The linear instability of the Akhmediev breather the ”missed” modes and the regular approach.

P. G. Grinevich[1], P.M. Santini[2]

[1]Steklov Mathematical Institute, Moscow, Russia
L.D. Landau Institute for Theoretical Physics, Chernogolovka, Russia,
Lomonosov Moscow State University, Russia.
The author was supported by RSF grants No 18-11-00316 and No 21-11-00331

[2]Dipartimento di Fisica, Università di Roma ”La Sapienza”, Italy
Istituto Nazionale di Fisica Nucleare, Sezione di Roma, Italy

Integrable Systems and Geometry, December 20, 2021.

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Plan of the talk.

1. Motivation of our research:
   - Rogue waves (anomalous waves, killing waves) in the ocean;
   - Rogue waves as a nonlinear phenomenon;
   - Anomalous waves in other nonlinear media;
   - Fermi-Pasta-Ulam-Tsingou recurrence of anomalous waves;
2. Mathematical model:
   - Focusing Nonlinear Schrödinger equation;
   - Special periodic Cauchy data;
   - Spectral transform for this special data. Almost degenerate Riemann surfaces;
   - Asymptotic formulas in terms of elementary functions;
   - Approximation of generic solutions by Akhmediev breathers;
3. Stability of Akhmediev breathers;
   - Effect of small loss or gain;
   - Squared eigenfunctions decomposition for linearized problem;
   - “Missed” unstable modes;
   - Regular derivation of “missed” modes.

Rogue waves

Rogue waves (freak waves, anomalous waves) in the ocean are great waves appearing from almost nowhere and disappearing to nowhere.

Figure: Akademik Ioffe ship, Drake Straight

Rogue waves

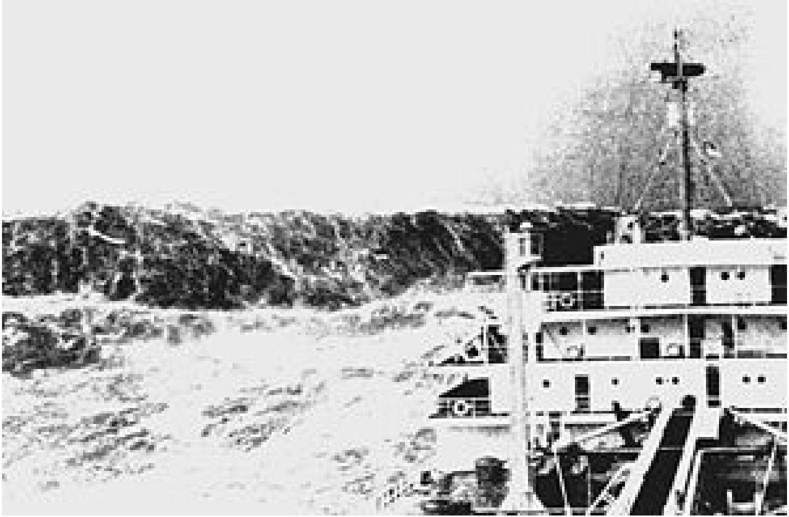


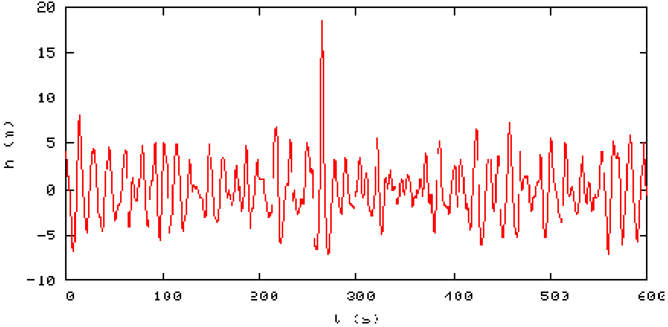
Figure: The photo from the Bay of Biscay.
https://commons.wikimedia.org/wiki/File:Wea00800,1.jpg

Rogue waves

Figure: Pyramidal wave near Japan, photo by Fukui Kuriyami

The Draupner wave

Anomalous waves were observed by sailors for long time, but these observations were treated as sailor's stories (like observations of mermaids).



The first confirmed observation of a rogue wave at the Draupner platform 160 kilometers southwest from the southern tip of Norway, January 1, 1995.

Rogue waves as nonlinear phenomena

The first idea was to apply linear model. We always have some waves in the ocean. In case of phase synchronization one can obtain a large localized wave. Let us do a simple numerical experiment:

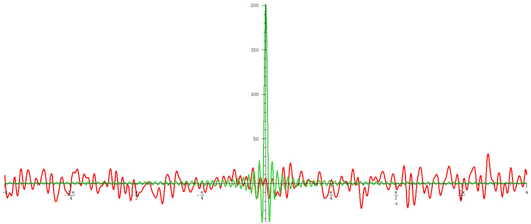


Figure: Graph of f_1 is shown in green, graph of f_2 is shown in red.

$$f_1(x) = \mathrm{Re}\left[\sum_{k=-100}^{100} \exp(ikx)\right], \quad f_2(x) = \mathrm{Re}\left[\sum_{k=-100}^{100} \exp(ikx + \phi_k)\right]$$

All phases of f_1 are synchronized at 0, the phases ϕ_k of f_2 – are random. **But physicists estimated the probability of such event. They will be too rare.**

Rogue waves as nonlinear phenomena

Common opinion: Linear theory could not explain this phenomenon. Non-linear mechanisms should be used, including **modulation instability.**

Example: Modulation instability in optics in a media with refractive index depending on the intensity of the light. Modulation instability takes place if the electric field **increases** the refractive index.

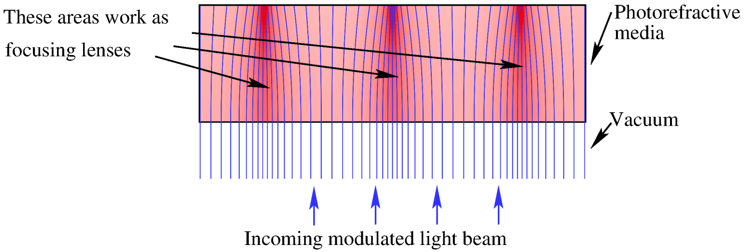


Figure: The density of rays mimics the intensity of light beam. The refraction index of media is marked by the intensity of the red color. Higher refraction index corresponds to more intensive red.

Rogue waves as nonlinear phenomena

Anomalous waves were also experimentally observed in other non-linear media:

- Fiber optics;
- Photorefractive medias;
- Bose-condensate with attraction;
- Langmuir waves in plasmas;
- ...

Many scientist work to apply **completely integrable (soliton) equation** to the theory of anomalous waves. Important results were obtained by the Zakharov's group.

Of course the exact models are non-integrable, therefore the prediction made using soliton equations are only approximate.

Fermi-Pasta-Ulam-Tsingou recurrence of anomalous waves in the experiments

Some works (experimental and theoretical) about

Van Simaeys, G., Emplit, P., Haelterman, M. “Experimental demonstration of the Fermi-Pasta-Ulam recurrence in a modulationally unstable optical wave”, *Phys. Rev. Lett.* **87** (2001) 033902.

Onorato, M., Residori, S., Bortolozzo, U., Montina, A., Arecchi F.T. “Rogue waves and their generating mechanisms in different physical contexts”, *Phys. Rep.* **528** (2013) 47–89.

Kimmoun O., Hsu H.C., Branger H., Li M. S., Chen Y.Y., Kharif C., Onorato M., Kelleher E.J.R., Kibler B., Akhmediev N., Chabchoub A. “Modulation Instability and Phase-Shifted Fermi-Pasta-Ulam Recurrence”, *Scientific Reports,* **6**:28516.

