On the Bojarskii–Meyers Estimates

Gregory A. Chechkin

Department of Differential Equations
Faculty of Mechanics and Mathematics
M.V.Lomonosov Moscow State University

.

Integrable Systems and Geometry

December 21, 2021

Short History

Higher integrability of the gradient or Bojarskii–Meyers estimate has the form

$$\int_{\Omega} |\nabla u|^{2+\delta} dx \leqslant C \int_{\Omega} |f|^{2+\delta}\, dx,$$

where u is a solution to a boundary value problem for the second order elliptic equation with right-hand side f, in bounded domain Ω.

Short History

The following paper
[1] B.V. Bojarskii, Generalized solutions to a system of first-order differential equations of elliptic type with discontinuous coefficients // Math. Sbornik, V. 43(85) (4, 1957). P. 451–503.
is the first publication in the topic. In this article the author showed, that the gradient of the solution to the Dirichlet problem for the divergent uniformly elliptic equations with measurable coefficients in bounded domain, is integrable in the power greater than two.

Short History

Later, in the multidimensional case for equations of the same type, the increased summability of the gradient of the solution of the Dirichlet problem in a domain with a sufficiently regular boundary was established in the work

[2] N. G. Meyers, An L^p–estimate for the gradient of solutions of second order elliptic divergence equations // Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3-e série. T. 17, (3, 1963). P. 189–206.

Subsequently, similar results were obtained for the Neumann problem.

Short History

We also note that the increased summability of the gradient of solutions to the Dirichlet problem in a domain with a Lishitz boundary for the p-Laplace equation with a variable exponent $p(x)$ satisfying special conditions on the modulus of continuity was obtained in the paper

[3] V.V. Zhikov, On some Variational Problems // Russian Journal of Mathematical physics, V. 5 (1, 1997). P. 105–116.

Note that V.V. Zhikov's study of the Meyers estimates was stimulated by the problem of a thermistor, which gives a joint description of the electric field potential and temperature. Systems of the same kind arise in the hydromechanics of quasi-Newtonian fluids.

Short History

Later, in the papers

[4] E. Acerbi, G. Mingione. Gradient estimates for the $p(x)$-Laplacian system. // J. Reine Angew. Math. 2005. V. 584. P. 117–148.

[5] L. Diening, S. Schwarzsacher. Global gradient estimates for the $p(.)$-Laplacian. // Nonlinear Anal. 2014. V. 106. P. 70–85.

this result was strengthened and extended to systems of elliptic equations with variable summability exponent.

Short History

For the Laplace equation, the mixed Zaremba problem formulated by W. Wirtinger, in a three-dimensional bounded domain with a smooth boundary and inhomogeneous Dirichlet and Neumann conditions was first considered in the work

[6] S. Zaremba. A mixed problem related to the Laplace equation. // Russian Mathematical Surveys. 1946. V.1. No 3–4 (13–14). P. 125–146.

The classical solvability of the problem was established by the methods of potential theory under the assumption that the boundary of the open set on which the Neumann data are given also has a certain smoothness.

Short History

The study of the properties of solutions to the Zaremba problem for second-order elliptic equations with variable regular coefficients goes back to the work

[7] G. Fichera. Sul problema misto per le equazioni lineari alle derivate parziali del secondo ordine di tipo ellittico (Italian) // Rev. Roumaine Math. Pures Appl. 1964. V. 9. P. 3–9.

In it, in particular, it was established that at the junction of the Dirichlet and Neumann data, the smoothness of the solutions is lost.

Short History

For divergent uniformly elliptic second-order equations with measurable coefficients, integral and pointwise estimates for solutions of the Zaremba problem under fairly general assumptions about the boundary of the domain are given in

[8] V.G. Mazya. Some estimates for solutions of second-order elliptic equations. // The USSR Academy of Sciences. Doklady. Mathematics. 1961. V. 137. No 5. P. 1057–1059.

Short History

Homogenization of rapidly oscillating Zaremba problem have been studied in the papers

[9] A. Damlamian, Li Ta-Tsien (Li Daqian). Boundary Homogenization for Elliptic Problems. // J.Math.Pure et Appl. 1987. V. 66. P. 351–361.

