HITCHIN SYSTEMS ON HYPERELLIPTIC CURVES: Separation of Variables

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Russian-Chinese Conference
«Integrable Systems and Geometry»
December 20–22, 2021

Hitchin systems – conventional set-up

G – classical simple Lie group $/\mathbb{C}$, Σ – cpt Riemann surface $/\mathbb{C}$

$\mathcal{N}$ – moduli space of holomorphic s/st vector G-bundles with the standard representation as a fibre

For $E \in \mathcal{N}$ let $\mathrm{End}(E)$ be a bundle with the fiber $\mathfrak{g}$ associated to E by means of the adjoint representation (Ad G)

Higgs field: a $\Phi \in H^0(\mathrm{End}(E) \otimes \mathcal{K})$, $\mathcal{K}$ – canonical bundle

χ – a degree d invariant polynomial χ on $\mathfrak{g}$, then

$$\chi(\Phi) \in H^0(\Sigma, \mathcal{K}^d).$$

Pick up a base $\{\Omega_j^d\}$ in $H^0(\Sigma, \mathcal{K}^d)$ and expand $\chi(\Phi)$ as:

$$\chi(\Phi) = \sum H_j(E, \Phi)\Omega_j^d.$$

THEOREM (HITCHIN, '87): $\{H_j\}$ give an integrable system on $T^*(\mathcal{N})$ (i.e. $\{H_i, H_j\} = 0$, and $\mathrm{card}\{H_j\} = \dim T^*(\mathcal{N})/2$)

Historical remarks

- 1987 Hitchin, Duke Math. J.
- 1994 Previato, van Geemen – geometry of $SL(2)$ case
- 1994 Simpson – Hitchin systems in higher dimension
- 1996 – 2000 Hurtubise – some observations on SoV and inverse spectral method
- 1998 Gawedzki – $SL(2)$, θ-functional formula
- 2001 Gorski–Nekrasov–Roubtsov: SoV in more effective terms ($G = GL(n)$)
- 2001 Krichever – Lax representation, Hamiltonian theory, inverse spectral method ($G = GL(n)$)
- 2018 Sh. – results of the present talk – simple classical structure groups, hyperelliptic surfaces, SoV in explicit terms

Spectral curve

Partial trivialization: $\Phi \mapsto \Phi/\xi, \quad \xi \in \Omega^1_{\text{hol}}(\Sigma)$

Spectral curve:

$$\det\left(\lambda - \frac{\Phi}{\xi}\right) = 0$$

Let $D = (\xi), \quad d_1, \ldots, d_n$ be the degrees of basis invariants of $\dfrac{\Phi}{\xi}$.

PROPOSITION: Basis degree d_j invariants of Φ/ξ run over $\mathcal{O}(-d_j D)$.

Hamiltonians: For $\forall j$, $H^0(\Sigma, \mathcal{K}^{d_j}) \simeq \mathcal{O}(-d_j D)\xi^{\otimes d_j}$. We pick up a base in $\mathcal{O}(-d_j D)$ rather than in $H^0(\Sigma, \mathcal{K}^{d_j})$.

Spectral curves of hyperelliptic A_n, B_n, C_n Hitchin systems (Sh'2018)

Let $\Sigma : y^2 = P_{2g+1}(x),\ \xi = \frac{dx}{y}$ $\big($then $D = 2(g-1)\cdot\infty\big)$.

PROPOSITION: A base in $\mathcal{O}(-d_j D)$ is formed by the functions $1, x, \dots, x^{d_j(g-1)}$, and $y, yx, \dots, yx^{(d_j-1)(g-1)-2}$.

PROPOSITION: (The affine part of) the spectral curve for a Hitchin system from the above list is a full intersection of the two surfaces in $\mathbb{C}^3$: $y^2 = P_{2g+1}(x)$ and $R(\lambda, x, y, H) = 0$ where

$$R = \lambda^d + \sum_{j=1}^{n} \left(\sum_{k=0}^{d_j(g-1)} H^{(0)}_{jk} x^k + \sum_{s=0}^{(d_j-1)(g-1)-2} H^{(1)}_{js} x^s y \right) \lambda^{d-d_j}$$

where $H^{(0)}_{jk}$, $H^{(1)}_{js}$ are parameters (Hamiltonians).

