Non-diagonalisable Hydrodynamic Type Systems, Integrable by Tsarev's Generalised Hodograph Method

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The Talk is based on joint works with my friends and colleagues:
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Tsarev's Generalised Hodograph Method

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$$W(\mathbf{u}) = I\, x + V(\mathbf{u})\, t,$$

where I is the $N \times N$ identity matrix, defines an implicit solution $u^k(x, t)$.

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Then such systems we call integrable by Tsarev's Generalised Hodograph Method.

A Nijenhuis tensor

Recall that, given an affinor $V_k^i(\mathbf{u})$, its Haantjes tensor is defined by the formula

$$H_{jk}^i = N_{pr}^i V_j^p V_k^r - N_{jr}^p V_p^i V_k^r - N_{rk}^p V_p^i V_j^r + N_{jk}^p V_r^i V_p^r,$$

where (here $\partial_p \equiv \partial/\partial u^p$)

$$N_{jk}^i = V_j^p \partial_p V_k^i - V_k^p \partial_p V_j^i - V_p^i (\partial_j V_k^p - \partial_k V_j^p)$$

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In a generic case all characteristic velocities μ^k are *pairwise distinct*. If all components of a Nijenhuis tensor vanish, then corresponding hydrodynamic type system

$$u_t^i = V_k^i(\mathbf{u}) u_x^k$$

can be reduced to the totally decoupled form

$$\tilde{u}_t^i = \mu^i(\tilde{u}^i) \tilde{u}_x^i$$

by an appropriate invertible point transformation $\tilde{u}^k(\mathbf{u})$.

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So a Haantjes tensor is defined by the formula

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In a generic case all characteristic velocities μ^k are *pairwise distinct*. If all components of a Haantjes tensor vanish, then corresponding hydrodynamic type system

$$u^i_t = V^i_k(\mathbf{u}) u^k_x$$

can be diagonalised, i.e. rewritten in the Riemann invariants

$$r^i_t = \mu^i(\mathbf{r}) r^i_x$$

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The **Statement**: *If a hydrodynamic type system is integrable by Tsarev's Generalised Hodograph Method, then all components of a Haantjes tensor vanish. Then this hydrodynamic type system can be reduced to a* **block-diagonal** *structure by an appropriate invertible point transformation* $\tilde{u}^k(\mathbf{u})$.

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The Darboux system is a 3D integrable system. Indeed, it is easy to see for every three distinct indices:

$$\partial_{r^1} \beta_{23} = \beta_{21} \beta_{13}, \quad \partial_{r^1} \beta_{32} = \beta_{31} \beta_{12},$$
$$\partial_{r^2} \beta_{13} = \beta_{12} \beta_{23}, \quad \partial_{r^2} \beta_{31} = \beta_{32} \beta_{21},$$
$$\partial_{r^3} \beta_{12} = \beta_{13} \beta_{32}, \quad \partial_{r^3} \beta_{21} = \beta_{23} \beta_{31}.$$

Special Example

Let us introduce the $N \times N$ matrix $\hat{\epsilon}$ with diagonal entries $r^1, \dots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries ϵ^{ik} =const, $k \neq i$.

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Indeed, differentiation of $\beta_{km} \epsilon^{mi} = -\delta^i_k$ with respect to any variable r^j implies

$$0 = \beta_{im} \partial_j \epsilon^{mk} + (\partial_j \beta_{im}) \epsilon^{mk}.$$

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Taking into account $\beta_{km}\epsilon^{mi} = -\delta^i_k$, finally we have

$$\partial_j \beta_{is} = \beta_{im} (\partial_j \epsilon^{mk}) \beta_{ks} \equiv \beta_{ij} \beta_{js}.$$

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Multiplying both sides of the above nonlinear system $\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}$ by ϵ^{pj} from the left, and by ϵ^{kq} from the right, we obtain

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This means: the general solution of the system $\partial_i \epsilon^{pq} = \delta^p_i \delta^q_i$ is determined by the $N \times N$ matrix $\hat{\boldsymbol{\epsilon}}$ with diagonal entries $r^1, \ldots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} =$const, $k \neq i$.

