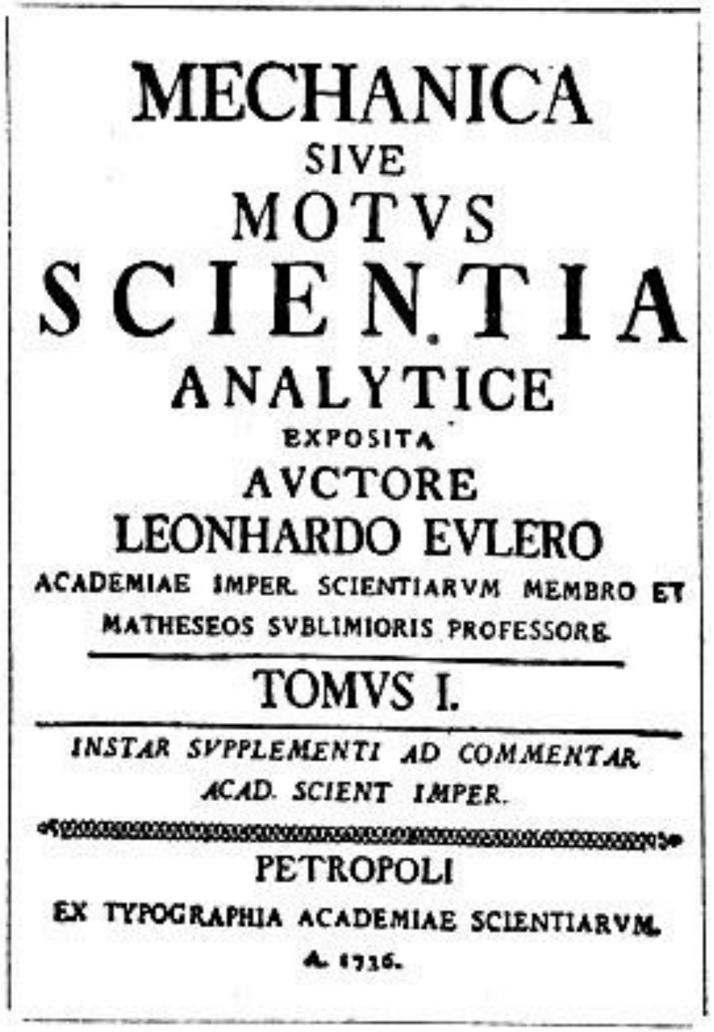
THE APPLICATION HISTORY OF THE ELLIPTIC FUNCTION THEORY TO ONE MECHANICAL PROBLEM

Anna O. Yulina

A.F. Mozhaysky Military-Space Academy

Leonhard Euler (1707–1783)

