Non-flat Frobenius manifolds: geometry and integrability

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1. The Witten-Dijkgraaf-Verlinde-Verlinde equations and the Dubrovin-Frobenius manifolds

$F(u^1, \dots, u^n)$ is a function of n variables,

η^{ij}, $1 \leq i, j \leq n$, is a constant non-degenerate symmetric matrix:

$\eta^{ij} = \mathrm{const}$ $\forall i, j$; $\det \eta^{ij} \neq 0$;

$\eta^{ij} = \eta^{ji}$

$\forall i, j, k, l$

$$\frac{\partial^3 F}{\partial u^i \partial u^j \partial u^s} \eta^{sp} \frac{\partial^3 F}{\partial u^p \partial u^k \partial u^l} = \frac{\partial^3 F}{\partial u^l \partial u^j \partial u^s} \eta^{sp} \frac{\partial^3 F}{\partial u^p \partial u^k \partial u^i}$$

(Witten, Dijkgraaf-Verlinde-Verlinde)

Dubrovin

overdetermined integrable system for F

$\forall$ matrix η^{ij}

Dubrovin – Frobenius manifolds

Definition

An n-dimensional pseudo-Riemannian manifold M^n with a metric g and a structure of a Frobenius algebra $(T_uM^n, \circ, g)$, $T_uM^n \times T_uM^n \xrightarrow{\circ} T_uM^n$, on each tangent space T_uM^n at any point $u \in M^n$ smoothly depending on the point is called Dubrovin–Frobenius if

1) the metric g is flat
2) the Frobenius algebra is commutative
$$X \circ Y = Y \circ X$$
$\forall$ vector fields X and Y on M^n
3) the Frobenius algebra is associative
$$(X \circ Y) \circ Z = X \circ (Y \circ Z)$$
$\forall$ vector fields X, Y and Z on M^n
4) the metric g is a non-degenerate invariant symmetric bilinear form on each tangent space T_uM^n:
$$g(X \circ Y, Z) = g(X, Y \circ Z)$$

It follow immediately from 2) and 4) that $F(X, Y, Z) = g(X \circ Y, Z)$ is a symmetric tensor

5) $A(X,Y,Z,W) = \nabla_W F(X,Y,Z)$

is also a symmetric tensor

In flat coordinates of the metric $g_{ij}(u)$, the metric is constant:

$$g_{ij}(u) = \eta_{ij},$$

η_{ij} is a constant non-degenerate symmetric matrix: $\eta_{ij} = \mathrm{const}\ \forall i,j,$

$\det \eta_{ij} \neq 0$, $\eta_{ij} = \eta_{ji}$

$$F_{ijk}(u) = \eta_{is}\, c^s_{jk}(u)$$

It follows from 5) that

$$\frac{\partial F_{ijk}}{\partial u^l} = \frac{\partial F_{ijl}}{\partial u^k}$$

$\Rightarrow \exists$ a function $F(u)$ such that

$$F_{ijk}(u) = \frac{\partial^3 F}{\partial u^i \partial u^j \partial u^k},$$

$$c^k_{ij}(u) = \eta^{ks} \frac{\partial^3 F}{\partial u^s \partial u^i \partial u^j},$$

$$\eta^{is}\eta_{sj} = \delta^i_j$$

Conditions 1), 2), 4), 5) are fulfilled and the condition 5) (the associativity of the Frobenius algebra) are equivalent to the Witten-Dijkgraaf-Verlinde-Verlinde equations.

Hessian metrics

In 1998 in my papers [1]–[3]

[1] O.I. Mokhov. Russian Mathematical Surveys, 53:2 (1998).

[2] O.I. Mokhov. Proceedings of the Steklov Institute of Mathematics (Moscow), Vol. 225 (1999).

[3] O.I. Mokhov. Reports on Math. Physics, Vol. 43, No. 1/2 (1999).

it was considered another important case when the metric $g_{ij}(u)$ is the Hessian of a function $\Phi(u)$:

$$g_{ij}(u) = \frac{\partial^2 \Phi}{\partial u^i \partial u^j}$$

In this case

$$\Gamma^i_{jk}(u) = \frac{1}{2} g^{is}(u) \frac{\partial^3 \Phi}{\partial u^s \partial u^j \partial u^k},$$

$$g^{is}(u) g_{sj}(u) = \delta^i_j, \quad g_{ij}(u) = \frac{\partial^2 \Phi}{\partial u^i \partial u^j}$$

and we have the following important formula (см. [2]):

$$(*) \quad R_{ijkl}(u) = \frac{1}{4} \left(\frac{\partial^3 \Phi}{\partial u^i \partial u^k \partial u^s} g^{sp}(u) \frac{\partial^3 \Phi}{\partial u^p \partial u^j \partial u^l} - \right.$$

$$\left. - \frac{\partial^3 \Phi}{\partial u^i \partial u^l \partial u^s} g^{sp}(u) \frac{\partial^3 \Phi}{\partial u^p \partial u^j \partial u^k} \right)$$

