Group Varieties and Group Structures of Algebraic Groups

Vladimir L. Popov

Steklov Mathematical Institute
Russian Academy of Sciences, Moscow
popovvl@mi.ras.ru

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Vladimir L. Popov, *Underlying varieties and group structures*, arXiv:2105.12861 (2021).

Notation:

- k is an algebraically closed field.
- G is a connected (not necessarily affine) algebraic (real Lie) group over k .
- $\underline{G}$ is the underlying variety (manifold) of G
- $\mathrm{Aut}(X)$ is the automorphism (diffeomorphism) group of an algebraic variety (differentiable manifold) X.

Main question

Since the group operations of G and the topology of $\underline{G}$ agree, there must be a dependence between G and $\underline{G}$. Whence the following

Question

To what extent is the group structure of G determined by $\underline{G}$?

Explicitly or implicitly, this question has long been considered in the literature, in particular, in the classical publications by A. Weil, C. Chevalley, A. Borel, A. Grothendieck, M. Rosenlicht, M. Lazard, ...

Example: Abelian varieties

A striking classical example of dependence of group structure on the geometric properties of group variety is the following

Theorem (C. Chevalley, A. Weil)

Let G be a connected algebraic group, whose underlying variety $\underline{G}$ is complete (i.e., G is an abelian variety). Then the group G is commutative.

Example: M. Lazard's theorem

Another classical example is

Example

$$\begin{array}{llcl} \text{C. Chevalley (1950)?:} & G \text{ unipotent} & \Longrightarrow & \underline{G} = \mathbb{A}^n, \\ \text{M. Lazard (1955):} & G \text{ unipotent} & \Longleftarrow & \underline{G} = \mathbb{A}^n \end{array}$$

Unipotent vs reductive groups

For connected **unipotent** algebraic groups, the dependence between G and $\underline{G}$ is weak:

For $n \geqslant 7$, the structures of (automatically unipotent) algebraic groups on $\mathbb{A}^n$ depend on parameters (moduli).

The case of connected **reductive** algebraic groups is an opposite pole:

There are only finitely many (up to isomorphism) groups of any fixed dimension that are products of tori and simply connected semisimple algebraic groups. Their centers are finitely generated Abelian groups. Therefore, there are no moduli in the case of reductive algebraic groups.

This leads to the following principle question of **uniqueness**:

Question

Is the structure of a connected reductive group on an irreducible variety endowed with such a structure unique (up to isomorphism)?

The following example shows that the answer is negative:

Example of nonuniqueness

Example

Let $G = \mathrm{GL}_n$, $n \geqslant 2$. Then $Z = \{\mathrm{diag}(t, \ldots, t) \mid t \in k^\times\}$ is the center of G (it is **connected**), $D = [G, G] = \mathrm{SL}_n$, and $C := Z \cap D = \{\mathrm{diag}(t, \ldots, t) \mid t^n = 1\}$ is the center of D.

Let $H := Z \times D$. Then $Z \times C$ is the center of H (it is **disconnected**). Hence G and H are **not isomorphic groups**. However, $\underline{G}$ and $\underline{H}$ are **isomorphic varieties**.

Proof.

$$\underline{H} \to \underline{G}, \quad (\mathrm{diag}(t, \ldots, t), A) \mapsto \mathrm{diag}(t, 1, \ldots, 1) \cdot A$$

is an isomorphism of varieties because

$$\underline{G} \to \underline{H}, \quad B \mapsto (\mathrm{diag}(\det(B), \ldots, \det(B)), (1/\det(B), 1, \ldots, 1) \cdot B)$$

is its inverse. $\square$

General case

This example turns out to be a manifestation of a **general phenomenon**. Namely, let

G be a connected reductive algebraic group,
$D := [G, G]$ (it is connected semisimple group),
Z be the identity component of the center of G (it is a torus).

The algebraic groups $D \times Z$ and G may be **nonisomorphic**:

The following properties are equivalent:

- $D \times Z$ and G are isomorphic;
- $D \cap Z = \{e\}$;
- the isogeny $D \times Z \to G$, $(d, z) \mapsto dz$, is an isomorphism.