Separating coordinates and Hamiltonians

Let $\mathfrak{g}$ be a complex Lie algebra of one of the types A_n, B_n, C_n.

We give the spectral curve by means of a tuple of points it passes through. Since coefficients of the spectral curve depend on $h = \dim \mathfrak{g}(g-1)$ parameters (Hamiltonians), every tuple consists of h points denoted by $(\gamma_1, \ldots, \gamma_h)$, $\gamma_i = (x_i, y_i, \lambda_i)$, $\forall i$. In these coordinates the system looks as follows.

Phase space: tuples $\{(\lambda_1, x_1, y_1), \ldots, (\lambda_h, x_h, y_h)\}$, $h = (\dim \mathfrak{g})(g-1)$, $\lambda_i, x_i, y_i \in \mathbb{C}$, $y_i^2 = P_{2g+1}(x_i)$ $(i = 1, \ldots, h)$

Poisson bracket is given by $\{\lambda_i, x_j\} = \delta_{ij} y_i$ (to be proven below)

Hamiltonians $H_{jk}^{(0)}$, $H_{js}^{(1)}$ are defined from the system of linear equations ($i = 1, \ldots, h$):

$$\lambda_i^d + \sum_{j=1}^{n} \left(\sum_{k=0}^{d_j(g-1)} H_{jk}^{(0)} x_i^k + \sum_{s=0}^{(d_j-1)(g-1)-2} H_{js}^{(1)} x_i^s y_i \right) \lambda_i^{d-d_j} = 0$$

Symplectic form

Action–angle variables (I, α) are defined by

$$\lambda dz = \sum_{a=1}^{h} I^a \Omega_a, \quad \alpha_a = \sum_{i=1}^{h} \int^{\gamma_i} \Omega_a, \quad dz = \frac{dx}{y}$$

where Ω_a are normalized Prym differentials, $h = \dim \mathfrak{g}(g-1)$

(follows from Gorski-Nekrasov-Rubtsov 2001, Hurtubise 1996).

For the **symplectic form** it follows (by def. and variation of the $\int$ in γ_i)

$$\omega = \sum_{a=1}^{h} \delta I^a \wedge \delta \alpha_a = \sum_{a=1}^{h} \delta I^a \wedge \sum_{i=1}^{h} \Omega_a(\gamma_i)$$

Also, $\lambda dz = \sum_{a=1}^{h} I^a \Omega_a \Longrightarrow \lambda_i \delta z_i = \sum_{a=1}^{h} I^a \Omega_a(\gamma_i) \Longrightarrow$

$\Longrightarrow \delta\lambda_i \wedge \delta z_i = \sum_{a=1}^{h} \delta I^a \wedge \Omega_a(\gamma_i)$, $\forall i$. Summing up in i gives

$$\boxed{\omega = \sum_{i=1}^{h} \delta\lambda_i \wedge \delta z_i = \sum_{i=1}^{h} \delta\lambda_i \wedge \frac{\delta x_i}{y_i}}$$

Example: $\mathfrak{g} = \mathfrak{sl}(2)$ ($\sim A_1$), genus 2 Hitchin system

(previous results E.Previato, 1994; Kz. Gawędzki, 1998)

Phase space:
triples $\{(\lambda_1, x_1, y_1), (\lambda_2, x_2, y_2), (\lambda_3, x_3, y_3)\}$ s.t.

$$\lambda_i^2 = H_0 + H_1 x_i + H_2 x_i^2, \quad y_i^2 = P_5(x_i) \; (i = 1, 2, 3)$$

In these coordinates

$$H_i = \frac{\Delta_i}{\Delta}, \; \Delta = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} \lambda_1^2 & x_1 & x_1^2 \\ \lambda_2^2 & x_2 & x_2^2 \\ \lambda_3^2 & x_3 & x_3^2 \end{vmatrix}, \quad \text{etc.}$$