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However, we can follow to an alternative strategy.
One can select any pair of particular solutions $\bar{H}_i$ and $\tilde{H}_i$ of the first linear system

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Here we remind that independent variables are r^k. So, $\partial_k \equiv \partial / \partial r^k$.

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Here we remind that independent variables are r^k. So, $\partial_k \equiv \partial/\partial r^k$.
Now we introduce an N component hydrodynamic type system

$$r^i_t = \mu^i(\mathbf{r}) r^i_x,$$

whose characteristic velocities

$$\mu^i(\mathbf{r}) = \frac{\tilde{H}_i}{\bar{H}_i}.$$

This hydrodynamic type system is integrable by Tsarev's Generalised Hodograph Method. In this construction: Riemann invariants r^k are functions of two independent variables x and t only.

Any diagonalisable hydrodynamic type system

$$r_t^i = \mu^i(\mathbf{r}) r_x^i, \quad i = 1, 2, ..., N$$

is integrable by Tsarev's Generalised Hodograph Method

$$x + \mu^i(\mathbf{r}) t = \zeta^i(\mathbf{r}),$$

if and only if the integrability condition (here $\partial_k \equiv \partial / \partial r^k$)

$$\partial_j \frac{\partial_k \mu^i}{\mu^k - \mu^i} = \partial_k \frac{\partial_j \mu^i}{\mu^j - \mu^i}, \quad i \neq j \neq k$$

is fulfilled. Here we remind that diagonal metric coefficients $g_{kk}(\mathbf{r}) = \bar{H}_k^2$ are determined by

$$\partial_k \ln \bar{H}_i = \frac{\partial_k \mu^i}{\mu^k - \mu^i}, \quad i \neq k,$$

while $\zeta^i(\mathbf{r})$ satisfy to the linear system

$$\partial_k \zeta^i = \frac{\partial_k \mu^i}{\mu^k - \mu^i} (\zeta^k - \zeta^i), \quad i \neq k.$$

Commuting Flows

Integrable N component hydrodynamic type system

$$r_t^i = \mu^i(\mathbf{r}) r_x^i$$

has infinitely many commuting flows (τ is the so called group parameter in the Lie group analysis, or an auxiliary time variable)

$$r_\tau^i = \zeta^i(\mathbf{r}) r_x^i.$$

This means, that the Riemann invariants r^i no longer depend on **two** independent variables x and t only. Now, the Riemann invariants r^i depend on three independent variables x, t, τ simultaneously.

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This means, that the Riemann invariants $r^i(x, t, \tau)$ solve two N component hydrodynamic type systems

$$r^i_t = \mu^i(\mathbf{r})r^i_x, \quad r^i_\tau = \zeta^i(\mathbf{r})r^i_x,$$

where the time variable τ is hidden in the first hydrodynamic type system, while the time variable t is hidden in the second hydrodynamic type system. Then both hydrodynamic type systems must commute with each other.

Commuting Flows

The compatibility conditions $(r_t^i)_\tau = (r_\tau^i)_t$ lead to the Tsarev conditions

$$\frac{\partial_k \mu^i}{\mu^k - \mu^i} = \frac{\partial_k \zeta^i}{\zeta^k - \zeta^i}, \quad i \neq k.$$

Taking into account the definition of the Lame coefficients

$$\partial_k \ln \bar{H}_i = \frac{\partial_k \mu^i}{\mu^k - \mu^i}, \quad i \neq k,$$

the Tsarev conditions show that both commuting hydrodynamic type systems

$$r_t^i = \mu^i(\mathbf{r}) r_x^i, \quad r_\tau^i = \zeta^i(\mathbf{r}) r_x^i$$

have the same diagonal metric $g_{kk}(\mathbf{r}) = \bar{H}_k^2$.