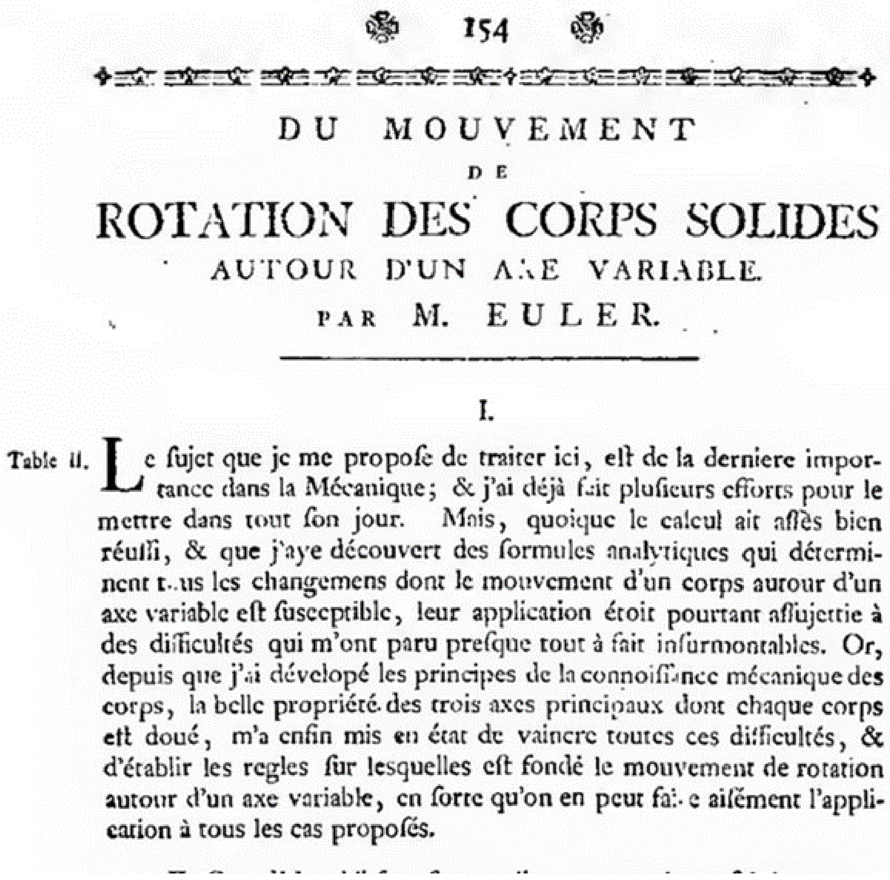
1736, 1758


MECHANICA
SIVE
MOTVS
SCIENTIA
ANALYTICE
EXPOSITA
AVCTORE
LEONHARDO EVLERO
ACADEMIAE IMPER. SCIENTIARVM MEMBRO ET
MATHESEOS SVBLIMIORIS PROFESSORE.

TOMVS I.

INSTAR SVPPLEMENTI AD COMMENTAR.
ACAD. SCIENT. IMPER.

PETROPOLI
EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
A. 1736.


154

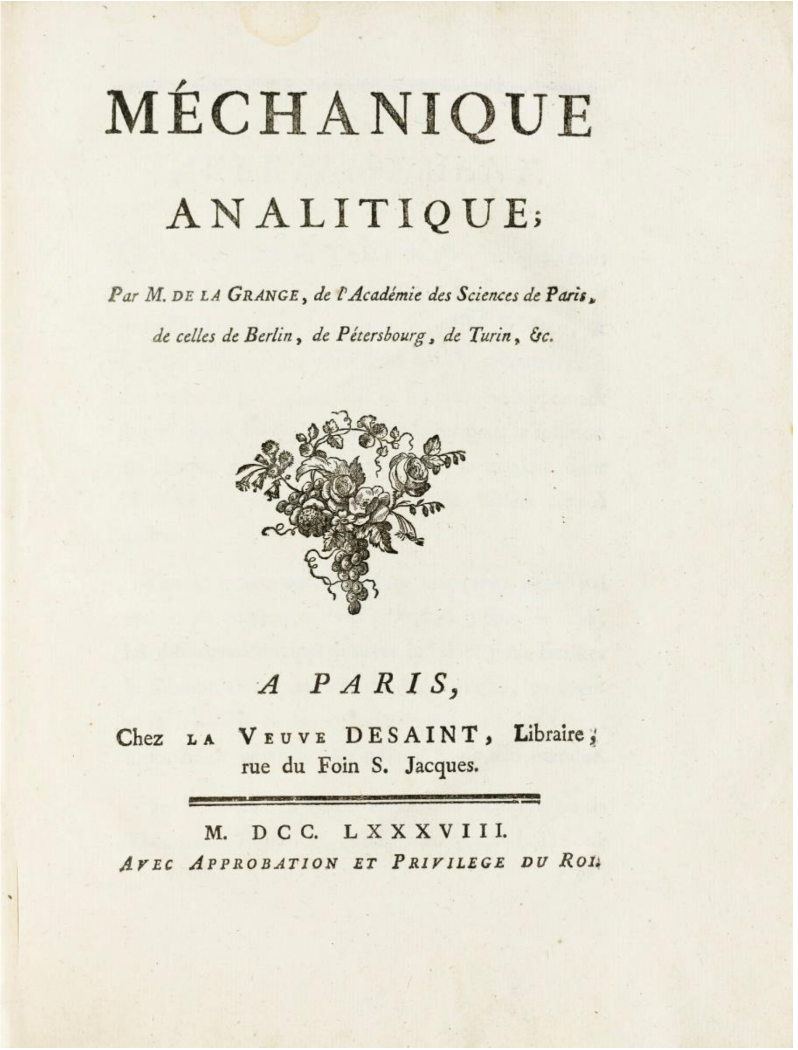
DU MOUVEMENT
DE
ROTATION DES CORPS SOLIDES
AUTOUR D'UN AXE VARIABLE.
PAR M. EULER.

I.

Table II. Le ſujet que je me propoſe de traiter ici, eſt de la derniere importance dans la Mécanique; & j'ai déjà fait pluſieurs efforts pour le mettre dans tout ſon jour. Mais, quoique le calcul ait aſſès bien réuſſi, & que j'aye découvert des formules analytiques qui déterminent tous les changemens dont le mouvement d'un corps autour d'un axe variable eſt ſusceptible, leur application étoit pourtant aſſujettie à des difficultés qui m'ont paru preſque tout à fait inſurmontables. Or, depuis que j'ai dévelopé les principes de la connoiſſance mécanique des corps, la belle propriété des trois axes principaux dont chaque corps eſt doué, m'a enfin mis en état de vaincre toutes ces difficultés, & d'établir les regles ſur leſquelles eſt fondé le mouvement de rotation autour d'un axe variable, en ſorte qu'on en peut faire aiſément l'application à tous les cas propoſés.