Symplectic form:

$$\sigma = d\lambda_1 \wedge \frac{dx_1}{y_1} + d\lambda_2 \wedge \frac{dx_2}{y_2} + d\lambda_3 \wedge \frac{dx_3}{y_3}$$

$\mathfrak{g} = \mathfrak{sl}(2)$: Hitchin equations in an explicit form

$$H_2 = \frac{\Delta_2}{\Delta}, \quad \Delta_2 = \begin{vmatrix} 1 & x_1 & \lambda_1^2 \\ 1 & x_2 & \lambda_2^2 \\ 1 & x_3 & \lambda_3^2 \end{vmatrix}, \ \Delta = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}$$

$$\{x_1, H_2\} = \frac{2\lambda_1 y_1}{\Delta}(x_3 - x_2),$$

$$\{\lambda_1, H_2\} = \frac{y_1}{(x_1 - x_2)^2(x_1 - x_3)^2}\left(\Delta_2 \frac{2x_1 - x_2 - x_3}{x_2 - x_3} - 1\right)$$

and cyclic permutations of indices for (x_2, λ_2), (x_3, λ_3)

Method of generating functions

Aim: to find out coordinates ϕ s.t. $\omega = \sum_j dH_j \wedge d\phi_j$.

Motivation: the flows are linearized in the coordinates (H, ϕ): $H_j = const$, $\phi_j = \delta_{ij} t_i + \phi_{0,j}$.

PROPOSITION (HAMILTON, JACOBI, ARNOLD, SKLYANIN):

$$\phi_j = \partial S / \partial H_j$$

where $S = \sum_{i=1}^h S_i$, S_i to be found from $R\left(\frac{\partial S_i}{\partial z_i}, z_i, H\right) = 0$.

(here, $S_i = S_i(\gamma_i)$, z_i is a pull back of z_i on Σ).

$$\frac{\partial R}{\partial \lambda}\frac{\partial}{\partial H_j}\frac{\partial S_i}{\partial z_i} + \frac{\partial R}{\partial H_j} = 0 \quad \Longrightarrow \quad \frac{\partial S_i}{\partial H_j} = -\sum_{j=1}^h \int^{\gamma_i} \frac{\partial R / \partial H_j}{\partial R / \partial \lambda} dz$$

Darboux coordinates – cases A_n, B_n, C_n

<u>Darboux coordinates:</u> $(H_{jk}^{(0)}, \phi_{jk}^{(0)})$, and $(H_{js}^{(1)}, \phi_{js}^{(1)})$

:

$$\phi_{jk}^{(0)} = \sum_{i=1}^{(\dim \mathfrak{g})(g-1)} \int^{(x_i, y_i, \lambda_i)} \frac{x^k \lambda^{d-d_j} dx}{R'_\lambda(x, y, \lambda)\, y}, \ 0 \le k \le d_j(g-1);$$

$$\phi_{js}^{(1)} = \sum_{i=1}^{(\dim \mathfrak{g})(g-1)} \int^{(x_i, y_i, \lambda_i)} \frac{x^s \lambda^{d-d_j} dx}{R'_\lambda(x, y, \lambda)}, \ 0 \le s \le (d_j - 1)(g-1) - 2$$

<u>THEOREM</u>: Integrands form a base of holomorphic differentials on the spectral curve for A_n $(n > 1)$, and a base of holomorphic Pryme differentials for the systems A_1, B_n, C_n (w.r.t involution $\lambda \to -\lambda$).

Remarks on basis differentials

Peculiarities of the case D_n:

- Separation relations $R(\lambda_i, x_i, y_i, H) = 0$ are quadratic in H (because the last coefficient of R is $\det L = (\mathrm{Pf}\, L)^2$);
- Basis differentials are the same for $j < n$, and are multiplied by $2(\mathrm{Pf}\, L)$ for $j = n$;
- They form a base of holomorphic Prym differentials on the normalization of the spectral curve.

Consecutive numbering:

$$\omega_j = \frac{\partial R/\partial H_j}{\partial R/\partial \lambda} \frac{dx}{y}, \quad j = 1, \dots, h$$

in all cases (A_n, B_n,C_n, D_n).