If the metric $g_{ij}(u) = \frac{\partial^2 \Phi}{\partial u^i \partial u^j}$ is flat, then we have an analogue of the WDVV equations:

$$\frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^s} g^{sp}(u) \frac{\partial^3 \Phi}{\partial u^p \partial u^k \partial u^l} = \frac{\partial^3 \Phi}{\partial u^l \partial u^j \partial u^s} g^{sp}(u) \frac{\partial^3 \Phi}{\partial u^p \partial u^k \partial u^i}$$

It was proved that these equations are also integrable and every solution of these equations generates a Dubrovin-Frobenius manifold with the multiplication $\circ$, defined by the structural functions

$$c^k_{ij}(u) = g^{ks}(u) \frac{\partial^3 F}{\partial u^s \partial u^i \partial u^j}$$

Conditions 1), 2), 4) and 5) are fulfilled and the condition 5) is equivalent to the equations for $\Phi(u)$ (the associativity equations).

At any Dubrovin-Frobenius manifold there exist local coordinates such that

$$g_{ij}(u) = \frac{\partial^2 \Phi}{\partial u^i \partial u^j}, \quad F_{ijk} = \frac{\partial^3 \Phi}{\partial u^i \partial u^j \partial u^k},$$

$$c^k_{ij}(u) = g^{ks}(u) \frac{\partial^3 \Phi}{\partial u^s \partial u^i \partial u^j}$$

$\Phi(u)$ is a new potential of the Dubrovin-Frobenius manifold.

2. Non-flat WDVV equations and non-flat Frobenius manifolds

Curved WDVV equations arise naturally in multidimensional supersymmetric mechanics as a generalization of the WDVV equations to curved space:

[4] N. Kozyrev, S. Krivonos, O. Lechtenfeld, A. Nersessian, A. Sutulin, Phys. Rev. D, 96, 101702 (R) (2017)

[5] N. Kozyrev, J. Physics: Conf. Series 1194 (2019) 012061

Besides, the non-flat WDVV equations arise naturally in the theory of submanifolds with potential of normals in pseudo-Euclidean spaces developed in

[6] O. I. Mokhov, Russian Mathematical Surveys, 63:2 (2008).

Consider an arbitrary pseudo-Riemannian metric $g_{ij}(u)$ and a symmetric tensor $F_{ijk}(u)$ such that

1. $\nabla_l F_{ijk} = \nabla_k F_{ijl}$,

2. $R_{ijkl} = F_{ilp} g^{ps} F_{sjk} - F_{ikp} g^{ps} F_{sjl}$

1 and 2 are non-flat generalization of the WDVV equations.

We introduce non-flat non-associative Frobenius manifolds, which are locally described by the non-flat WDVV equations and generalize the flat Dubrovin-Frobenius manifolds.

Non-flat non-associative Frobenius manifolds

Def An n-dimensional pseudo-Riemannian manifold M^n with a metric g and a structure of a Frobenius algebra $(T_u M^n, \circ, g)$, $T_u M^n \times T_u M^n \xrightarrow{\circ} T_u M^n$, on each tangent space $T_u M^n$ at any point $u \in M^n$ smoothly depending on the point is called a non-flat non-associative Frobenius manifold if

1) g is a pseudo-Riemannian metric
2) the Frobenius algebra is commutative:
$X \circ Y = Y \circ X$
$\forall$ vector fields X and Y on M^n

3) the Frobenius algebra is non-associative, the following relation is satisfied:

$$(X \circ Y) \circ Z - X \circ (Y \circ Z) = c\, R(X,Y)Z,$$

$c = \text{const}$

$\forall$ vector fields X, Y and Z on M^n

4) the metric g is a non-degenerate invariant symmetric bilinear form on each tangent space $T_u M^n$:

$$g(X \circ Y, Z) = g(X, Y \circ Z)$$

(the Frobenius property)

It again follows from 2) and 4) that $F(X,Y,Z) = g(X \circ Y, Z)$ is a symmetric tensor

5) $A(X,Y,Z,W) = \nabla_W F(X,Y,Z)$ is a symmetric tensor

$$c^k_{ij}(u) = g^{ks}(u)\, F_{sij}(u)$$

are structure functions of a non-flat non-associative Frobenius algebra

Locally, non-flat Frobenius manifolds are described by the non-flat WDVV equations

Theorem 1. The considered above flat and non-flat WDVV equations are natural reductions of the fundamental equations of the theory of submanifolds in pseudo-Euclidean spaces.

Theorem 2. Any Dubrovin-Frobenius manifold can be realized as a special flat submanifold with flat normal bundle in a pseudo-Euclidean space.

Theorem 3. Any non-flat Frobenius manifold can be realized as a special submanifold with potential of normals in a pseudo-Euclidean space.

Submanifolds with potential of normals in pseudo-Euclidean spaces

We consider totally non-isotropic submanifolds in pseudo-Euclidean spaces:

$$\vec{r}(u) = (x^1(u^1, \ldots, u^n), \ldots, x^N(u^1, \ldots, u^n)),$$

$$\operatorname{rank}\left(\frac{\partial x^i}{\partial u^j}\right) = n, \quad n < N$$

Special class of submanifolds [6]

1) $N=2n$
2) There exist a vector function $\vec{m}(u)$ on the submanifold such that $\left(\frac{\partial \vec{m}}{\partial u^1}, \dots, \frac{\partial \vec{m}}{\partial u^n}\right)$ is a basis in the normal spaces of the submanifold.