El's Nonlocal Kinetic Equation

El's integro-differential kinetic equation for dense soliton gas (2003)

$$f_t + (sf)_x = 0,$$

$$s(\eta) = S(\eta) + \int_0^\infty G(\mu,\eta) f(\mu)[s(\mu) - s(\eta)]\, d\mu,$$

where $f(\eta) = f(\eta, x, t)$ is a distribution function and $s(\eta) = s(\eta, x, t)$ is the associated transport velocity. Here the variable η is the spectral parameter in the Lax pair; the function $S(\eta)$ (free soliton velocity) and the kernel $G(\mu,\eta)$ (phase shift due to pairwise soliton collisions) are independent of x and t. The kernel $G(\mu,\eta)$ is assumed to be symmetric: $G(\mu,\eta) = G(\eta,\mu)$. This system describes the evolution of a dense soliton gas and represents a broad generalisation of Zakharov's kinetic equation for rarefied soliton gas. In this case

$$S(\eta) = 4\eta^2, \qquad G(\mu,\eta) = \frac{1}{\eta\mu} \log\left|\frac{\eta-\mu}{\eta+\mu}\right|,$$

the above system was derived by G. El as thermodynamic limit of the KdV Whitham equations

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (an **iso-spectral** case, 2010, G.A. El, A.M. Kamchatnov, MVP, S.A. Zykov),

$$f(\eta, x, t) = \sum_{i=1}^{N} u^i(x, t)\ \delta(\eta - \eta^i),$$

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where $v^i = -s(\eta^i, x, t)$ can be recovered from the linear system (here $\xi^i = -S(\eta^i)$)

$$v^i = \xi^i + \sum_{m \neq i} \epsilon^{mi} u^m (v^m - v^i), \qquad \epsilon^{ki} = G(\eta^k, \eta^i),\ k \neq i.$$

Parametrisation

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In these dependent variables r^i, the quasilinear system

$$u_t^i = (u^i v^i)_x,$$

reduces to a diagonal form

$$r_t^i = v^i r_x^i,$$

where velocities v^i can be expressed in terms of Riemann invariants as follows.

Tsarev's Generalised Hodograph Method

Let us introduce the $N \times N$ matrix $\hat{\epsilon}$ with diagonal entries $r^1, \ldots, r^N$ (so that $\epsilon^{ii} = r^i$) and off-diagonal entries $\epsilon^{ik} = G(\eta^i, \eta^k), \; k \neq i$.

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$$x + \xi_i t = P_i(r^i) - r^i P_i'(r^i) - \sum_{m \neq i} \epsilon^{mi} P_m'(r^m), \quad i = 1, 2, ..., N,$$

where $P_i(r^i)$, $i = 1, \dots, N$, are arbitrary functions.

Linearly Degenerate Diagonalisable Hydrodynamic Type Systems

We call a diagonal system

$$r_t^i = v^i r_x^i, \quad i = 1, 2, ..., N$$

linearly degenerate if (for every index i. Here: no summation!)

$$\partial_i v^i = 0, \quad i = 1, 2, ..., N.$$

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If a linearly degenerate hydrodynamic type system is semi-Hamiltonian, then such a system is the so called **Darboux integrable**. Their classification was made by E.V. Ferapontov (1991).

Tsarev's Generalised Hodograph Method

Under the re-parametrization

$$P_k''(\xi) = -\frac{\phi_k(\xi)}{f(\xi)}$$

the generalized hodograph solution

$$x + \xi_i t = P_i(r^i) - r^i P_i'(r^i) - \sum_{m \neq i} \epsilon^{mi} P_m'(r^m), \quad i = 1, 2, ..., N,$$

becomes

$$x + \xi_i t = \int^{r^i} \frac{\xi \phi_i(\xi)}{f(\xi)} d\xi + \sum_{m \neq i} \epsilon^{mi} \int^{r^m} \frac{\phi_m(\xi)}{f(\xi)} d\xi \, .$$