General case

Theorem

There is an injective homomorphism of algebraic groups $\iota\colon Z \hookrightarrow G$ *such that the mapping*

$$\underline{D \times Z} \to \underline{G},\ (d, z) \mapsto d \cdot \iota(z),$$

is an isomorphism of algebraic varieties (but, in general, not a homomorphism of algebraic groups).

Example

If $G = \mathrm{GL}_n$, then $Z = \{\mathrm{diag}(t, \ldots, t) \mid t \in k^\times\}$ and

$$\iota\colon Z \hookrightarrow G, \quad \mathrm{diag}(t, \ldots, t) \mapsto \mathrm{diag}(t, 1, \ldots, 1).$$

Question of B. Kunyavsky

The following answers the question of B. Kunyavsky:

Corollary

The varieties $\underline{G}$ and $\underline{Z \times D}$ are isomorphic.

In particular, the underlying variety of any connected reductive nonsemisimple group of dimension $\geqslant 2$ always **splits into a product of two varieties of positive dimension**.

Question

This leads to the question of the **existence of splitting of the underlying variety of an algebraic group into a product of varieties of positive dimension**.

Here are some results in this direction.

Varieties with invertible functions as direct factors

Theorem

An algebraic variety on which there is a nonconstant invertible regular function, cannot be a direct factor of the underlying variety of a connected semisimple algebraic group.

Field k and topology

Below, unless otherwise stated, $k = \mathbb{C}$. By the Lefschetz principle, the results below are valid for any field k of characteristics zero. Topological terms refer to the classical topology; homology and cohomology are singular.

Curves as direct factors

Theorem

An algebraic curve cannot be a direct factor of the underlying variety of a connected semisimple algebraic group.

Property of direct factors

Theorem

If a d-dimensional algebraic variety X is a direct factor of the underlying variety of a connected reductive algebraic group, then

$$H_d(X,\mathbb{Z}) \simeq \mathbb{Z} \quad \text{and} \quad H_i(X,\mathbb{Z}) = 0 \ \text{ for } i > d.$$

Contractible varieties as direct factors

Corollary

A contractible algebraic variety (in particular, $\mathbb{A}^d$) of positive dimension cannot be a direct factor of the underlying variety of a connected reductive algebraic group.

Surfaces as direct factors

Theorem

An algebraic surface cannot be a direct factor of the underlying variety of a connected semisimple algebraic group.

Properties of G determined by properties of $\underline{G}$

We now turn back to discussing the main question.

Let G be a connected (not necessarily affine) algebraic group.

Theorem (C. Chevalley)

G contains the largest connected affine normal subgroup $G_{\rm aff}$, and the group $G/G_{\rm aff}$ is an abelian variety.

Definition (M. Rosenlicht)

G is called *toroidal* group (nowdays called *semi-abelian variety*) if $G_{\rm aff}$ is a torus.

In the following theorem, no restrictions on k are imposed.

Theorem (Semi-abelianness criterion)

The following two properties of a connected algebraic group G are equivalent:

- *G is a semi-abelian variety;*
- *$\underline{G}$ does not contain subvarieties isomorphic to $\mathbb{A}^1$.*

Notation:

- X is an irreducible algebraic variety.
- $k[X]^\times$ is the multiplicative group of invertible regular functions on X.

Theorem (M. Rosenlich)

Rosenlicht's theorem: $k[X]^\times/k^\times$ is a finitely generated free abelian group.

Definition

$$\mathrm{units}(X) := \mathrm{rank}(k[X]^\times/k^\times).$$

Properties of $\mathrm{units}(X)$

Theorem (M. Rosenlicht)

- *If X and Y are irreducible algebraic varieties, then*

$$\mathrm{units}(X \times Y) = \mathrm{units}(X) + \mathrm{units}(Y).$$

- *If G is a connected algebraic group, then*

$$\mathrm{units}(\underline{G}) = \mathrm{rank}\big(\mathrm{Hom}(G, \mathbb{G}_m)\big).$$

Theorem

If G is a connected algebraic group, then

(i) $\mathrm{units}(G) \leqslant \dim(\underline{G})$;

(ii) *the equality in* (i) *holds if and only G is a torus.*

Notation: invariant $\mathrm{mh}(X)$

Definition

$$\mathrm{mh}(X) := \max\{d \in \mathbb{Z}_{\geqslant 0} \mid H_d(G, \mathbb{Q}) \neq 0\}.$$