[10] G.A. Chechkin. On Boundary — Value Problems for a second — order Elliptic Equation with Oscillating Boundary Conditions. // Nonclassical Partial Differential Equations, Ed. Vladimir N.Vragov. Novosibirsk: IM SOAN SSSR, 1988, P. 95–104. (Reported in Referent. Math., 1989, 12B442, p.62)

[11] M. Lobo, M.E. Pérez. Asymptotic Behavior of an Elastic Body With a Surface Having Small Stuck Regions. // Math Modelling Numerical Anal. V. 22. № 4. 1988. P. 609–624.

Short History

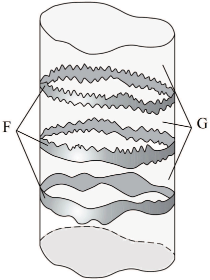
The question of Meyers-type estimates for solutions of the Zaremba problem has practically not been studied. In the papers

[12] Yu.A. Alkhutov, G.A. Chechkin. Increased Integrability of the Gradient of the Solution to the Zaremba Problem for the Poisson Equation. // Russian Academy of Sciencies. Doklady Mathematics 103 (2, 2021): 69–71.

[13] Yu.A. Alkhutov, G.A. Chechkin, The Meyer's Estimate of Solutions to Zaremba Problem for Second-order Elliptic Equations in Divergent Form // C R Mécanique, T. 349 (2, 2021). P. 299–304.

for the elliptic equation of the second order, an estimate is obtained for the increased integrability of the gradient of the solution to the Zaremba problem in a domain with a Lipschitz boundary and a rapid change of the Dirichlet and Neumann boundary conditions.

Examples of the Domains



Cylinder with "rings"

Examples of the Domains

Spots

Examples of the Domains

Fractals

Setting of the problem

This work is connected with estimates of solutions to the Zaremba problem for elliptic equation in bounded Lipschitz domain $D \in \mathbb{R}^n$, where $n > 1$, of the form

$$\mathcal{L}u := \operatorname{div}(a(x)\nabla u) \tag{1}$$

with uniformly elliptic measurable and summetric matrix $a(x) = \{a_{ij}(x)\}$, i.e. $a_{ij} = a_{ji}$ and

$$\alpha^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \alpha|\xi|^2 \text{ for almost all } x \in D \text{ and all } \xi \in \mathbb{R}^n. \tag{2}$$

We assume that $F \subset \partial D$ is closed and $G = \partial D \setminus F$.

Setting of the problem

Consider the Zaremba problem

$$\begin{cases} \mathcal{L}u = \operatorname{div} f \quad \text{in} \quad D, \\ u = 0 \quad \text{on} \quad F, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad G, \end{cases} \tag{3}$$

where $\frac{\partial u}{\partial \nu}$ is the outer conormal derivative of u, and the components of the vector-function $f = (f_1, \dots, f_n)$ are functions from $L_2(D)$.

Setting of the problem

Denote by $W_2^1(D, F)$ the completion of the set of infinitely differentiable in the closure of D functions vanishing in the vicinity of F, by the norm

$$\| u \|_{W_2^1(D,F)} = \left(\int_D u^2 \, dx + \int_D |\nabla u|^2 \, dx \right)^{1/2}.$$

Setting of the problem

By the solution of the problem (3) we mean the function $u \in W_2^1(D, F)$ for which the integral identity

$$\int\limits_D a\nabla u \cdot \nabla\varphi\, dx = \int\limits_D f \cdot \nabla\varphi\, dx \tag{4}$$

holds for all test-functions $\varphi \in W_2^1(D, F)$.

Auxiliaries

We are interested in the question of increased summability (integrability) of the gradient of solutions to the problem (3). The conditions on the structure of the set of the Dirichlet data support F playes the key role.