Tsarev's Generalised Hodograph Method

Now we consider the particular choice of $f(\xi)$ defined as $f(\xi) = \sqrt{R_K(\xi)}$, where

$$R_K(\xi) = \prod_{m=1}^{K} (\xi - E_m),$$

and $E_1 < E_2 < \cdots < E_K$ are real constants ($K = 2N + 1$ and $K = 2N + 2$ for odd and even number of branch points of this hyperelliptic curve of a genus N); and $\phi_k(\xi)$ being arbitrary polynomials in ξ of degrees less than N.

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describes quasiperiodic solutions of the form

$$x + \xi_i t = \int^{r^i} \frac{\xi \phi_i(\xi) d\xi}{\sqrt{R_K(\xi)}} + \sum_{m \neq i} \epsilon^{mi} \int^{r^m} \frac{\phi_m(\xi) d\xi}{\sqrt{R_K(\xi)}}, \quad i = 1, 2, ..., N.$$

Hydrodynamic Reductions. Dirac Delta-Functional Ansatz

Under a delta-functional ansatz (a **non-isospectral** case, 2012, G.A. El, V.B. Taranov, MVP),

$$f(\eta, x, t) = \sum_{i=1}^{N} u^i(x, t)\ \delta(\eta - \eta^i(x, t)),$$

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$$s(\eta) = S(\eta) + \int_0^\infty G(\mu, \eta) f(\mu)[s(\mu) - s(\eta)]\ d\mu,$$

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where $v^i = -s(\eta^i, x, t)$ can be recovered from the linear system (here $\xi^i = -S(\eta^i)$)

$$v^i = \xi^i + \sum_{m \neq i} \epsilon^{mi} u^m (v^m - v^i), \qquad \epsilon^{ki} = G(\eta^k, \eta^i),\ k \neq i.$$

Block-Diagonal Hydrodynamic Type Systems

Introducing new field variables

$$r^i = -\frac{1}{u^i}\left(1 + \sum_{m \neq i} \epsilon^{mi} u^m\right),$$

this $2N \times 2N$ quasilinear system

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this $2N \times 2N$ quasilinear system

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can be rewritten in a block-diagonal form

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x,$$

where

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \ v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}, \ p^i = \frac{1}{u^i}\left(\sum_{m\neq i} \epsilon^{mi}_{,\eta^i} (v^m - v^i) u^m + (\xi^i)'\right).$$

Block-Diagonal Hydrodynamic Type Systems

Now we study integrability aspects of quasilinear systems

$$u_t^i = V_k^i(\mathbf{u})u_x^k,$$

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Their commuting flows $u^i_y = W^i_k(\mathbf{u})u^k_x$ are in the same form (2021, E.V. Ferapontov, MVP)

$$r^i_y = w^i r^i_x + q^i \eta^i_x,$$
$$\eta^i_y = w^i \eta^i_x.$$

Then unknown expressions $w^i(\mathbf{r},\boldsymbol{\eta})$, $q^i(\mathbf{r},\boldsymbol{\eta})$ can be found from the compatibility conditions $(r^i_y)_t = (r^i_t)_y$, $(\eta^i_y)_t = (\eta^i_t)_y$, $i = 1,2,\dots,N$.

Block-Diagonal Hydrodynamic Type Systems

For the given block-diagonal hydrodynamic type system

$$r^i_t = v^i r^i_x + p^i \eta^i_x,$$
$$\eta^i_t = v^i \eta^i_x,$$

we introduce necessary definitions

$$a_i = \frac{v^i_{r^i}}{p^i}, \quad b_i = \frac{v^i_{\eta^i} - p^i_{r^i}}{p^i};$$

$$a_{ij} = \frac{v^i_{r^j}}{v^j - v^i}, \quad b_{ij} = \frac{v^i_{\eta^j} - a_{ij} p^j}{v^j - v^i}, \quad c_{ij} = \frac{p^i_{r^j} + a_{ij} p^i}{v^j - v^i}, \quad d_{ij} = \frac{p^i_{\eta^j} + b_{ij} p^i - c_{ij} p^j}{v^j - v^i}.$$