Property:

If X is a nonsingular affine algebraic variety, then

$$\mathrm{mh}(X) \leqslant \dim(X).$$

$\dim(\mathrm{Rad}_u(G))$ and $\dim(\mathrm{Rad}(G))$ are expressed in terms of the properties of $\underline{G}$

Theorem

If G is a connected affine algebraic group, then

$$\dim(\mathrm{Rad}_u(G)) = \dim(\underline{G}) - \mathrm{mh}(\underline{G}),$$
$$\dim(\mathrm{Rad}(G)) = \dim(\underline{G}) - \mathrm{mh}(\underline{G}) + \mathrm{units}(\underline{G}).$$

Reductivity criterion for G in terms of the properties of $\underline{G}$

Theorem (Reductivity criterion)

The following properties of a connected affine algebraic group G are equivalent:

- *G is reductive;*
- $\dim(\underline{G}) = \mathrm{mh}(\underline{G})$.

Semisimplicity criterion for G in terms of the properties of $\underline{G}$

Theorem (Semisimplicity criterion)

The following properties of a connected affine algebraic group G are equivalent:

- *G is semisimple;*
- $\dim(\underline{G}) = \mathrm{mh}(\underline{G}) - \mathrm{units}(\underline{G})$.

The previous corollaries show that the properties of a connected affine algebraic group to be **reductive** or semisimple are expressed in terms the geometric properties of its underlying variety. The next result **generalizes M. Lazard's theorem** and shows that the same is true for the property of an a connected affine algebraic group to be **solvable**.

In it, is used the following

Notation:

$\mathbb{A}^n_*$ is the product of n copies of the variety $\mathbb{A}^1 \setminus \{0\}$.

Theorem (Solvability criterion)

The following properties of a connected affine algebraic group S are equivalent:

- *S is solvable;*
- $\mathrm{mh}(\underline{S}) = \mathrm{units}(\underline{S})$*;*
- *there are non-negative $t, r \in \mathbb{Z}$ such that $\underline{S}$ is isomorphic to $\mathbb{A}_*^t \times \mathbb{A}^r$; wherein automatically $t = \mathrm{units}(\underline{S})$.*

If these properties hold, then $\mathrm{units}(\underline{S})$ is the dimension of maximal tori of S and $\dim(\mathscr{R}_u(S)) = \dim(\underline{S}) - \mathrm{units}(\underline{S})$.

Unipotency criterion for G in terms of the properties of $\underline{G}$

Theorem (Unipotency criterion)

The following properties of a connected affine algebraic group G are equivalent:

- *G is unipotent;*
- $\mathrm{mh}(\underline{G}) = \mathrm{units}(\underline{G}) = 0$;
- *$\underline{G}$ is isomorphic to $\mathbb{A}^{\dim(\underline{U})}$.*

Toricity criterion for G in terms of the properties of $\underline{G}$

Theorem (Toricity criterion)

The following properties of a connected affine algebraic group G are equivalent:

- *G is a torus;*
- $\dim(\underline{G}) = \mathrm{units}(\underline{G})$;
- *$\underline{G}$ is isomorphic to $\mathbb{A}_*^{\dim(\underline{G})}$.*

Different group structures on the same variety

For some types of groups the uniqueness holds:

Theorem

Let G_1 and G_2 be algebraic groups, one of which is **semi-abelian**. *The following properties are equivalent:*

- *G_1 are G_2 isomorphic algebraic groups;*
- *$\underline{G_1}$ and $\underline{G_2}$ are isomorphic algebraic varieties.*

Corollary

Isomorphness of algebraic groups, among which there is either a torus or an abelian variety, is equivalent to isomorphness of their group varieties.

In particular, this yields the following statement that is well-known since long ago

Corollary (A. Weil, 1948)

Abelian varieties are isomorphic if and only if their underlying varieties are isomorphic.