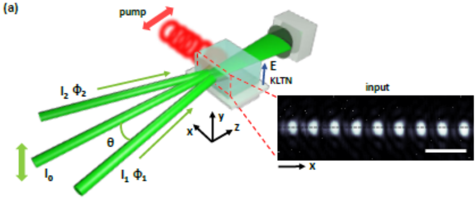
Mussot A., Naveau C., Conforti M., Kudlinski A., Copie F., Szriftgiser P., Trillo S. “Fibre multi-wave mixing combs reveal the broken symmetry of Fermi-Pasta-Ulam recurrence”, *Nature Photonics,* **12** (MAY 2018), 303–308.

Fermi-Pasta-Ulam-Tsingou recurrence of anomalous waves in the experiments

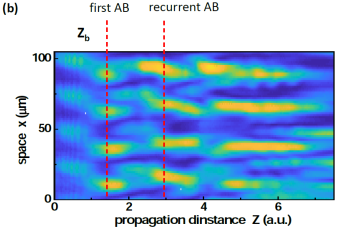
Pierangeli D., Flammini M., Zhang L., Marcucci G., Agranat A.J., Grinevich P.G., Santini P.M., Conti C., DelRe E. "Observation of Fermi-Pasta-Ulam-Tsingou recurrence and its exact dynamics", *Physical Review X*, **8**:4 (2018), p. 041017 (9 pages); doi:10.1103/PhysRevX.8.041017;

Naveau C., Szriftgiser P., Kudlinski A., Conforti M., Trillo S., Mussot A. "Experimental characterization of recurrences and separatrix crossing in modulational instability", *Optics Letters,* **44**:22 (15 November 2019).

Pierangeli D., Flammini M., Zhang L., Marcucci G., Agranat A.J., Grinevich P.G., Santini P.M., Conti C., DelRe E.



The symmetric 3-wave interferometric scheme used to generate the background wave with a single-mode perturbation propagating in a pumped photorefractive KLTN (potassium-lithium-tantalate-niobate) crystal.
Since NLS is supposed to describe the above physics only at the leading order, one expects that the exact NLS RW recurrence be replaced by a "Fermi-Pasta-Ulam-Tsingou" - type recurrence, before thermalization destroys the pattern.


(b)
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recurrent AB
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100
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space x (μm)
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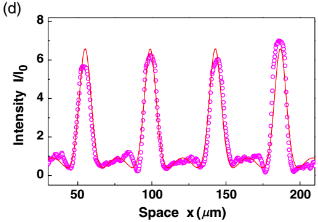
propagation dinstance Z (a.u.)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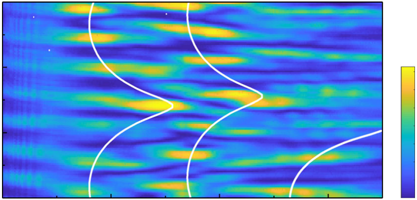
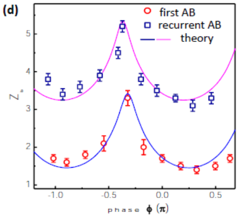
Space x (μm)




(d)
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theory
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phase ϕ (π)

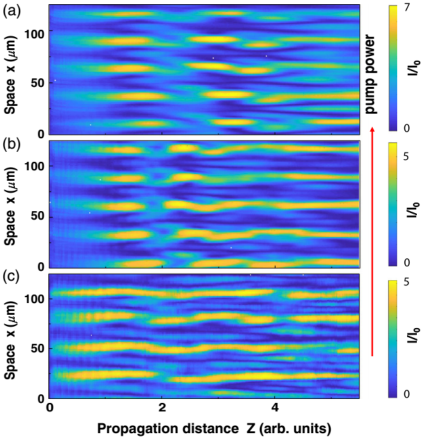


Figure: Transition between integrable and non-intergable regime.

Fermi-Pasta-Ulam-Tsingou experiment

E. Fermi, J. Pasta, S. Ulam "Studies of nonlinear problems", Los Alamos preprint LA-1940 (7 November 1955):
"We thank Miss Mary Tsingou for efficient coding of the problem and for running the computations on the Los Alamos MANIAC machine".

Figure: Mary Tsingou-Menzel at the Los Alamos National Laboratory. Photo by: Los Alamos National Laboratory.

Fermi-Pasta-Ulam-Tsingou experiment

The main axiom of classical equilibrium thermodynamics: each degree of freedom contains energy $kT/2$, i.e. in the equilibrium state the energy is uniformly distributed between all degrees of freedom. Thermalization means that starting from a generic state we shall obtain equilibrium distribution.

In linear systems the energy in each mode in conserved, and thermalization is impossible. There was an attempt to model the thermalization by adding non-linear interaction. Three basic model were considered:

$$\ddot{x}_i = (x_{i+1} + x_{i-1} - 2x_i) + \alpha[(x_{i+1} - x_i)^2 - (x_i - x_{i-1})^2],$$

$$\ddot{x}_i = (x_{i+1} + x_{i-1} - 2x_i) + \beta[(x_{i+1} - x_i)^3 - (x_i - x_{i-1})^3],$$

$$\ddot{x}_i = \delta_1(x_{i+1} - x_i) - \delta_2(x_i - x_{i-1}) + c.$$

In the last case $\delta_1(x)$, $\delta_2(x)$ are continuous piecewise linear functions.

Periodic boundary conditions with period 64 were used:

$$i = 1, \ldots, 64, \quad x_0 = x_{64}$$

In these experiments the thermalization was much slower than expected.

Moreover, there was some energy transfer between linear modes, but at some moments the system returned to states, **very close to the starting one.**

Later this phenomenon was explained using the soliton theory. It turned out that after adding nonlinear terms the system remains **integrable in the leading order,** i.e. close to integrable.

Fermi-Pasta-Ulam-Tsingou recurrence: Reappearance of coherent structures in nonlinear systems, close to integrable.

Of course, in real physical experiments integrable equations like KdV, KP or NLS arise only as approximate models, and higher order correction shall be taking into account. Therefore, it is important to develop perturbation theory near exact solutions.

3 classical paper about modulation instability in nonlinear medias

1) V. I. Bespalov and V. I. Talanov, “Filamentary structure of light beams in nonlinear liquids”, JETP Letters. **3** (12), 307, 1966.

2) T. B. Benjamin, J. E. Feir, “The disintegration of wave trains on deep water”. Part I. Theory, Journal of Fluid Mechanics 27 (1967) 417-430.

3) V. E. Zakharov, “Stability of period waves of finite amplitude on surface of a deep fluid”, Journal of Applied Mechanics and Technical Physics, 9(2) (1968) 190-194.

In papers 1) and 3) the **Nonlinear Schrödinger equation (NLS)**

$$iu_t + u_{xx} \pm 2u^2\bar{u} = 0, \quad u = u(x,t) \in \mathbb{C}, \quad (x,t) \in \mathbb{R}^2, \qquad (1)$$

was used as the basic mathematical model.

Nonlinear Schrödinger equation

Solutions of two real forms of NLS:

$$iu_t + u_{xx} + 2u^2\bar{u} = 0, \quad \text{self-focusing NLS,} \tag{2}$$

$$iu_t + u_{xx} - 2u^2\bar{u} = 0, \quad \text{defocusing NLS,} \tag{3}$$

have different analytic properties. In the theory of anomalous waves self-focusing NLS (2) is used.

The literature dedicated to the applications of NLS equations in the anomalous waves theory is enormous.

Of course, the question of NLS applicability to real physics is very delicate. Nevertheless, in some situations NLS approximation works good enough.

Mathematical model – Focusing NLS

We study the anomalous waves on the focusing NLS equation (SfNLS) with periodic boundary conditions:

$$iu_t + u_{xx} + 2u^2\bar{u} = 0$$

We use the following Cauchy data (anomalous waves Cauchy problem):

$$u(x,0) = a + \epsilon v(x), \quad v(x+L) \equiv v(x), \quad |\epsilon| \ll 1,$$

$$v(x) = \sum_{j\geq 1}\left(c_j e^{ik_j x} + c_{-j}e^{-ik_j x}\right), \quad k_j = \frac{2\pi}{L}j, \quad |c_j| = O(1),$$

To simplify calculations we also assume that the period L is generic: $L \neq \pi n$, $n \in \mathbb{Z}$.