For the compact $K \subset \mathbb{R}^n$ we define the capacity $C_p(K)$, $1 < p < n$, by the formula

$$C_p(K) = \inf \left\{ \int\limits_{\mathbb{R}^n} |\nabla \varphi|^p \, dx : \ \varphi \in C_0^\infty(\mathbb{R}^n), \ \varphi \geqslant 1 \text{ on } K \right\}. \quad (5)$$

Auxiliaries

Suppose $B_r^{x_0}$ is an open ball of the radius r centered in x_0, and $mes_{d-1}(E)$ is $(d-1)$-measure of the set E. Assume also that $p = 2n/(n+2)$ as $n > 2$ and $p = 3/2$ as $n = 2$. We suppose one of the following conditions is fulfilled: for an arbitrary point $x_0 \in F$ as $r \leqslant r_0$ the inequality

$$C_p(\partial F \cap \overline{B}_r^{x_0}) \geqslant c_0 r^{n-p} \tag{6}$$

holds true or the inequality

$$mes_{n-1}(\partial F \cap \overline{B}_r^{x_0}) \geqslant c_0 r^{n-1} \tag{7}$$

holds, in which the positive constant c_0 does not depend on x_0 and r.

Auxiliaries

The condition (7) is stronger, than (6), but it is clearer. Note that under any of these conditions, the functions $v \in W_2^1(D, F)$ satisfy the Friedrichs inequality

$$\int\limits_D v^2\,dx \leqslant K \int\limits_D |\nabla v|^2\,dx,$$

which, by the Lax-Milgram theorem, implies the unique solvability of the problem (3).

Main result

Theorem

If $f \in L_{2+\delta_0}(D)$, where $\delta_0 > 0$, then there exist positive constants $\delta(n, \delta_0) < \delta_0$ and C, such that for a solution to the problem (3) the estimate

$$\int\limits_D |\nabla u|^{2+\delta} dx \leqslant C \int\limits_D |f|^{2+\delta} \, dx, \qquad (8)$$

holds, where C depends only on δ_0, the dimension n, constant c_0 from (6) and (7), and also the constant r_0.

How to prove

The proof of this statement is based on the inner and boundary bounds for the increased integrability of the gradient of solutions to the problem (3). First, an estimate for the increased integrability is established in a neighborhood of the boundary of the domain D. Here the technique of local straightening of the boundary ∂D is used.

How to prove

In a vicinity of an arbitrary point $x_0 \in \partial D$ we consider a cube

$$Q_{R_0} = \{x : \ |x_i - x_{0i}| < R_0, \ i = 1, \dots, n\}$$

and change the variables straightening the boundary

The transformation of the cube Q_{R_0} to the $\widetilde{Q}_{R_0}$ and inscribed cube K_{R_0}

How to prove

Lemma

The domain $\widetilde{Q}_{R_0}$ *contains the cube*

$$K_{R_0} = \{y : \ |y_i| < (1 + \sqrt{n-1}L)^{-1} R_0, \ i = 1, \dots, n\}. \qquad (9)$$

How to prove

The given problem becomes a problem in the semi-cube $K^+_{R_0} = K_{R_0} \cap \widetilde{D}$, and has the form

$$\begin{cases} \widetilde{\mathcal{L}} v = \operatorname{div} \widetilde{f} \quad \text{in} \quad K^+_{R_0}, \\ v = 0 \text{ on } \widetilde{F} \cap K_{R_0}, \\ \frac{\partial v}{\partial \widetilde{\nu}} = 0 \text{ on } \widetilde{G} \cap K_{R_0}. \end{cases} \tag{10}$$

Here $\widetilde{\mathcal{L}} v := \operatorname{div}(b(y)\nabla v)$ is an elliptic operator.

How to prove

Let us continue the solution v of the problem (10) even with respect to the hyperplane $\{y: \ y_n = 0\}$. Retaining the notation for the extended function, we obtain the problem

$$\begin{cases} \mathcal{L}_1 v = \operatorname{div} h \quad \text{in} \quad K_{R_0} \setminus (\widetilde{F} \cap K_{R_0}), \\ v = 0 \quad \text{on} \quad \widetilde{F} \cap K_{R_0}. \end{cases} \tag{11}$$

How to prove

Here

$$\widetilde{\mathcal{L}}_1 u := \operatorname{div}(c(y)\nabla u),$$

positive definite matrix $c = \{c_{ij}(y)\}$ is such, that the elements $c_{jn}(y) = c_{jn}(y)$ for $j \neq n$ are odd continuation of the elements $b_{jn}(y)$, and all other elements of $c_{ij}(y)$ are an even continuation of $b_{ij}(y)$. The vector-function $h = (h_1, \dots, h_n)$ in (11) is defined by similar equalities: its components $h_i(y)$ for $i = 1, \dots, n-1$ are even extensions of the components $\widetilde{f}_i(y)$ from (10), and $h_n(y)$ is an odd extension of $\widetilde{f}_n(y)$.