Block-Diagonal Hydrodynamic Type Systems

Then the compatibility conditions

$$(r^i_y)_t = (r^i_t)_y, \quad (\eta^i_y)_t = (\eta^i_t)_y, \quad i = 1, 2, ..., N$$

of both commuting flows

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \qquad r^i_y = w^i r^i_x + q^i \eta^i_x,$$
$$\eta^i_t = v^i \eta^i_x, \qquad \eta^i_y = w^i \eta^i_x.$$

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of both commuting flows

$$\begin{aligned} r^i_t &= v^i r^i_x + p^i \eta^i_x, & \qquad r^i_y &= w^i r^i_x + q^i \eta^i_x, \\ \eta^i_t &= v^i \eta^i_x, & \qquad \eta^i_y &= w^i \eta^i_x. \end{aligned}$$

lead to the set of linear equations

$$w^i_{r^i} = a_i q^i, \quad w^i_{\eta^i} = b_i q^i + q^i_{r^i},$$

$$w^i_{r^j} = a_{ij}(w^j - w^i), \quad w^i_{\eta^j} = b_{ij}(w^j - w^i) + a_{ij} q^j,$$

$$q^i_{r^j} = c_{ij}(w^j - w^i) - a_{ij} q^i, \quad q^i_{\eta^j} = d_{ij}(w^j - w^i) + c_{ij} q^j - b_{ij} q^i.$$

Integrability Conditions I

The list of integrability conditions for every pair of distinct indices is

$$a_{i,r^j}=0,\qquad a_{ij,r^i}=a_{ij}a_{ji}+a_ic_{ij};$$

$$a_{i,\eta^j}=0,\qquad b_{ij,r^i}=b_{ij}a_{ji}+a_{ij}c_{ji}+a_id_{ij};$$

$$b_{i,r^j}=2a_{ij}a_{ji}+2a_ic_{ij},$$

$$a_{ij,\eta^i}=a_{ij}b_{ji}-c_{ij}a_{ji}+b_ic_{ij}+c_{ij,r^i};$$

$$b_{i,\eta^j}=2a_{ij}c_{ji}+2b_{ij}a_{ji}+2a_id_{ij},$$

$$b_{ij,\eta^i}=b_{ij}b_{ji}+a_{ij}d_{ji}-d_{ij}a_{ji}-c_{ij}c_{ji}+b_id_{ij}+d_{ij,r^i};$$

$$a_{ij,r^j}=b_ja_{ij}-a_jb_{ij}-a_{ij}^2,\qquad a_{ij,\eta^j}=b_{ij,r^j};$$

$$c_{ij,r^j}=b_jc_{ij}-a_jd_{ij}-2a_{ij}c_{ij},\qquad c_{ij,\eta^j}=d_{ij,r^j}.$$

Integrability Conditions II

The list of integrability conditions for every triad of distinct indices is

$$a_{ij,r^k} = a_{ij}a_{jk} + a_{ik}a_{kj} - a_{ij}a_{ik}.$$

$$a_{ij,\eta^k} = a_{ij}b_{jk} + a_{ik}c_{kj} + b_{ik}a_{kj} - a_{ij}b_{ik},$$
$$b_{ij,r^k} = b_{ij}a_{jk} + a_{ik}b_{kj} + a_{ij}c_{jk} - a_{ik}b_{ij}.$$

$$b_{ij,\eta^k} = a_{ij}d_{jk} + a_{ik}d_{kj} + b_{ij}b_{jk} + b_{ik}b_{kj} - b_{ij}b_{ik}.$$