Main classes of analytic NLS solutions:

1. Spatially localized solutions obtained using Riemann-Hilbert problem;
2. Explicit solutions in terms of elementary functions – Darboux transformation or other equivalent methods;
3. Finite-gap periodic (quasiperiodic) solutions.

Finite-gap method – S.P. Novikov, B.A. Dubrovin, V.B. Matveev, A.R. Its, P. Lax, H, McKean, P. van Moerbeke, I.M. Krichever, ... Finte-gap complexified NLS solutions:

Its, A.R., Kotljarov, V.P., "Explicit formulas for solutions of a nonlinear Schrödinger equation", *Dokl. Akad. Nauk Ukrain. SSR Ser. A,* **1051** (1976), 965–968.

Selection of spectral data, corresponding to self-focusing NLS (very non-trivial):

Cherednik, I.V., "Reality conditions in finite-zone integration", *Soviet Phys. Dokl.,* **25**:6 (1980), 450–452.

1. The Riemann-Hibert problem method provides the answer in terms of solutions of a linear integral equation. To use these solutuions serious extra work is required;
2. Exact elementary solutions correspond to very special Cauchy data. For example, Akhmediev breathers appears only once, but generic periodic Cauchy data results in recurrent appearance of anomalous waves;
3. Finite-gap formulas looks simple, but the Riemann theta-functions are very nontrivial, and all parameters of the theta-functions are transcendental expressions of the spectral data.

Our idea was the following: for special Cauchy data (small perturbations of spatially constant solutions) the corresponding spectral curves are almost degenerate, therefore all components of the finite-gap solutions can be explicitly calculated in the leading order, therefore we obtain simple approximate formulas, describing strongly nonlinear behaviour.

Linear perturbation theory near the background

Consider the unstable background: ($\epsilon = 0$):

$$u_0(x,t) \;=\; a e^{2i|a|^2 t}.$$

Denoting

$$u(x,t) = \exp(2|a|^2 it)U(x,t),$$

we obtain

$$iU_t + U_{xx} + 2|U|^2 U - 2|a|^2 U = 0.$$

Consider a perturbation of the background:

$$U(x,t) = a + \delta U(x,t), \;\; \bar{U}(x,t) = \bar{a} + \delta\bar{U}(x,t).$$

In is convenient to assume that δu and $\delta\bar{u}$ are independent functions. Then we obtain the complexified linearization near the background:

$$\left\{ \begin{array}{r} i\,\delta U + \delta U_{xx} + 2|a|^2 \delta U + 2a^2\,\delta\bar{U} = 0, \\ -i\,\delta\bar{U} + \delta\bar{U}_{xx} + 2|a|^2\,\delta U + 2\bar{a}^2\,\delta U = 0. \end{array} \right.$$

Linear perturbation theory near the background

The basis solutions of linearized NLS can be written as:

$$\delta U = e^{\sigma t}\left(\alpha e^{ikx} + \beta e^{-ikx}\right), \quad \delta \bar{U} = e^{\sigma t}\left(\beta' e^{ikx} + \alpha' e^{-ikx}\right),$$

Substituting into linearized NLS we obtain:

$$\sigma = \pm |a| k \sqrt{4|a|^2 - k^2}.$$

We see that σ is real iff $|k| \le 2|a|$, therefore a harmonic perturbation

$$U(x,0) = a + \epsilon e^{ikx}$$

is **unstable** if $|k| < 2|a|$ and **stable** if $|k| \ge 2|a|$.
To obtain solutions of "normal" linearized NLS it is sufficient to take "real" linear combinations of δu, $\delta \bar{u}$:

$$\delta U_1 = \delta U + \bar{\delta} U, \quad \text{and} \quad \delta U_2 = i[\delta U - \bar{\delta} U].$$

Then δU_1, δU_2 satisfy linearized NLS:

$$i\,\delta U + \delta U_{xx} + 2|a|^2 \delta U + 2a^2\,\delta \bar{U} = 0,$$

Linear stability of the background

We study periodic problem, therefore only L-periodic perturbations are considered:

$$k = k_j = \frac{2\pi}{L} j, \quad j \in \mathbb{Z}.$$

The first N harmonics are unstable, where

$$N = \left[\frac{|a|L}{\pi} \right]$$

with the growing factor in the linear mode:

$$\sigma_j = |a| k_j \sqrt{4|a|^2 - k_j^2}, \quad 1 \le j \le N,$$

All other modes are stable. They give only small corrections and we discard them.

Zero-curvature representation

Integrability of self-focusing NLS equation (SfNLS)

$$iu_t + u_{xx} + 2u^2\bar{u} = 0, \quad u = u(x,t)$$

is based on the zero-curvature representation (Zakharov-Shabat):

$$\vec{\Psi}_x(\lambda,x,t) = U(\lambda,x,t)\vec{\Psi}(\lambda,x,t), \quad \vec{\Psi}_t(\lambda,x,t) = V(\lambda,x,t)\vec{\Psi}(\lambda,x,t),$$

$$U = \begin{bmatrix} -i\lambda & iu(x,t) \\ \overline{iu(x,t)} & i\lambda \end{bmatrix},$$

$$V(\lambda,x,t) = \begin{bmatrix} -2i\lambda^2 + iu(x,t)\overline{u(x,t)} & 2i\lambda u(x,t) - u_x(x,t) \\ 2i\lambda\overline{u(x,t)} + \overline{u_x(x,t)} & 2i\lambda^2 - iu(x,t)\overline{u(x,t)} \end{bmatrix},$$

where

$$\vec{\Psi}(\lambda,x,t) = \begin{bmatrix} \Psi^1(\lambda,x,t) \\ \Psi^2(\lambda,x,t) \end{bmatrix}.$$

One unstable mode

We derived approximate formulas for finite number N of unstable mode. In particular, we proved that solutions at each time are well-approximated by n-breather solutions ($n \leq N$), but different approximations at different time intervals shall be used.

Let us discuss the first non-trivial case $N = 1$.

We assume: $\pi/|a| < L < 2\pi/|a|$ i.e. we have exactly one unstable mode. It corresponds to the following perturbation of the background:

$$u(x,0) = a\left(1 + \epsilon\left(c_1 e^{k_1 x} + c_{-1} e^{-ik_1 x}\right)\right), \quad k_1 = \frac{2\pi}{L}, \quad \epsilon \ll 1,$$

where c_1 and c_{-1} are arbitrary $O(1)$ complex parameters.

Problem: Calculate the time of the first rogue wave appearance and its position. Calculate the periodicity of appearances in terms of the Cauchy data.

Akhmediev breathers

The unstable mode is described by Riemann theta functions of 2 variables.

But for this special Cauchy data it admits a good approximation as a sequence of Akhmediev breathers (Grinevich–Santini).