Semi-abelian varieties are commutative. If $\mathrm{char}(k) = 0$, the following generalization of the above theorem has been proved recently

Theorem (G. A. Dill (arXiv:2107.14667 30 Jul 2021))

Let $\mathbf{char}(\boldsymbol{k}) = \mathbf{0}$ *and let* G_1 *and* G_2 *be two connected* **commutative** *algebraic groups. The following properties are equivalent:*

- G_1 *are* G_2 *isomorphic algebraic groups;*
- $\underline{G_1}$ *and* $\underline{G_2}$ *are isomorphic algebraic varieties.*

Remark

It is well-known that if $\mathrm{char}(k) > 0$, then there exist nonisomorphic commutative connected unipotent algebraic groups of the same dimension. Therefore, in the above theorem the condition $\mathrm{char}(k) = 0$ is essential.

We have seen above that **for reductive groups the uniqueness does not hold**. However, the following results show that for them **the nonuniqueness is rather limited**.

Maximal reductive subgroups

Theorem

Let G_1 and G_2 be connected affine algebraic groups, and let R_i be a maximal reductive algebraic subgroup of G_i, $i = 1, 2$.

If $\underline{G_1}$ and $\underline{G_2}$ are isomorphic algebraic varieties, then R_1 and R_2 are connected algebraic groups, and $\mathrm{Lie}(R_1)$ and $\mathrm{Lie}(R_2)$ are isomorphic Lie algebras.

Theorem

Let R be a connected reductive algebraic group.

(a) *If G is an algebraic group such that $\underline{G}$ and $\underline{R}$ are isomorphic algebraic varieties, then*

- *G is connected and reductive, and $\mathrm{Lie}\,(R)$ and $\mathrm{Lie}\,(G)$ are isomorphic Lie algebras;*
- *if R is semisimple and simply connected, then R and G are isomorphic algebraic groups.*

(b) *There are only finitely many (up isomorphism) algebraic groups G such that $\underline{G}$ and $\underline{R}$ are isomorphic algebraic varieties.*

The proof of statement (b) of the previous theorem is based on the following **finiteness theorem for connected reductive groups** (in which no constraints on the filed k are imposed):

Finiteness theorem for connected reductive groups

Theorem F

There are only finitely many (up to isomorphism) connected reductive algebraic groups of every fixed rank.

Comment: Theorem F is a fundamental fact in the theory of algebraic groups and the theory of compact Lie groups. For semisimple groups this theorem is well known and follows from the finiteness of the centers of such groups. However, in full generality (i.e., for reductive and not only semisimple groups), I failed to find it in the literature.

Application 1: Since connected reductive groups are classified by their root data, Theorem F yields

Theorem

There are only finitely many (up to isomorphism) root data of every fixed rank.

Application 2: In view of the correspondence between connected reductive algebraic groups and connected compact real Lie groups, given by passing to a compact real form, Theorem F yields

Theorem

There are only finitely many (up to isomorphism) connected compact real Lie groups of every fixed rank.

Nonsemisimple reductive groups vs semisimple

The first theorem of this talk provides a way to construct nonisomorphic connected reductive **nonsemisimple** groups with isomorphic underlying varieties. This theorem does not yield a way to construct **semisimple** groups with this property.

However, **nonisomorphic semisimple groups with isomorphic underlying varieties do exist**. We turn to describing **a method that allows one to construct them** (in this construction, no constraints are imposed on the field k).

Notation:

- n a positive integer.
- A an abstract group.
- $G := A^{\times n} := A \times \cdots \times A \quad (n \text{ factors})$.
- $\mathscr{C}(G)$ the center of G.
- F_n a free group of rank n with the identity element 1.
- $x_1, \ldots, x_n$ a free system of generators of F_n.
- If $g = (a_1, \ldots, a_n) \in G$, $a_j \in A$, and $w \in F_n$, then $w(g) \in A$ is obtained from w by replacing x_j with a_j for every j.
- Every $\sigma \in \mathrm{End}(F_n)$ determines the map

$$\sigma_G \colon G \to G, \quad g \mapsto (\sigma(x_1)(g), \ldots, \sigma(x_n)(g)).$$

(in general, σ_G is **not** a group homomorphism!)

Proposition

Let σ and $\tau \in \mathrm{End}(F_n)$. Then

(a) $(\sigma \circ \tau)_G = \tau_G \circ \sigma_G$.

(b) $\underline{1} = \mathrm{id}$.