$$c_{ij,r^k} = c_{ij}a_{jk} + c_{ik}a_{kj} - c_{ij}a_{ik} - c_{ik}a_{ij}.$$

$$c_{ij,\eta^k} = c_{ij}b_{jk} + c_{ik}c_{kj} + d_{ik}a_{kj} - a_{ij}d_{ik} - c_{ij}b_{ik},$$
$$d_{ij,r^k} = d_{ij}a_{jk} + c_{ij}c_{jk} + c_{ik}b_{kj} - a_{ik}d_{ij} - c_{ik}b_{ij}.$$

$$d_{ij,\eta^k} = c_{ij}d_{jk} + c_{ik}d_{kj} + d_{ij}b_{jk} + d_{ik}b_{kj} - b_{ij}d_{ik} - b_{ik}d_{ij}.$$

Commuting Flows

The block-diagonal system

$$r_t^i = v^i r_x^i + p^i \eta_x^i, \quad \eta_t^i = v^i \eta_x^i,$$

where

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \; v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}, \; p^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (v^m - v^i) u^m + (\xi^i)' \right),$$

possesses infinitely many commuting block-diagonal flows

$$r_y^i = w^i r_x^i + q^i \eta_x^i, \quad \eta_y^i = w^i \eta_x^i,$$

where

$$w^i = \frac{1}{u^i} \sum_{m=1}^{N} \varphi^m \beta_{mi}, \qquad q^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (w^m - w^i) u^m - r^i \mu^i + \varphi^i_{,\eta^i} \right).$$

Here $\mu^i(\eta^i)$ are N arbitrary functions of one variable and the functions $\varphi^i(\eta^1, \dots, \eta^N)$ satisfy the relations $\partial_{\eta^k} \varphi^i = \epsilon^{ki} \mu^k, \; k \neq i$. The general commuting flow depends on $2N$ arbitrary functions of one variable: N functions $\mu^i(\eta^i)$ plus extra N functions coming from φ^i.

Conservation Laws

Conservation laws $h_t = g_x$ provide an alternative way to derive integrability conditions for the block-diagonal system

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x.$$

Their existence leads to a system of second-order linear PDEs

$$h_{r^i r^i} = b_i h_{r^i} - a_i h_{\eta^i}, \quad h_{r^i \eta^j} = a_{ji} h_{\eta^j} + c_{ji} h_{r^j} + b_{ij} h_{r^i},$$

$$h_{r^i r^j} = a_{ij} h_{r^i} + a_{ji} h_{r^j}, \quad h_{\eta^i \eta^j} = d_{ij} h_{r^i} + d_{ji} h_{r^j} + b_{ij} h_{\eta^i} + b_{ji} h_{\eta^j},$$

where $g_{r^i} = v^i h_{r^i}$, $g_{\eta^i} = p^i h_{r^i} + v^i h_{\eta^i}$.

The general conservation law has the form ($\sigma^i(\eta^i)$ are arbitrary functions)

$$\left(\sum_{m=1}^N u^m \psi^m(\eta) + \sum_{m=1}^N \sigma^m(\eta^m)\right)_t = \left(\sum_{m=1}^N u^m v^m \psi^m(\eta) + \sum_{m=1}^N \tau^m(\eta^m)\right)_x,$$

where $(\tau^i)' = (\sigma^i)'\xi^i$ and $\psi^i_{,\eta^k} = (\sigma^j)'\epsilon^{ik}$, $k \neq i$. This general conservation law depends on $2N$ arbitrary functions of one variable: N functions $\sigma^i(\eta^i)$, plus extra N functions coming from ψ^i.

Tsarev's Generalised Hodograph Method

We remind: If the hydrodynamic type system $u_t = V(u)u_x$ has a commuting flow $u_y = W(u)u_x$, where $V(u)$ and $W(u)$ are $N \times N$ matrices (the commutativity conditions $u_{ty} = u_{yt}$ impose differential constraints on V and W),

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$$W(u) = I\,x + V(u)\,t,$$

where I is the $N \times N$ identity matrix, defines an implicit solution $u(x,t)$. Note that, due to the commutativity conditions, only N out of the above N^2 relations will be functionally independent.