How to prove

Let $Q_R^{y_0}$ be a cube centered in y_0 with edges of the length $2R$, parallel to the coordinate axes and

$$y_0 \in K_{\frac{R_0}{2}} \setminus \partial K_{\frac{R_0}{2}}, \text{ where } R \leqslant \frac{1}{2} dist(y_0, \partial K_{\frac{R_0}{2}}).$$

Denote

$$\fint_{Q_R^{y_0}} w\, dx = \frac{1}{|Q_R^{y_0}|} \int_{Q_R^{y_0}} w\, dx, \qquad \lambda = \fint_{Q_{\frac{3R}{2}}^{y_0}} v,\ dy.$$

where $|Q_R^{y_0}|$ is n-dimensional volume of the cube $Q_R^{y_0}$.

How to prove

Assume that $Q^{y_0}_{\frac{3R}{2}} \subset K_{R_0}$. We take the test-function

$$\varphi = (v - \lambda)\eta^2$$

in the integral identity of the given problem.
Here the cut-off function $\eta \in C_0^\infty(Q^{y_0}_{\frac{3R}{2}})$ satisfies

$$0 < \eta \leqslant 1, \quad \eta = 1 \text{ in } Q^{y_0}_R \qquad \text{and} \qquad |\nabla \eta| \leqslant \frac{C}{R}. \qquad (12)$$

How to prove

Lemma

For the solution v to the problem (11) the Caccioppoli inequality of the form

$$\int_{Q_R^{y_0}} |\nabla v|^2\,dy \leqslant C(n,\alpha,L)\left(\frac{1}{R^2}\int_{Q_{\frac{3R}{2}}^{y_0}} (v-\lambda)^2\,dy + \int_{Q_{\frac{3R}{2}}^{y_0}} |h|^2\,dy\right) \quad (13)$$

is valid

Here α is the ellipticity constant.

How to prove

Then, using the Poincaré–Sobolev inequality

$$\left(\fint_{Q^{y_0}_{\frac{3R}{2}}} (v-\lambda)^2\,dx \right)^{1/2} \leqslant C(n,p)R \left(\fint_{Q^{y_0}_{\frac{3R}{2}}} |\nabla v|^p\,dx \right)^{1/p}$$

with $p \geqslant \frac{2n}{n+2}$ and the Caccioppoli inequality (13), we derive

$$\left(\fint_{Q^{y_0}_{R}} |\nabla v|^2\,dy \right)^{1/2} \leqslant C\left(\left(\fint_{Q^{y_0}_{2R}} |\nabla v|^p\,dy \right)^{1/p} + \left(\fint_{Q^{y_0}_{2R}} |h|^2\,dy \right)^{1/2} \right). \tag{14}$$

How to prove

If $Q^{y_0}_{\frac{3R}{2}} \cap (\widetilde{F} \cap K_{R_0}) \neq \emptyset$, then we apply the Friedrichs–Sobolev inequality

$$\left(\fint_{Q^{y_0}_{2R}} v^2\, dy \right)^{1/2} \leqslant C(n, p, L, c_0) R \left(\fint_{Q^{y_0}_{2R}} |\nabla v|^p\, dy \right)^{1/p}. \qquad (15)$$