(c) $\sigma_G(S^{\times n}) \subseteq S^{\times n}$ *for any subgroup S of A.*

(d) *If B is a group, $H := B^{\times n}$, and $\gamma\colon A \to B$ a homomorphism, then $\gamma_n\colon G \to H$, $(g_1, \ldots, g_n) \mapsto (\gamma(g_1), \ldots, \gamma(g_n))$, is $\mathrm{End}(F_n)$-equivariant, i.e., $\gamma_n \circ \sigma_G = \sigma_H \circ \gamma_n$.*

(e) $\sigma_G(gc) = \sigma_G(g)\sigma_G(c)$ *for all $g \in G$, $c \in \mathscr{C}(G)$.*

(f) *σ_G commutes with the diagonal action of A on G by conjugation.*

(g) *If A is an algebraic group (a real Lie group), then σ_G is a morphism (differentiable mapping).*

Action of $\mathrm{Aut}(F_n)$ on $\underline{G}$

Corollary

$$\mathrm{Aut}(F_n) \to \mathrm{Sym}(G), \quad \sigma \mapsto (\sigma^{-1})_G,$$

is a group homomorphism defining an **action** *of* $\mathrm{Aut}(F_n)$ *on the* **set** G.

If A *(hence* G*) is* **an algebraic group (real Lie group)**, *then* σ_G *for every* $\sigma \in \mathrm{Aut}(F_n)$ *is* **an automorpism (diffeomorphism) of the variety (manifold)** $\underline{G}$.

Construction of non-isomorphic semisimple algebraic groups with isomorphic underlying varieties

G a connected semisimple algebraic (compact real Lie) group.
$\sigma \in \mathrm{Aut}(F_n)$.
C a subgroup of $\mathscr{C}(G)$.
Consider **two actions** of C on the variety (manifold) $\underline{G}$:

$$c(g) := \begin{cases} c \cdot g & \text{for the first action,} \\ \sigma_G(c) \cdot g & \text{for the second action} \end{cases} \qquad c \in C, g \in G.$$

Property (e) implies that the variety (manifold) automorphism

$$\sigma_G \colon \underline{G} \to \underline{G}$$

is C-**equivariant** if one considers the **first** action of C on the **left** copy of $\underline{G}$ and the **second** action on the **right** copy.

Construction of non-isomorphic semisimple algebraic groups with isomorphic underlying varieties

Therefore, this automorphism **descends to the quotients** inducing their **isomorphism (diffeomorphism)**:

$$\begin{array}{ccc}
\underline{G} & \xrightarrow{\sigma_G} & \underline{G} \\
\downarrow & & \downarrow \\
\underline{G/C} & \xrightarrow{\simeq} & \underline{G/\sigma_G(C)}
\end{array}$$

Thus G/C and $G/\sigma_G(C)$ are **connected semisimple algebraic (compact real Lie) groups whose underlying varieties (manifolds) are isomorphic (diffeomorphic)**.

Construction of non-isomorphic semisimple algebraic groups with isomorphic underlying varieties

However, in general, algebraic (real Lie) groups G/C and $G/\sigma_G(C)$ are **not isomorphic**.

Theorem

If G (equivalently, A) is simply connected, then the following properties are equivalent:

(i) *G/C and $G/\sigma_G(C)$ are isomorphic semisimple algebraic (real compact Lie) groups;*

(ii) *C and $\sigma_G(C)$ lie in the same orbit of the natural action of $\mathrm{Aut}(G)$ on the set of subgroups of $\mathscr{C}(G)$.*

Remark. The action of $\mathrm{Aut}(G)$ in (ii) is reduced to that of $\mathrm{Out}(G)$, which is isomorphic to the automorphism group of the Dynkin diagram of G.

Example

Example

• Let A be a **simply connected simple** algebraic (compact real Lie) group with **non-trivial center** $\mathscr{C}(A)$.