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where I is the $N \times N$ identity matrix, defines an implicit solution $u(x,t)$. Note that, due to the commutativity conditions, only N out of the above N^2 relations will be functionally independent. For commuting block-diagonal systems

$$r^i_t = v^i r^i_x + p^i \eta^i_x, \quad \eta^i_t = v^i \eta^i_x,$$
$$r^i_y = w^i r^i_x + q^i \eta^i_x, \quad \eta^i_y = w^i \eta^i_x,$$

the hodograph formula becomes

$$w^i(r,\eta) = x + v^i(r,\eta)\,t, \qquad q^i(r,\eta) = p^i(r,\eta)\,t,$$

which is a system of $2N$ implicit relations for the $2N$ dependent variables.

Tsarev's Generalised Hodograph Method

Denote β_{ik} the matrix elements of $\hat{\beta}$ (indices i and k are allowed to coincide). Then we obtain the following formulae for u^i, v^i and p^i:

$$u^i = \sum_{m=1}^{N} \beta_{mi}, \ v^i = \frac{1}{u^i} \sum_{m=1}^{N} \xi^m \beta_{mi}, \ p^i = \frac{1}{u^i} \left(\sum_{m \neq i} \epsilon^{mi}_{,\eta^i} (v^m - v^i) u^m + (\xi^i)' \right).$$

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$$r^i = \frac{\varphi^i_{,\eta^i} - (\xi^i)'\, t}{\mu^i}, \qquad \varphi^i(\eta^1, \ldots, \eta^N) = x + \xi^i(\eta^i)\, t;$$

where $\mu^i(\eta^i)$ are arbitrary functions of their arguments and the functions $\varphi^i(\eta^1, \ldots, \eta^N)$ satisfy the relations $\varphi^i_{,\eta^k} = \epsilon^{ki}(\eta^i, \eta^k)\, \mu^k(\eta^k)$, $i \neq k$. The last N above equations define $\eta^i(x,t)$ as implicit functions of x and t; then the first N equations define $r^i(x,t)$ explicitly.

Block-Diagonal Hydrodynamic Type Systems

Now we come back to integrability aspects of quasilinear systems

$$u_t^i = V_k^i(\mathbf{u})u_x^k,$$

whose matrix V consists of N Jordan blocks of size 2×2:

$$\begin{aligned} r_t^i &= v^i r_x^i + p^i \eta_x^i, \\ \eta_t^i &= v^i \eta_x^i, \end{aligned}$$

$i = 1, \ldots, N$, where the coefficients $v^i(r, \eta)$ and $p^i(r, \eta)$ are functions of the N dependent variables $r = (r^1, \ldots, r^N)$ and N dependent variables $\eta = (\eta^1, \ldots, \eta^N)$.

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We introduced necessary definitions

$$a_i = \frac{v_{r^i}^i}{p^i}, \quad b_i = \frac{v_{\eta^i}^i - p_{r^i}^i}{p^i};$$

$$a_{ij} = \frac{v_{r^j}^i}{v^j - v^i}, \quad b_{ij} = \frac{v_{\eta^j}^i - a_{ij}p^j}{v^j - v^i}, \quad c_{ij} = \frac{p_{r^j}^i + a_{ij}p^i}{v^j - v^i}, \quad d_{ij} = \frac{p_{\eta^j}^i + b_{ij}p^i - c_{ij}p^j}{v^j - v^i}.$$

Linearly-Degenerate and Genuinely nonlinear Hydrodynamic Type Systems

We introduced necessary definitions

$$a_i = \frac{v^i_{r^i}}{p^i}, \quad b_i = \frac{v^i_{\eta^i} - p^i_{r^i}}{p^i};$$

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for block-diagonalisable integrable hydrodynamic type systems

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Linearly-Degenerate and Genuinely nonlinear Hydrodynamic Type Systems

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Now we can split this class of integrable systems to two distinguish families: genuinely nonlinear and linearly-degenerate.