Akhmediev breathers:

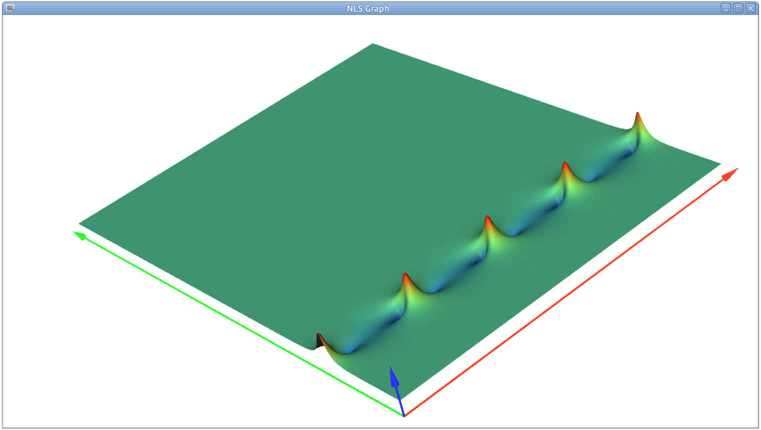
N. N. Akhmediev, V. M. Eleonskii, and N. E. Kulagin, “Exact first order solutions of the Nonlinear Schdinger equation”, *Theor. Math. Phys,* **72**, 809 (1987).

$$\mathcal{A}(x,t;\theta,X,T) =$$
$$= a\, e^{2i|a|^2 t} \cdot \frac{\cosh[\sigma(\theta)(t-T)+2i\theta] + \sin\theta\cos[k(\theta)(x-X)]}{\cosh[\sigma(\theta)(t-T)] - \sin\theta\cos[k(\theta)(x-X)]},$$

$$k_1 = k(\theta) = 2|a|\cos\theta,\ \sigma(\theta) = k(\theta)\sqrt{4|a|^2 - k^2(\theta)} = 2|a|^2\sin(2\theta),$$

Akhmediev breathers

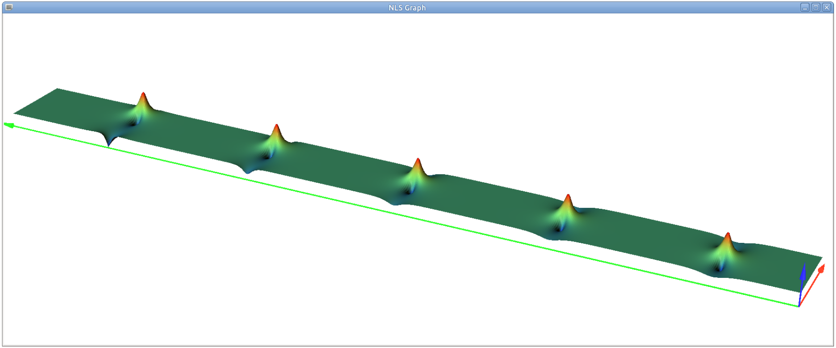
They are spatially periodic and localized in time:



The x coordinate axis marked red, the t coordinate axis marked green. In the future we draw only one period of solution with respect to x.

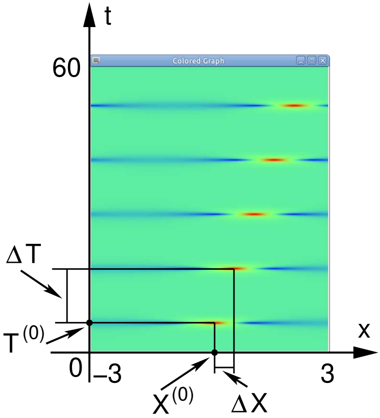
One unstable mode

Generic solution for one unstable mode is well-approximated by a sequence of Akhmediev breathers:



Recurrence of Akhmediev breathers for one unstable mode ($L = 6$).
Here we draw exactly one period in the x-variable.

One unstable mode



Recurrence of Akhmediev breathers for one unstable mode ($L = 6$).

Essential parameters:
First appearance time $T^{(0)}$, position of maximum at first appearance $X^{(0)}$, interval between subsequent appearances ΔT , phase shift between subsequent appearances ΔX.

One unstable mode

Approximation of the genus 2 solution:

$$u(x,t) = \sum_{m=0}^{n} \mathcal{A}\Big(x,t;\phi_1,x^{(m)},t^{(m)}\Big)e^{i\rho^{(m)}} - \frac{1-e^{4in\phi_1}}{1-e^{4i\phi_1}}ae^{2i|a|^2 t}, \quad x\in[0,L],$$

where:

$$x^{(m)} = X^{(1)} + (m-1)\Delta X, \quad t^{(m)} = T^{(1)} + (m-1)\Delta T,$$

$$X^{(1)} = \frac{\arg\alpha}{k_1} + \frac{L}{4}, \quad \Delta X = \frac{\arg(\alpha\beta)}{k_1}, \quad (\bmod L),$$

$$T^{(1)} = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^2}{2|a|^4\epsilon|\alpha|}\right), \quad \Delta T = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^4}{4|a|^8\epsilon^2|\alpha\beta|}\right),$$

$$\rho^{(m)} = 2\phi_1 + (m-1)4\phi_1, \quad n = \left[\frac{T-T^{(1)}}{\Delta T} + \frac{1}{2}\right],$$

$$\cos\phi_1 = \frac{\pi}{L|a|}, \quad k_1 = \frac{2\pi}{L} = 2|a|\cos(\phi_1), \quad \sigma_1 = k_1\sqrt{4|a|^2-k_1^2} = 2|a|^2\sin(2\phi_1),$$

$$\alpha = e^{-i\phi_1}\overline{c_1} - e^{i\phi_1}c_{-1}, \quad \beta = e^{i\phi_1}\overline{c_{-1}} - e^{-i\phi_1}c_1.$$

One unstable mode

The spectra curve has genus $g = 2$ and 6 branch points: E_0, E_1, E_2, $\bar{E}_0$, $\bar{E}_1$, $\bar{E}_2$. The pair E_1, E_2 is obtained as a results of splitting the resonant point $\lambda_1 = i|a|\sin\phi_1$:

$$E_l = \lambda_1 + (-1)^l \frac{\epsilon |a|^2}{2\lambda_1}\sqrt{\alpha\beta} + O(\epsilon^2), \quad l = 1, 2,$$

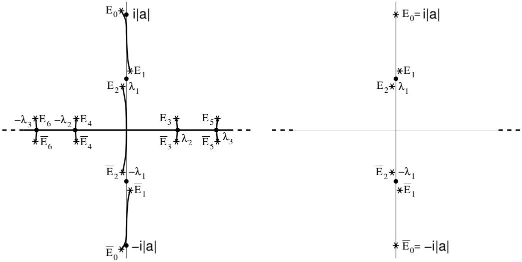


Figure: Right: the exact spectrum; Left: the approximating curve.

Two special symmetric configurations:

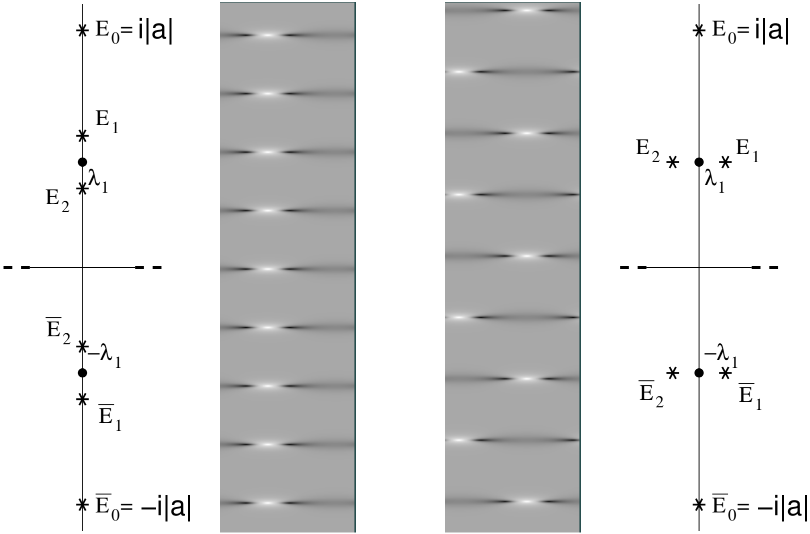


Figure: Left: vertical gap. Right: horisontal gap.

Two special symmetric configurations:

Remark: In these two special cases theta-functions of genus 2 can be reduced to genus 1.