• Take $n = 2$, i.e.,

$$G = A \times A. \tag{1}$$

• Take $\sigma \in \mathrm{Aut}(F_2)$ defined by

$$\sigma(x_1) = x_1, \quad \sigma(x_2) = x_1 x_2^{-1}$$

• Let S be a **non-trivial subgroup** of $\mathscr{C}(A)$. Take

$$C := \{(s, s) \mid s \in S\}. \tag{2}$$

Then

$$\sigma_G(C) = \{(s, e) \mid s \in S\}. \tag{3}$$

Example

In view of the simplicity of A, the elements of $\mathrm{Out}(G)$ carry out permutations of the factors on the right-hand side of equality (1). This, (2), and (3) imply that C and $\sigma_G(C)$ **do not lie in the same** $\mathrm{Out}(G)$**-orbit**. Therefore,

$$G/C = (A \times A)/C \quad \text{and} \quad G/\sigma_G(C) = (A/S) \times A$$

are **nonisomorphic algebraic (compact real Lie) groups, whose underlying varieties (manifolds) are isomorphic (diffeomorphic)**.

Example (algebraic groups)

For example, let $A = \mathrm{SL}_d$, $d \geqslant 2$, and $S = \langle z \rangle$, where $z = \mathrm{diag}(\varepsilon, \ldots, \varepsilon) \in A$, $\varepsilon \in k$ is a primitive d-th root of 1.

This yields **nonisomorphic algebraic groups**

$$G/C = (\mathrm{SL}_d \times \mathrm{SL}_d)/\langle (z, z) \rangle, \quad G/\sigma_G(C) = \mathrm{PSL}_d \times \mathrm{SL}_d,$$

whose underlying varieties are isomorphic.

Example (algebraic groups)

If $d = 2$, then $G = \mathrm{Spin}_4$, $G/C = \mathrm{SO}_4$. Hence **the algebraic groups**

$$\mathrm{SO}_4 \;\text{ and }\; \mathrm{PSL}_2 \times \mathrm{SL}_2$$

are not isomorphic but their underlying varieties $\underline{\mathrm{SO}_4}$ **and** $\underline{\mathrm{PSL}_2 \times \mathrm{SL}_2}$ **are isomorphic**.

Example (compact real Lie groups)

Taking $A = \mathrm{SU}_d$ and the same group S as above, we obtain that

$$G/C = K_1 := (\mathrm{SU}_d \times \mathrm{SU}_d)/C, \quad G/\sigma_G(C) = K_2 := \mathrm{PU}_d \times \mathrm{SU}_d$$

are **nonisomorphic compact real Lie groups whose underlying manifolds $\underline{K_1}$ and $\underline{K_2}$ are diffeomorphic**.

Remark

For $d = p^r$ with prime p, this was proved in 1965 by P. F. Baum, W. Browder, who deduced the nonisomorphism of K_1 and K_2 from the nonisomorphism of their Pontryagin rings $H_*(K_1, \mathbb{Z}/p\mathbb{Z})$ and $H_*(K_2, \mathbb{Z}/p\mathbb{Z})$ (discribing these rings is a nontrivial problem).

Example (compact real Lie groups)

If $d = 2$, then $K_1 = \mathrm{SO}_4$, $K_2 = \mathrm{SO}_3 \times \mathrm{SU}_2$. Hence **the compact real Lie groups**

$$\mathrm{SO}_4 \ \text{ and } \ \mathrm{SO}_3 \times \mathrm{SU}_2.$$

are not isomorphic but their underlying manifolds $\underline{\mathrm{SO}_4}$ **and** $\underline{\mathrm{SO}_3 \times \mathrm{SU}_2}$ **are diffeomorphic**.

Remark

This is an example from the textbooks where a diffemorphism between $\underline{\mathrm{SO}_4}$ and $\underline{\mathrm{SO}_3 \times \mathrm{SU}_2}$ is constructed by means of quaternions (see A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002).

The constructed above nonisomorphic connected semisimple algebraic (compact real Lie) groups with isomorphic (diffeomorphic) underlying varieties (manifolds) are **not simple**. The following theorems show that this is not accidental:

Connected simple algebraic groups

Theorem

Let G_1 and G_2 be algebraic groups, one of which is connected and **simple**. *The following properties are equivalent:*

- *G_1 and G_2 are isomorphic algebraic groups;*
- *$\underline{G_1}$ and $\underline{G_2}$ are isomorphic varieties;*

Connected simple compact real Lie groups

Theorem

Two connected compact **simple** *real Lie groups are isomorphic if and only if their underlying manifolds are homotopy equivalent.*

Application

The above construction, used to find nonisomorphic algebraic groups with isomorphic underlying varieties, finds also an application in exploration of the automorphism groups of algebraic varieties, which has become a trend over the last decade.