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Here we remind that two first integrability conditions are

$$a_{i,r^j} = 0, \qquad a_{i,\eta^j} = 0.$$

Taking into account that the class of block-diagonalisable hydrodynamic systems is invariant under changes of variables of the form

$$r^i \to f^i(r^i, \eta^i), \qquad \eta^i \to g^i(\eta^i),$$

we can choose $a_i = 1$ or $a_i = 0$ for genuinely nonlinear and linearly-degenerate systems respectively.

Linearly-Degenerate Hydrodynamic Type Systems

This means: if we choose $a_i = 0$, then

$$v^i_{r^i} = 0$$

from the first integrability conditions

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We believe that these hydrodynamic type systems possess global solutions.

Genuinely nonlinear Hydrodynamic Type Systems

If we choose $a_i = 1$, we call these block-diagonal hydrodynamic type systems as genuinely nonlinear. In this case, the first integrability condition

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Then all computations will be significantly simplified. In this case we obtain $q^i = w^i_{r^i}$, and the linear system describing commuting flows take the form

$$w^i_{\eta^i} = w^i_{r^i r^i} + b_i w^i_{r^i}, \quad w^i_{r^j} = a_{ij}(w^j - w^i), \quad w^i_{\eta^j} = a_{ij} w^j_{r^j} + b_{ij}(w^j - w^i).$$

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Also, all conservation law densities $h(\mathbf{r}, \boldsymbol{\eta})$ satisfy adjoint linear system

$$h_{r^i r^i} = b_i h_{r^i} - h_{\eta^i}, \quad h_{r^i r^j} = a_{ij} h_{r^i} + a_{ji} h_{r^j}.$$

The compatibility conditions of both above linear systems can be written in the form

$$\partial_{r^k} a_{ij} = a_{ik} a_{kj} + a_{ij} a_{jk} - a_{ij} a_{ik},$$

$$\partial_{\eta^i} a_{ij} = \partial_{r^i}(\partial_{r^i} a_{ij} - 2a_{ij} a_{ji} + b_i a_{ij}) - a_{ij} \partial_{r^i} b_i, \quad \partial_{\eta^j} a_{ij} = \partial_{r^j} b_{ij}, \quad \partial_{\eta^k} a_{ij} = \partial_{r^j} b_{ik},$$

$$\partial_{r^j} b_i = 2\partial_{r^i} a_{ij}, \quad \partial_{\eta^j} b_i = 2\partial_{r^i} b_{ij},$$

where $b_{ij} = b_j a_{ij} - a^2_{ij} - \partial_{r^j} a_{ij}$. This provides a compact formulation of integrability conditions.

Genuinely nonlinear Hydrodynamic Type Systems

Thus, we come to observation: any particular solution $v^i(\mathbf{r}, \boldsymbol{\eta})$ of the Modified KP hierarchy

$$w^i_{\eta^i} = w^i_{r^i r^i} + b_i w^i_{r^i}, \quad w^i_{r^j} = a_{ij}(w^j - w^i), \quad w^i_{\eta^j} = a_{ij} w^j_{r^j} + b_{ij}(w^j - w^i)$$

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determines the block-diagonal hydrodynamic type system (we remind that $p^i = v^i_{r^i}$)

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Any other particular solution $w^i(\mathbf{r}, \boldsymbol{\eta})$ of the Modified KP hierarchy determines corresponding particular solution (by virtue of Tsarev's Generalised Hodograph Method)

$$w^i(r, \eta) = x + v^i(r, \eta)\, t, \qquad w^i_{r^i} = t v^i_{r^i},$$

together with the commuting block-diagonal hydrodynamic type system (we remind that $q^i = w^i_{r^i}$)

$$r^i_y = w^i r^i_x + w^i_{r^i} \eta^i_x, \quad \eta^i_y = w^i \eta^i_x.$$

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