1. Elliptic solutions:

   Akhmediev N.N., Eleonskii V.M, Kulagin N.E., "Exact first-order solutions of the nonlinear Schrödinger equation", *Theoret. and Math. Phys.*, **72**:2 (1987), 809–818;

2. Reduction for generic parameters:

   Smirnov A.O., "Periodic two-phase "Rogue waves"", *Mathematical Notes*, **94** (2013), 897–907;

3. Reduction in terms of σ-functions:

   Ayano T., Buchstaber V.M., "Relationships between hyperelliptic functions of genus 2 and elliptic functions", arXiv:2106.06764.

Effect of small loss/gain

As we mentioned above, in real physics it is necessary to take into account small corrections to the NLS equation.
The effect of Hamiltonian perturbations vanishes in the leading order. In contrast, effect of non-Hamiltonian perturbations is non-trivial in the leading order.

Effect of small loss/gain

$$iu_t + u_{xx} + 2u^2\bar{u} = -i\nu u, \quad u = u(x,t), \quad \nu \in \mathbb{R}, \quad |\nu| \ll 1.$$

was recently analytically studied in:

Coppini F., Grinevich P.G., Santini P.M. “The effect of a small loss or gain in the periodic NLS anomalous wave dynamics. I,” *Phys. Rev. E,* **101**:3 (2020), 032204, 8 pages, - Published 6 March 2020; doi:10.1103/PhysRevE.101.032204.

Small loss/gain – experiments and numerics

Our aim was to explain the results of experimental and numerical observations:

O. Kimmoun, H.C. Hsu, H. Branger, M.S. Li, Y.Y. Chen, C. Kharif, M. Onorato, E.J.R. Kelleher, B. Kibler, N. Akhmediev, A. Chabchoub, "Modulation Instability and Phase-Shifted Fermi-Pasta-Ulam Recurrence", *Scientific Reports*, **6**, Article number: 28516 (2016), doi:10.1038/srep28516.

J.M. Soto-Crespo, N. Devine, and N. Akhmediev, "Adiabatic transformation of continuous waves into trains of pulses", *PHYSICAL REVIEW A*, **96**, 023825 (2017).

Small loss/gain – experiments and numerics

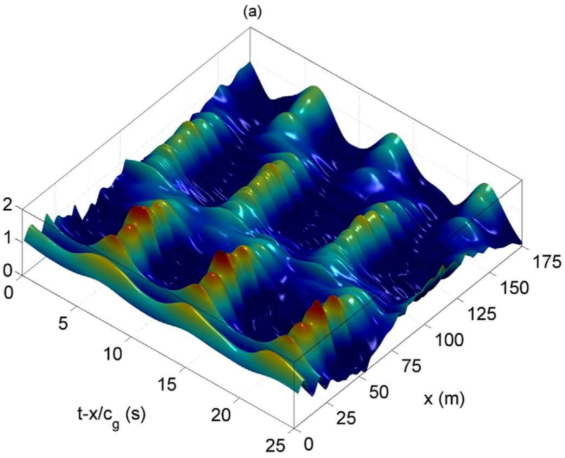


Figure: Measured AB envelope along the large wave facility. The picture was presented in the paper by O. Kimmoun at al, doi:10.1038/srep28516. The phase shift between subsequent appearances of anomalous waves is equal to the semi-period of the wave.

Generic initial data:

A numeric experiment with **very small** losses:

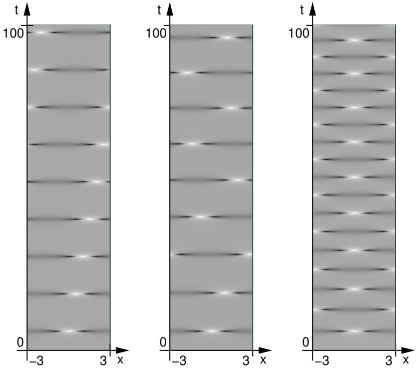


Figure: $-L/2 \le x \le L/2$, $0 \le t \le 100$, $L = 6$, $\epsilon = 10^{-4}$, generic initial data: $c_1 = 0.5$ and $c_{-1} = 0.15 - 0.2i$. Form left to right: $\nu = 0$, $\nu = 10^{-9}$, and $\nu = 10^{-5}$. The first appearance is essentially the same in all the three cases.

Symmetric initial data:

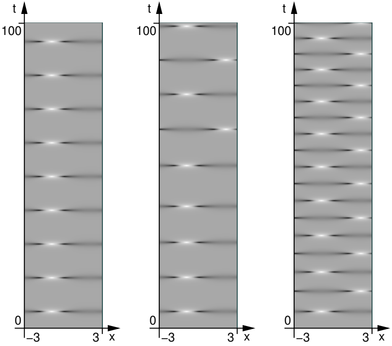


Figure: The density plot of $|u(x,t)|$ with $-L/2 \le x \le L/2$, $0 \le t \le 100$, $L = 6$, $\epsilon = 10^{-4}$, for a real initial condition ($c_{-j} = \overline{c_j}$, $\forall j$), with $c_1 = 0.3 + 0.4i$. Consequently $\alpha\beta > 0$. Left picture: $\nu = 0$, then $\Delta X = 0$. Center picture: $\nu = 10^{-9}$; then for $\tilde{m} = 6$, Q_m changes its sign, from positive to negative values; correspondingly, ΔX_m switches from 0 to $L/2$. Right picture: $\nu = 10^{-5}$; then all Q_m are negative and $\Delta X_m = L/2$ $\forall m$. The first appearance is essentially the same in all the three cases.

Analytic formulas.

We have the following approximate formulas:
the spectral curve is not time–invariant, but it changes each time we have an anomalous wave:

$$(E_1 - E_2)^2\Big|_{t=0} = -\frac{\epsilon^2 |a|^2 \alpha\beta}{\sin^2\phi_1},$$

$$(E_1^{(m)} - E_2^{(m)})^2 = -\frac{\epsilon^2 |a|^2 \alpha\beta}{\sin^2\phi_1} + 4m\nu\cot\phi_1, \quad m \geq 0,$$

where $E_1^{(m)}, E_2^{(m)}$ are the branch points after the m^{th}-th breather.
Therefore:

$$\Delta X_m := \tilde{x}^{(m+1)} - \tilde{x}^{(m)} = \frac{\arg(Q_m)}{k_1} \pmod{L},$$
$$\Delta T_m := \tilde{t}^{(m+1)} - \tilde{t}^{(m)} = \frac{1}{\sigma_1}\log\left(\frac{\sigma_1^4}{4\epsilon^2|Q_m|}\right),$$

$$\epsilon^2 Q_m = \epsilon^2\alpha\beta - \frac{\nu\sigma_1}{|a|^4}m, \quad m \geq 1, \tag{4}$$

Evolution of the branch points

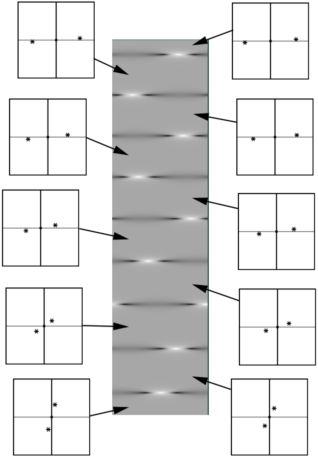
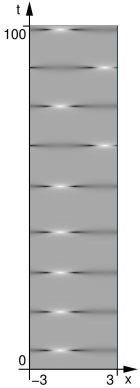


Figure: Evolution of the branch points

Symmetric initial data numerics vs analytics:



$\tilde{t}^{(1)} = 5.51209$ (theory) $\tilde{t}^{(1)} = 5.51208$ (numerics)

$\Delta T_1 = 11.18230$ (theory) $\Delta T_1 = 11.18230$ (numerics)

$\Delta T_2 = 11.40337$ (theory) $\Delta T_2 = 11.40338$ (numerics);

$\Delta T_3 = 11.77375$ (theory) $\Delta T_3 = 11.77376$ (numerics);

$\Delta T_4 = 13.31847$ (theory) $\Delta T_4 = 13.31848$ (numerics);

$\Delta T_5 = 11.84989$ (theory) $\Delta T_5 = 11.84988$ (numerics);

$\Delta T_6 = 11.44140$ (theory) $\Delta T_6 = 11.44142$ (numerics);

$\Delta T_7 = 11.20765$ (theory) $\Delta T_7 = 11.20766$ (numerics);

$\Delta T_8 = 11.04319$ (theory) $\Delta T_8 = 11.04320$ (numerics)

Stability of Akhmediev breathers

Our analytic formulas as well as numerical simulations provide the strong evidence that:

1. The process of Akhmediev breather generation is very stable;
2. In contrast, the Fermi-Pasta-Ulam-Tsingou recurrence is very sensitive to small perturbation. For example, **small perturbations of solutions generate recurrence.** Moreover, the solutions demonstrate the highest unstability when they reach the maximal value.

