [\alpha, 1]$ s.t. $\lambda(D) \to \int_0^1 c(\alpha P(s) + y_\alpha, s)\mathrm{d}s$;

(c,d) $\alpha > 1$: $\lambda(D) \to \int_0^1 c(\tilde{P}_\alpha(s), s)\mathrm{d}s$.

Thank you!