Any flat and non-flat Frobenius manifold can be realized as a submanifold with potential of normals in a pseudo-Euclidean space.

Flat case

In flat coordinates $g_{ij}(u) = \eta_{ij} = const$

Gram matrix of basis in the normal spaces of the submanifold

$$h_{ij}(u) = c\eta_{ij}, \quad c = const \neq 0$$

n second fundamental forms:

$$b^k_{ij}(u) = \eta^{ks} \frac{\partial^3 F}{\partial u^s \partial u^i \partial u^j}, \quad F(u)$$

torsion forms

$$d^k_{ij}(u) = 0$$

Theorem. A submanifold with such $g_{ij}(u), h_{ij}(u), b^k_{ij}(u), d^k_{ij}(u)$ exists if and only if the matrix η_{ij} and the function $F(u)$ satisfy the WDVV equations.

$$c^k_{ij}(u) = \eta^{ks} \frac{\partial^3 F}{\partial u^s \partial u^i \partial u^j} = b^k_{ij}(u)$$

Frobenius structure is given by the first and second fundamental forms of the submanifold.

Theorem. The Gauss and Weingarten decompositions for the constructed submanifolds are the representation of zero curvature for the WDVV equations:

In Hessian coordinates

$$g_{ij}(u) = \frac{\partial^2 \Phi}{\partial u^i \partial u^j}$$

$$h_{ij}(u) = c \frac{\partial^2 \Phi}{\partial u^i \partial u^j}, \quad c = \text{const} \neq 0$$

$$b^k_{ij}(u) = g^{ks}(u) \frac{\partial^3 \Phi}{\partial u^s \partial u^i \partial u^j}$$

In this case we proved analogous statements

Non-flat case

We have $g_{ij}(u)$ and $F_{ijk}(u)$.
Let us consider a submanifold with potential of normals in a pseudo-Euclidean space such that $g_{ij}(u)$ is the first fundamental form of the submanifold,
$b^k_{ij}(u) = g^{ks}(u) F_{sij}(u)$ are the second fundamental forms of the submanifold,
$h_{ij}(u) = c\, g_{ij}(u)$, $c = \mathrm{const} \neq 0$, is the Gram matrix in normal spaces of the submanifold.

<u>Theorem</u>. A submanifold with potential of normals and with such $g_{ij}(u)$, $b^k_{ij}(u)$ and $h_{ij}(u)$ exists if and only if the non-flat WDVV equations are satisfied.

The Gauss and Weingarten decompositions for the submanifolds are the representation of zero curvature for the non-flat WDVV equations.

Duality in the class of submanifolds with potential of normals in pseudo-Euclidean spaces

If $(\vec{r}(u), \vec{n}(u))$ is a submanifold with potential of normals in a pseudo-Euclidean space, then $(\vec{n}(u), \vec{r}(u))$ is also a submanifold with potential of normals in the pseudo-Euclidean space.
Moreover, there is a duality:
Gauss and Weingarten decompositions, metrics $g_{ij}(u)$ and $h_{ij}(u)$,
Gauss and Ricci equations,
the Peterson-Mainardi-Kodazzi equations are self-dual.
The most remarkable case, when Gauss equations and Ricci equations coincide, is the case of non-flat WDVV equations, which coincide in this case with the Gauss equations and the Ricci equations for the submanifolds.
If submanifold is flat, we have the case of classical WDVV equations.

This is the case of flat and non-flat Frobenius submanifolds.

There is also a connection with local and non-local Hamiltonian structures of hydrodynamic type and integrable hierarchies related to the Frobenius manifolds.

$$P^{ij} = g^{ij}(u)\frac{d}{dx} + b^{ij}_k(u)\, u^k_x +$$

$$+ \sum_{m,p} \mu^{mp} (w_m)^i_k(u)\, u^k_x \left(\frac{d}{dx}\right)^{-1} (w_p)^j_s\, u^s_x$$

$$L^{ij} = \eta^{ij}\frac{d}{dx} + \sum_{m,e} \eta^{me}\eta^{ip}\eta^{jr}\frac{\partial^3 F}{\partial u^p \partial u^m \partial u^k} u^k_x \circ$$

$$\circ \left(\frac{d}{dx}\right)^{-1} \frac{\partial^3 F}{\partial u^r \partial u^e \partial u^s} u^s_x$$

the WDVV equations

$$\eta^{ij}\frac{d}{dx},$$

$$\sum_{m,e} \eta^{me}\eta^{ip}\eta^{jr}\frac{\partial^3 F}{\partial u^p \partial u^m \partial u^k} u^k_x \left(\frac{d}{dx}\right)^{-1} \frac{\partial^3 F}{\partial u^r \partial u^e \partial u^s} u^s_x$$

are compatible and generate integrable hierarchies of hydrodynamic type related to the Dubrovin-Frobenius manifolds.