At the next slide we illustrate this conclusion by a numeric example.

Stability of Akhmediev breathers – numerical test

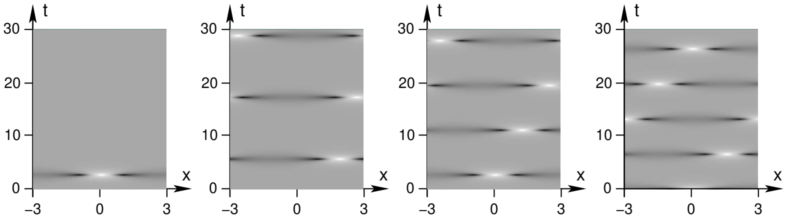


Figure: The figures are enumerated from left to right. We use the recurrence times to measure the effect of perturbation. Smaller recurrence times mean stronger instability. At Figures 2-4 we apply the same perturbation $\delta u(x.0) = 10^{-4}[(0.1 - 0.5i)e^{ik_1 x} + (0.1 + 0.1i)e^{-ik_1 x}]$.

1. Fig.1: Exact Akhmediev breather;
2. Fig.2: Perturbation of the background;
3. Fig.3: Perturbation of Akhmediev breather applied 2.7 seconds before the peak;
4. Fig.4: Perturbation of Akhmediev breather applied at the peak time.

When we started to present these results, we got criticism that our results contradict to the linear stability of N-Akhmediev breather.

There was the following common believe in the literature. Let the NLS background be unstable with respect to the first N modes. Then

1. If $M < N$ unstable modes are exited, then the corresponding M-breather solution is linearly unstable;
2. If all N unstable modes are exited, then the corresponding N-breather solution is neutrally stable, due to "saturation of instabilities".

Let us recall the arguments. To study the linear perturbation theory the squared eigenfunctions expansion of the linearized equation is used.

A. Calini, C.M. Schober, "Dynamical criteria for rogue waves in nonlinear Schrödinger models", *Nonlinearity*, **25**:12 (2012) R99–R116; doi:10.1088/0951-7715/25/12/R99.
A. Calini, C.M. Schober, "Observable and reproducible rogue waves", J. Opt. 15 (2013) 105201 (9pp).
A. Calini, C.M. Schober, "Numerical investigation of stability of breather-type solutions of the nonlinear Schrödinger equation", Nat. Hazards Earth Syst. Sci., 14, 14311440, 2014 www.nat-hazards-earth-syst-sci.net/14/1431/2014/doi:10.5194/nhess-14-1431-2014.

Let us recall the main formulas.

Linear perturbation theory near Akhmediev breather

To simplify formulas we use the following gauge transformation.

$$u(x,t) \to \exp(2it)u(x,t), \quad \vec{\psi} \to \exp(i\sigma_3 t)\vec{\psi},$$

The new function $u(x,t)$ satisfy:

$$iu_t + u_{xx} + 2|u|^2 u - 2u = 0.$$

Assume for a moment that δu and $\delta\bar{u}$ are independent functions.
Important fact: Squared eigenfunctions

$$\delta u = \psi_1(\lambda,x,t)\varphi_1(\lambda,x,t), \quad \delta\bar{u} = \overline{\psi_2(\lambda,x,t)\varphi_2(\lambda,x,t)}$$

satisfy the complexified linearized NLS equation:

$$\begin{cases} i\,\delta u + \delta u_{xx} + 4u\bar{u}\,\delta u - 2\delta u + 2u^2\,\delta\bar{u} = 0, \\ -i\,\delta\bar{u} + \delta\bar{u}_{xx} + 4u\bar{u}\,\delta\bar{u} - 2\delta\bar{u} + 2\bar{u}^2\,\delta u = 0. \end{cases}$$

(Complexified means exactly that δu and $\delta\bar{u}$ are treated as independent functions.)

To construct solutions of “normal”, not complexified linearized NLS?, it is sufficient to consider “real”linear combinations:'

$$\left\langle \vec{\psi}(\lambda,x,t),\vec{\varphi}(\lambda,x,t)\right\rangle_{+} := \psi_1(\lambda,x,t)\varphi_1(\lambda,x,t)+\overline{\psi_2(\lambda,x,t)\varphi_2(\lambda,x,t)},$$
$$\left\langle \vec{\psi}(\lambda,x,t),\vec{\varphi}(\lambda,x,t)\right\rangle_{-} := i\left[\psi_1(\lambda,x,t)\varphi_1(\lambda,x,t)-\overline{\psi_2(\lambda,x,t)\varphi_2(\lambda,x,t)}\right]$$

They satisfy the linearized NLS equation

$$iw_t + w_{xx} + 4|u|^2 w + 2u^2\bar{w} - 2w = 0.$$

Of course, we have to select spatially-perioic solutions with the period L.

Linear perturbation theory near Akhmediev breather

In the aforementioned papers it was shown that if we have N unstable modes and nonlinear superpositions of M Akhmediev breathers, $M \leq N$ then

1. If not all unstable modes are exited ($M < N$) then there exist x-periodic squared eigenfunctions exponentially growing in t;
2. If all unstable modes are exited ($M = N$) then all x-periodic squared eigenfunctions are bounded in t;

Therefore the conclusion about linear stability was made.

But we see the instability

Linear perturbation theory near Akhmediev breather

We obtained the following resolution of the paradox (we studied the case $M = N = 1$):

Due to presence of non-removable double points the spectral decomposition of linerized NLS solutions includes not only x-periodic squared eigenfunctions, but also some special combinations of derivatives with respect to the spectral parameter.

The fact that it is necessary to use derivatives with respect to the spectral parameter can be extracted from the paper
I.M. Krichever, Spectral theory of two-dimensional periodic operators and its applications, *Russ. Math. Surv.*, **44**(2), 145225 (1989)

The presence of non-removable double points means that the Lax operator has Jordan cells for $\lambda = \pm\lambda_1$.

We study the stability in the class of smooth x-periodic perturbation with the spatial period L. Let us demonstrate the "missed modes".