Action–angle variables via (H, ϕ)

Let $A_j^a = \oint_a \omega_j$, $A = (A_j^a)$, then $\omega_j = \sum_{a=1}^h A_j^a \Omega_a$, hence

for angle variables: $$\boxed{\alpha_a = \sum (A^{-1})_a^j \phi_j}$$

Problem of finding the matrix A descends to the problem of finding the branch points of the spectral curve, i.e. to the system of algebraic equations $R(\lambda, x, y) = 0$, $R'_\lambda(\lambda, x, y) = 0$. The last is solvable in radicals only for $\mathfrak{g} = \mathfrak{sl}(2), \mathfrak{so}(4)$.

Defining relation for action variables is: $\lambda \frac{dx}{y} = \sum_{a=1}^h I^a \Omega_a$. Plug $\Omega_a = \sum (A^{-1})_a^j \omega_j$, $\omega_j = \frac{\partial R / \partial H_j}{R'_\lambda} \frac{dx}{y}$, and $\lambda = \lambda_i$, $x = x_i$, $y = y_i$. Then I^a are to be found out of the linear system of equations

$$\lambda_i R'_\lambda(x_i, y_i, H) = \sum_{a=1}^h I^a \sum_{j=1}^h (A^{-1})_a^j \frac{\partial R(x_i, y_i, H)}{\partial H_j}, \quad i = 1, \ldots, h$$

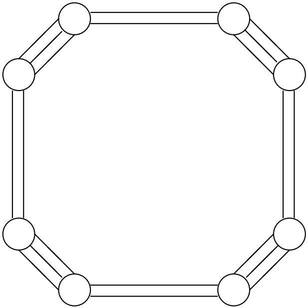
$\mathfrak{so}(4)$, $g = 2$ case (P.Borisova, Sh'19)

Spectral curve: $R(\lambda, x, y, H) = \lambda^4 + \lambda^2 p + q^2 = 0$

where $p = H_0 + xH_1 + x^2 H_2$, $q = H_3 + xH_4 + x^2 H_5$.

THEOREM (P.BORISOVA): Separation equations and equations for branching points are solvable in radicals.

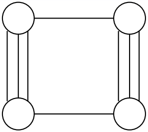
Normalized spectral curve has 16 branching points. By Riemann–Hurwitz $\hat{g} = 13$. Involution $\sigma : \lambda \to -\lambda$ is a rotation by π around the center of the picture. No fixed points. 8 preimages of 4 singular points are located in the middles of horizontal lines (2 at each one). Normalization map glues



the points at the opposite horizontal lines.

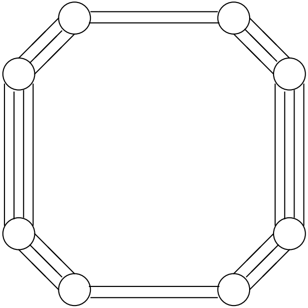
$\mathfrak{sl}(2)$, $\mathfrak{sp}(4)$, $\mathfrak{so}(5)$, $g = 2$ cases

Case $\mathfrak{sl}(2)$ Spectral curve:
$\lambda^2 p + H_0 + xH_1 + x^2 H_2 = 0$
has 4 branching points.
By Riemann–Hurwitz $\hat{g} = 5$.
Involution $\sigma : \lambda \to -\lambda$ is a rotation by π around the vertical axis of the picture.



Fixed points=branching points with $\lambda = 0$

Case $\mathfrak{sp}(4)$ Spectral curve:
$\lambda^4 + \lambda^2 p + q = 0$,
$p = H_0 + xH_1 + x^2 H_2$, $q = H_3 + xH_4 + \ldots x^4 H_7 + yH_8 + xyH_9$,
has 24 branching points.
By Riemann–Hurwitz $\hat{g} = 17$.
Involution $\sigma : \lambda \to -\lambda$ is a reflection in the vertical axis.



Case $\mathfrak{so}(5)$ Spectral curve is the same as for $\mathfrak{sp}(4)$

Thank you!