How to prove

Now, applying the Gehring Lemma, we get

$$\int\limits_{K_{\frac{R_0}{4}}} |\nabla v|^{2+\delta}\, dy \leqslant C(n,\alpha,\delta_0,c_0,L,R_0) \int\limits_{K_{\frac{R_0}{2}}} |h|^{2+\delta}\, dy$$

if $h \in L_{2+\delta_0}(K_{R_0})$, $\delta_0 > 0$, or

$$\int\limits_{K^+_{\frac{R_0}{4}}} |\nabla v|^{2+\delta}\, dy \leqslant C(n,\alpha,\delta_0,c_0,L,R_0) \int\limits_{K^+_{\frac{R_0}{2}}} |\widetilde{f}|^{2+\delta}\, dy. \qquad (16)$$

How to prove

Returning to the original variables, we get

$$\int\limits_{D\cap Q^{x_0}_{\mu R_0}} |\nabla u|^{2+\delta}\,dx \leqslant C(d,\alpha,\delta_0,c_0,L,R_0) \int\limits_{D\cap Q^{x_0}_{R_0}} |f|^{2+\delta}\,dx.$$

We take the finite subcovering and sum over such domains.

The estimates inside the domain are obtained in a standard way. Summing up all the estimates, we obtain the required inequality (22).

How to apply

Denote by M_ε the number of the Dirichlet parts F^j, $F = \bigcup_{j=1}^{M_\varepsilon} F^j$.

Consider in D the problem

$$\begin{cases} -\Delta u = f \quad \text{in} \quad D, \\ \frac{\partial u}{\partial n} + au = 0 \quad \text{on} \quad G, \\ u = 0 \quad \text{on} \quad F \end{cases} \tag{17}$$

and the limit problem

$$\begin{cases} -\Delta u_0 = f \quad \text{in} \quad D, \\ \frac{\partial u_0}{\partial n} + au_0 = 0 \quad \text{on} \quad \partial D. \end{cases} \tag{18}$$

How to apply

We estimate the rate of convergence $u \to u_0$ as $\varepsilon \to 0$.

1) The family $\|u\|$ is bounded, hence there exists a weak limit $u \rightharpoonup u_0$.

2) Cut–off $\psi_\varepsilon = \prod\limits_k \psi_\varepsilon^k$, $\psi_\varepsilon^k = \psi\left(\frac{|\ln \varepsilon|}{|\ln r_k|}\right)$, $\psi(s) = \begin{cases} 0, s \leqslant 1, \\ 1, s \geqslant 1 + \sigma. \end{cases}$

3) Take $\varphi_\varepsilon = \varphi\psi_\varepsilon$ as a test-function, subtract one integral identity from another. We have

$$\int\limits_D (\psi_\varepsilon \nabla u - \nabla u_0) \cdot \nabla\varphi \, dx + \int\limits_{\partial D} a(u - u_0)\varphi \, ds =$$

$$= \int\limits_D f \cdot \nabla\varphi(\psi_\varepsilon - 1) \, dx + \int\limits_D \nabla u \cdot \nabla\psi_\varepsilon \varphi \, dx + \int\limits_D f \cdot \nabla\psi_\varepsilon \varphi \, dx. \tag{19}$$

How to apply

Keeping in mind the equivalence of the norms in the Sobolev space, we derive

$$\|u - u_0\|^2_{W_2^1(D)} \leqslant C\left(\int\limits_D f \cdot \nabla\varphi(\psi_\varepsilon - 1)\, dx + \int\limits_D \nabla u \cdot \nabla \psi_\varepsilon\, dx \right). \tag{20}$$

The first term in the right hand side of the inequality (20) is estimated by

$$K\, M_\varepsilon^{\frac{1}{2}}\, \varepsilon^{\frac{1}{1+\sigma}}.$$

Here $\varepsilon^{\frac{1}{1+\sigma}}$ is the diameter of the ball, where $\psi_\varepsilon - 1 \neq 0$.

4) Next, we estimate $\int\limits_D (\nabla u, \nabla \psi_\varepsilon)\; dx$.