The "missed modes"

The resonant point is:

$$\lambda_1 = \sqrt{\mu_1^2 - 1}, \quad \mu_1 = \frac{k_1}{2} = \frac{\pi}{L},$$

$$k = k_1 = 2\mu_1, \quad \sigma = \sigma_1 = -4i\lambda_1\mu_1, \quad \theta(\lambda_1) = \frac{1}{2}\left(ikx - \sigma t\right).$$

Consider the following eigenfunctions:

$$\vec{q}(\lambda) = \begin{bmatrix} q_1(\lambda) \\ q_2(\lambda) \end{bmatrix} = \begin{bmatrix} \sqrt{\mu - \lambda}\, e^{\theta(\lambda)} + \sqrt{\mu + \lambda}\, e^{-\theta(\lambda)} \\ \sqrt{\mu + \lambda}\, e^{\theta(\lambda)} - \sqrt{\mu - \lambda}\, e^{-\theta(\lambda)} \end{bmatrix},$$

$$\vec{r}(\lambda) = \begin{bmatrix} r_1(\lambda) \\ r_2(\lambda) \end{bmatrix} = \begin{bmatrix} \sqrt{\mu - \lambda}\, e^{\theta(\lambda)} - \sqrt{\mu + \lambda}\, e^{-\theta(\lambda)} \\ \sqrt{\mu + \lambda}\, e^{\theta(\lambda)} + \sqrt{\mu - \lambda}\, e^{-\theta(\lambda)} \end{bmatrix},$$

$$\vec{\phi}(\lambda) = \begin{bmatrix} \phi_1(\lambda) \\ \phi_2(\lambda) \end{bmatrix} = \begin{bmatrix} 1 \\ (\mu + \lambda) \end{bmatrix} e^{\theta(\lambda)}$$

$$\vec{q} = \vec{q}(\lambda_1), \quad \vec{r} = \vec{r}(\lambda_1),$$

The Darboux transformation operator is defined by:

$$\mathfrak{D}(\lambda) = (\lambda - \lambda_1)E + \frac{2\lambda_1}{|q_1|^2 + |q_2|^2} \begin{bmatrix} -\overline{q_2} \\ \overline{q_1} \end{bmatrix} [-q_2, q_1],$$

The "missed modes"

Operator $\mathfrak{D}$ maps the background eigenfunctions to the eigenfunctions for the Akhmediev breather.

$$\vec{\psi}(\lambda) = \mathfrak{D}(\lambda)\vec{\psi}_0(\lambda)$$

Denote:

$$D_\mu = \partial_\mu + \frac{\partial\lambda}{\partial\mu}\partial_\lambda = \partial_\mu + \frac{\mu}{\lambda}\partial_\lambda.$$

We have

$$(D_\mu + D_{\bar{\mu}}) \left\langle \chi_+(\lambda), \chi_-(\lambda) \right\rangle_\pm \Big|_{\lambda=\lambda_1} = \left\langle \chi_+^{(1)}, \chi_-^{(0)} \right\rangle_\pm,$$

$$(D_\mu + D_{\bar{\mu}})^2 \left\langle \chi_+(\lambda), \chi_-(\lambda) \right\rangle_\pm \Big|_{\lambda=\lambda_1} = \left\langle \chi_+^{(2)}, \chi_-^{(0)} \right\rangle_\pm + 2\left\langle \chi_+^{(1)}, \chi_-^{(1)} \right\rangle_\pm,$$

$$(D_\mu + D_{\bar{\mu}}) \left\langle \tilde{\phi}(\lambda), \tilde{\phi}(\lambda) \right\rangle_\pm \Big|_{\lambda=\lambda_1} = 2\left\langle \tilde{\phi}^{(1)}, \tilde{\phi}^{(0)} \right\rangle_\pm.$$

The “missed modes”

Theorem The following combinations of the derivatives of the squared eigenfunctions with respect to the spectral parameter:

$$Sym_1 = 2l_0^2\left[m_0\left(\left\langle\chi_+^{(2)},\chi_-^{(0)}\right\rangle_+ + 2\left\langle\chi_+^{(1)},\chi_-^{(1)}\right\rangle_+\right) - 4\left\langle\chi_+^{(1)},\chi_-^{(0)}\right\rangle_+ - 16\left\langle\phi_+^{(1)},\phi_-^{(0)}\right\rangle_+\right],$$

$$Sym_2 = 2l_0^2\left(\left\langle\chi_+^{(2)},\chi_-^{(0)}\right\rangle_- + 2\left\langle\chi_+^{(1)},\chi_-^{(1)}\right\rangle_- - \frac{2m_0}{l_0^2}\left\langle\chi_+^{(1)},\chi_-^{(0)}\right\rangle_-\right),$$

$$\text{where} \qquad \lambda_1 = il_0, \;\; \mu_1 = m_0.$$

- They are x-periodic with period L;
- They exponentially grow as $t \to \pm\infty$;
- They are solutions of the linearized NLS near the Akhmediev breather.

Therefore they represent the “missed modes”.

Make the shift: $x \to x - L/4$. Let

$$\widehat{Sym}_1(x,t) = \frac{1}{2}\left(Sym_1(x,t) - Sym_1(-x,t)\right)$$

Then:

$$\widehat{Sym}_1(x,t) = k\,\frac{\widehat{Num}_1(x,t)}{\mathcal{D}(x,t)}$$

$$\widehat{Num}_1(x,t) = \Big[\big[48\sigma k^4 - 8\sigma k^6\big]\cosh(\sigma t) + \big[192ik^4 + 8ik^8 - 80ik^6\big]\sinh(\sigma t)\Big] t\sin(kx) +$$
$$+\Big[\big[-16i\sigma + 16i\sigma k^2\big]\cosh(3\sigma t) + \big[-64i\sigma + 40i\sigma k^2 - ik^4\sigma\big]\cosh(\sigma t) +$$
$$+\big[16k^4 - 48k^2\big]\sinh(3\sigma t) + \big[-64k^2 - k^6 + 24k^4\big]\sinh(\sigma t)\Big]\sin(kx) +$$
$$+\Big[\big[-48ik^3 + 64ik + 8ik^5\big]\cosh(2\sigma t) + \big[32\sigma k - 8\sigma k^3\big]\sinh(2\sigma t) +$$
$$+\big[8ik^5 - 48ik^3 + 64ik\big]\Big]\sin(2kx) +$$
$$+\Big[\big[8i\sigma k^2 - i\sigma k^4 - 16i\sigma\big]\cosh(\sigma t) + \big[8k^4 - 16k^2 - k^6\big]\sinh(\sigma t)\Big]\sin(3kx),$$

$$\mathcal{D}(x,t) = 4\left[4\,k\cosh^2(\sigma\,t) - 4\,\sigma\,\cosh(\sigma\,t)\,\cos(k\,x) + k(4-k^2)\cos^2(kx)\right].$$

Sketch of the proof of completeness

The method was developed in the papers:

For eigenfunctions expansion:
Krichever, I.M., “The spectral theory of “finite-gap” non-stationary Schrödinger operators. The non-stationary Peierls model”, *Functional Anal. Appl.* **20** (1986), 203–214.

For eigenfunctions and squared eigenfunctions expansions:
Krichever, I.M., “Spectral theory of two-dimensional periodic operators and its applications”, *Russian Math. Surveys,* **44**:2 (1989), 145–225.

(The main example was the Kadomtsev-Petviashvili equation.)

For decaying at infinity boundary conditions the completeness of squared eigenfunctions for focusing NLS was proved in:

Kaup, D.J., “Closure of the Squared Zakharov-Shabat Eigenstates”, *Journal of Mathematical Analysis and Applications*, **54** (1976), 849–864.

Sketch of the proof of completeness

How to prove the convergence of the standard Fourier series?

Let us recall how it is done in the calculus textbooks.
Let $u(x) = u(x + 2\pi)$ be a 2π periodic sufficiently regular function.
The partial Fourier series:

$$u_n(x) = \sum_{k=-n}^{n} \hat{u}_j e^{ikx}$$

can be written as:

$$u_n(x) = \int_0^{2\pi} K_n(x-y)u(y)dy, \ \ K_n(x) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{ikx} = \frac{1}{2\pi}\frac{e^{(n+1/2)ix} - e^{-(n+1/2)ix}}{e^{(1/2)ix} - e^{-(1/2)ix}}$$

The Dirichlet kernel $K_n(x-y)$ admits the following representation:

$$K_n(x-y) = \frac{1}{2\pi} \oint_{|z|=n+1/2} \frac{e^{izx}e^{-izy}dz}{e^{2\pi iz} - 1}, \ \ z \in \mathbb{C}.$$

As analogous representation can be obtained in terms of squared NLS eigenfunctions.