Namely, maintain the above notation:

$$\begin{gathered} A \ \text{ a group,} \\ G := A \times \cdots \times A \quad (n \text{ factors}), \\ \mathrm{Aut}(F_n) \to \mathrm{Sym}(G), \quad \sigma \mapsto (\sigma^{-1})_G \end{gathered} \tag{4}$$

Solvable vs nonsolvable A

Theorem

- *If A is* **solvable** *and $n \geqslant 3$, then homomorphism (4) is* **not an embedding**.
- *If A is* **nonsolvable** *and any of the following properties hold:*
  - *A is a connected (not necessarily affine) algebraic group;*
  - *A is a connected real Lie group,*

  then homomorphism (4) **is an embedding**.

Corollary

Let A be either a connected (not necessarily affine) **algebraic group** *or a connected* **real Lie group**. *Assume that A is* **nonsolvable**, *and $n \geqslant 1$ and $1 \leqslant s \leqslant n$ are the natural integers. Then*

- $\mathrm{Aut}(\underline{G})$ **contains the group** $\mathrm{Aut}(F_s)$;
- $\mathrm{Aut}(\underline{G})$ *for $n \geqslant 3$* **contains the braid group** B_n;
- $\mathrm{Aut}(\underline{G})$ *for $n \geqslant 3$ is a* **nonlinear group**;
- $\mathrm{Aut}(\underline{G})$ *for $n \geqslant 2$ is a* **nonamenable group**.

Notation:

Cr_d is the **Cremona group of rank** d over k (i.e., $\mathrm{Aut}_k(K)$ for a purely transcendental field extension K/k of the transcendence degree d).

Corollary

For any integers $n > 0$, $d \geqslant 3n$, and $0 < s \leqslant n$,

- Cr_d *contains the group* $\mathrm{Aut}(F_s)$*;*
- Cr_d *for* $n \geqslant 3$ *contains the braid group* B_n*.*

<u>**Notation**</u>:

b_n is the **minimal rank of the Cremona groups containing the braid group** B_n.

Corollary

$$b_n \leqslant 3n \quad \textit{for } n \geqslant 8.$$

Remark

This upper bound is stronger than $b_n \leqslant n(n-1)/2$ following from D. Krammer, *Braid groups are linear*, Annals of Math. **155** (2002), 131–156.

Descending action of $\mathrm{Aut}(F_n)$ to the quotient

Generalization:

Let A be a connected (not necessarily affine) algebraic group and let S be its **finite subgroup**. Then S acts of $\underline{G}$ **diagonally by conjugation** and

- the **geometric quotient** $\pi\colon \underline{G} \to \underline{G}/S$ **exists**;
- the actions of S and $\mathrm{Aut}(F_n)$ on $\underline{G}$ commute.

Therefore, the action of $\mathrm{Aut}(F_n)$ on $\underline{G}$ **descends to the action of** $\mathrm{Aut}(F_n)$ **on** $\underline{G}/S$, thereby defining a group homomorphism

$$\mathrm{Aut}(F_n) \to \mathrm{Aut}(\underline{G}/S). \tag{5}$$

If $S = \{e\}$, then we return back to the previous construction, and (5) turns into (4).

Theorem

Let A be a nonsolvable connected (not necessarily affine) algebraic group, and let S be its finite noncentral subgroup. Then

- *homomorphism* (5) *is an embedding (thereby,* $\mathrm{Aut}(\underline{G}/S)$ *contains the group* $\mathrm{Aut}(F_n)$*);*
- $\mathrm{Aut}(\underline{G}/S)$ *for* $n \geqslant 3$ *contains the braid group;*
- $\mathrm{Aut}(\underline{G}/S)$ *for* $n \geqslant 3$ *is a nonlinear group;*
- $\mathrm{Aut}(\underline{G}/S)$ *for* $n \geqslant 2$ *is a nonamenable group.*

It can be shown that among varieties $\underline{G}/S$ there are **affine** and **nonaffine** ones, among affine ones there are **smooth** and **nonsmooth**, **rational** and **nonrational**.