How to apply

I

$$\int\limits_D (\nabla u, \nabla \psi_\varepsilon)\, dx \leqslant \Big(\int\limits_D |\nabla u|^2\, dx\Big)^{\frac{1}{2}} \Big(\int\limits_D |\nabla \psi_\varepsilon|^2\, dx\Big)^{\frac{1}{2}} \leqslant$$

$$\leqslant K_1 M_\varepsilon^{\frac{1}{2}} |\ln \varepsilon| \Big(\int\limits_{\varepsilon}^{\varepsilon^{\frac{1}{1+\sigma}}} |\ln r|^{-4} d \ln r \Big)^{\frac{1}{2}} \leqslant K_2 M_\varepsilon^{\frac{1}{2}} |\ln \varepsilon|^{-\frac{1}{2}}.$$

$$M_\varepsilon = |\ln \varepsilon|^{1-\theta}, \qquad 0 < \theta < 1.$$

How to apply

II $\qquad p_1 = 2+\delta > 2, \quad p_2 = \frac{2+\delta}{1+\delta} < 2.$

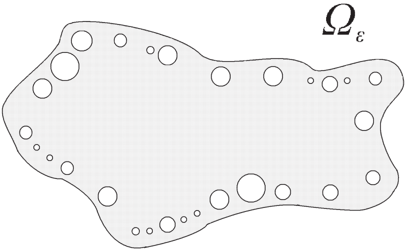
$$\int\limits_D (\nabla u, \nabla \psi_\varepsilon)\, dx \leqslant \Big(\int\limits_D |\nabla u|^{p_1}\, dx\Big)^{\frac{1}{p_1}} \Big(\int\limits_D |\nabla \psi_\varepsilon|^{p_2}\, dx\Big)^{\frac{1}{p_2}} \leqslant$$

$$\leqslant K_1 M_\varepsilon^{\frac{1}{p_2}} \varepsilon^{\frac{2-p_2}{p_2(1+\sigma)}} |\ln \varepsilon| \Big(\int\limits_\varepsilon^{\varepsilon^{\frac{1}{1+\sigma}}} |\ln r|^{-2p_2} d \ln r \Big)^{\frac{1}{p_2}} \leqslant K_2 M_\varepsilon^{\frac{1}{p_2}} \varepsilon^{\frac{2-p_2}{p_2(1+\sigma)}} |\ln \varepsilon|^{\frac{1}{p_2}-1}$$

$$M_\varepsilon = \varepsilon^{-\frac{\delta}{(1+\delta)(1+\sigma)}} |\ln \varepsilon|^{\frac{1}{1+\delta}-\theta}, \qquad 0 < \theta < \frac{1}{1+\delta}.$$

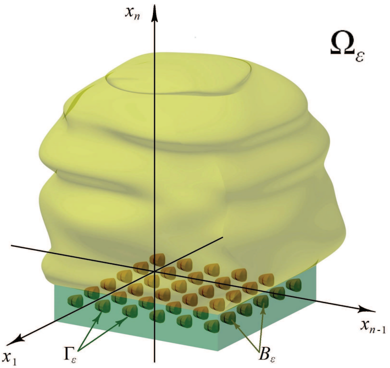
Domains perforated along the boundary

We consider the analogous problem in the domain $\Omega_\varepsilon \in \mathbb{R}^n$, where $n > 1$, perforated along the boundary.



Perforated domain

Domains perforated along the boundary



Perforated domain

Settings

Consider the following problem in perforated domain $\Omega_\varepsilon := \Omega \setminus \overline{B_\varepsilon}$:

$$\Delta u_\varepsilon = \operatorname{div} f \text{ in } \Omega_\varepsilon, \quad u_\varepsilon = 0 \text{ on } \partial B_\varepsilon, \quad \frac{\partial u_\varepsilon}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{21}$$

Main result

Theorem

If $f \in L_{2+\delta_0}(\Omega)$, *where* $\delta_0 > 0$, *then there exist positive constants* $\delta(n, \delta_0) < \delta_0$ *and* C, *such that for a solution to the problem* (21) *the estimate*

$$\int\limits_{\Omega_\varepsilon} |\nabla u|^{2+\delta} dx \leqslant C \int\limits_{\Omega_\varepsilon} |f|^{2+\delta}\, dx, \qquad (22)$$

holds, where C *depends only on* δ_0, *the dimension* n, *constant* c_0 *from* (6) *and* (7), *and also the constant* r_0.

Thanks for the attention!