Sketch of the proof of completeness

We consider symmetrically normalized eigenfunctions for the Lax pair:

$$\begin{bmatrix}\phi_1(\gamma,x,t)\\ \phi_2(\gamma,x,t)\end{bmatrix} = \frac{1}{\psi_1(\gamma,0,0)+\psi_2(\gamma,0,0)}\begin{bmatrix}\psi_1(\gamma,x,t)\\ \psi_2(\gamma,x,t)\end{bmatrix},$$

which are meromorphic on rational Riemann surface Γ_A with 2 double points. Γ_A is defined by equation

$$\nu^2 = (\lambda^2 + |a|^2)(\lambda^2 - \lambda_1^2)^2. \quad (5)$$

A point $\gamma \in \Gamma$ is a pair of complex numbers $\gamma = (\lambda, \nu) \in \mathbb{C}^2$, satisfying (5).

We also denote:

$$\nu = (\lambda^2 - \lambda_1^2)\,\mu, \quad \text{where} \;\; \mu^2 = \lambda^2 + |a|^2.$$

Let σ and τ be the following involutions on Γ_A

$$\sigma(\lambda,\nu) = (\lambda,-\nu), \;\; \tau(\lambda,\nu) = (\bar\lambda,\bar\nu).$$

Spectral curve for the Akhmediev breather

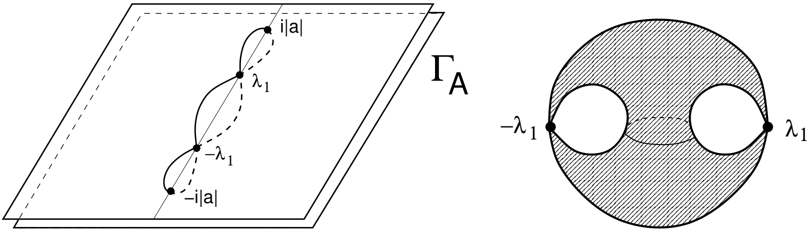


Figure: The curve Γ_A as a two-sheeted covering of the complex plane (left) and its topological model (right).

Γ_A is a two-sheeted covering of the λ-plane:

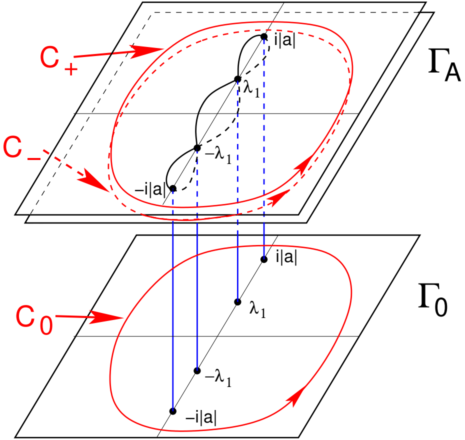
$$\Gamma_A \to \mathbb{C}: \quad (\lambda, \nu) \to \lambda.$$

It has two branch points

$$E_0: \ \lambda = i|a|, \quad \bar{E}_0: \ \lambda = -i|a|,$$

and two double points

Spectral curve for the Akhmediev breather squared eigenfunctions



$\tilde{\Gamma}_A = \Gamma_A \cup \Gamma_0$.
Γ_A is the Akhmediev breather spectral curve. $\Gamma_0 = \mathbb{C}P^1$ is the Riemann sphere.

Blue lines connect the glued pairs of points.
$(\lambda = i, \nu = 0) \leftrightarrow \lambda = i$,
$(\lambda = -i, \nu = 0) \leftrightarrow \lambda = -i$,
$(\lambda = \lambda_1, \nu = 0) \leftrightarrow \lambda = \lambda_1$,
$(\lambda = -\lambda_1, \nu = 0) \leftrightarrow \lambda = -\lambda_1$.

We integrate over $C_+ \cup C_-$. Integrals over C_0 are equal to 0.

Figure: The spectral curve for the squared eigenfunctions $\tilde{\Gamma}_A$ is obtained by gluing Γ_A with the Riemann sphere Γ_0.

The squared eigenfunctions

A vector-function $\vec{\Phi}(\gamma) = \vec{\Phi}(\gamma, x)$ on $\tilde{\Gamma}_A$ is defined by:

$$\vec{\Phi}(\gamma) = \begin{bmatrix} \Phi_1(\gamma) \\ \Phi_2(\gamma) \end{bmatrix} = \begin{cases} \begin{bmatrix} \phi_1^2(\gamma) \\ \phi_2^2(\gamma) \end{bmatrix} & \text{if } \gamma \in \Gamma_A, \\ \begin{bmatrix} \phi_1(\lambda,\nu)\phi_1(\lambda,-\nu) \\ \phi_2(\lambda,\nu)\phi_2(\lambda,-\nu) \end{bmatrix} & \text{if } \gamma \in \Gamma_0. \end{cases}$$

We calculated explicitly the corresponding Cherednik differential we have:

$$\tilde{\Omega}(\gamma) = \begin{cases} \frac{\left[\mu(\lambda^2-\lambda_1^2)+\lambda_1^2+(\mu_1^2+\lambda_1^2)\lambda^2\right]^2}{\mu^2(\lambda^2-\lambda_1^2)^2} d\lambda & \text{if } \gamma \in \Gamma_A, \\ -2\,\frac{\mu^2(\lambda^2-\lambda_1^2)^2+\left[\lambda_1^2+(\mu_1^2+\lambda_1^2)\lambda^2\right]^2}{\mu^2(\lambda^2-\lambda_1^2)^2} d\lambda & \text{if } \gamma \in \Gamma_0. \end{cases}$$

Let us define an analog of the Dirichlet kernel in the Fourier theory.

$$K^{(n)}(x,y) = \frac{1}{\pi} \oint\limits_{C_+\cup C_-} \begin{bmatrix} \Phi_1(\gamma,x) \\ \Phi_2(\gamma,x) \end{bmatrix} \left[-\bar{\Phi}_1(\tau\gamma,y), \quad \bar{\Phi}_2(\tau\gamma,y) \right] \frac{\tilde{\Omega}}{e^{2i\mu L}-1}, \quad (6)$$

$$C_+ \cup C_- = \{(\lambda,\nu) \in \Gamma_A : |\lambda| = R_n\}, \quad R_n = \sqrt{(n\pi/L)^2 - 1} + 1/2.$$

If $n \to \infty$, $K^{(n)}(x,y)$ coincide with the n-th Dirichlet kernel up to $O(1/n)$ corrections. On the other hand, $K^{(n)}(x,y)$ can be calculated as the sum of residues. The integrand in (6) has:

1. First-order poles at the resonant points $\gamma_m = (\lambda_m, \mu_m) = = (\pm\sqrt{(m\pi/L)^2 - 1}, m\pi/L)$, $2 \le m \le n$. The residues at these points are $\Phi_k(\gamma_m, x)\bar{\Phi}_k(\tau\gamma_m, y)$ times normalization constants;
2. Second-order poles at the branch points $\lambda = \pm i, \mu = 0$. The residues at these points are linear combinations of these products and their first derivatives with respect to the spectral parameter;
3. Third-order poles at the double points $\lambda = \pm\lambda_1, \nu = 0$. The residues at these points are linear combinations of these products and their first and second derivatives with respect to the spectral parameter.

Periodicity of the Dirichlet-type kernel

Where we use the properties of the Cherednik differential. We have to check that

$$K^{(n)}(x,y) = \frac{1}{\pi}\oint\limits_{C_+\cup C_-} \begin{bmatrix}\Phi_1(\gamma,x)\\ \Phi_2(\gamma,x)\end{bmatrix}\left[-\bar{\Phi}_1(\tau\gamma,y), \quad \bar{\Phi}_2(\tau\gamma,y)\right]\frac{\tilde{\Omega}}{e^{2i\mu L}-1},$$

is L=periodic in x and y. But:

$$K^{(n)}_{lm}(x+L,y) - K^{(n)}_{lm}(x,y) = \frac{(-1)^m}{\pi}\oint\limits_{\gamma\in\Gamma_A, |\lambda|=R_N} \Phi_l(\gamma,x)\bar{\Phi}_m(\tau\gamma,y)\tilde{\Omega},$$