Joseph Louis Lagrange (1736–1813)

1788


MÉCHANIQUE

ANALITIQUE;

Par M. DE LA GRANGE, de l'Académie des Sciences de Paris, de celles de Berlin, de Pétersbourg, de Turin, &c.

A PARIS,

Chez LA VEUVE DESAINT, Libraire, rue du Foin S. Jacques.

M. DCC. LXXXVIII.

AVEC APPROBATION ET PRIVILEGE DU ROI.

Giulio Carlo de' Toschi di Fagnano (1682−1766)

1750

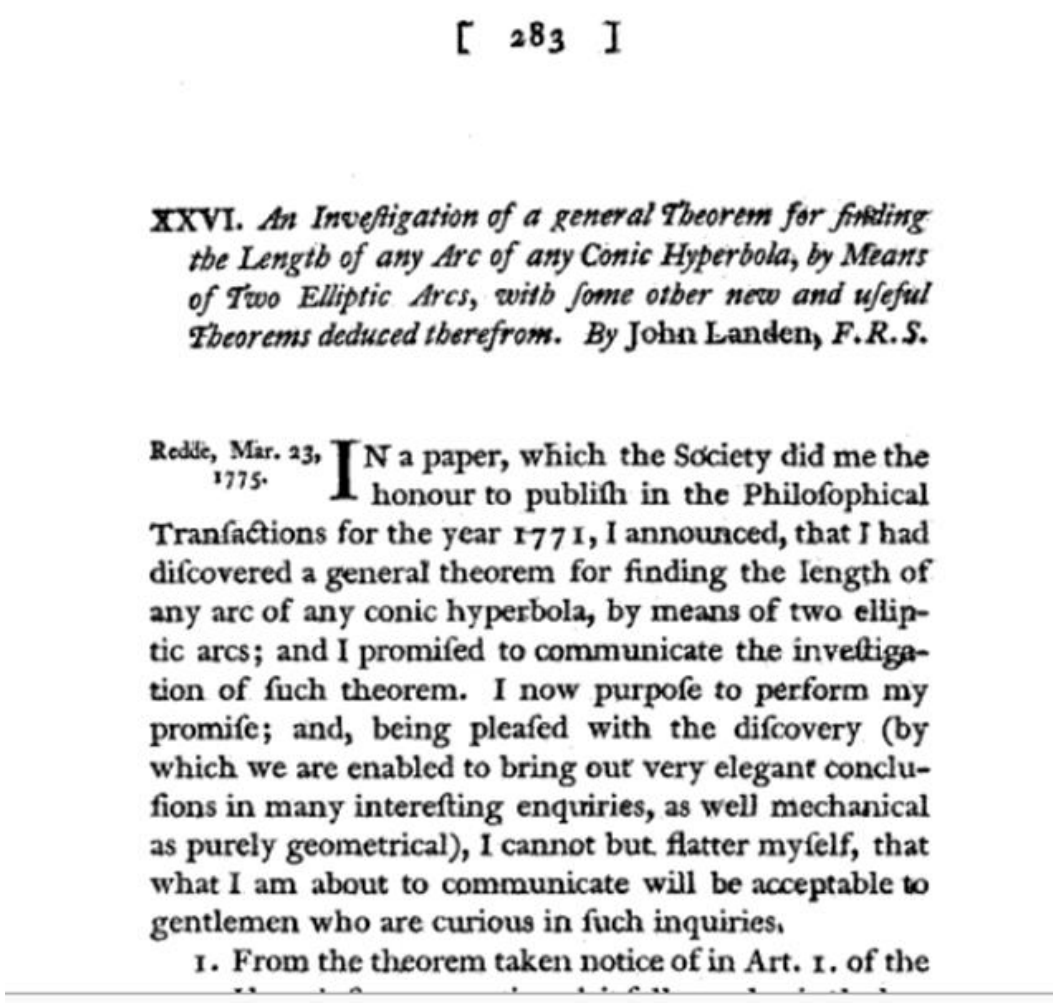

PRODUZIONI
MATEMATICHE
DEL CONTE GIULIO CARLO
DI FAGNANO,
MARCHESE DE' TOSCHI,
E DI SANT' ONORIO
NOBILE ROMANO, E PATRIZIO SENOGAGLIESE
ALLA SANTITA' DI N. S.
BENEDETTO XIV.
PONTEFICE MASSIMO.
TOMO PRIMO.

IN PESARO

L' ANNO DEL GIUBBILEO M. DCC. L.
NELLA STAMPERIA GAVELLIANA
CON LICENZA DE' SUPERIORI.

JOHN LANDEN (1719–1790)

1775



[283]

XXVI. *An Inveſtigation of a general Theorem for finding the Length of any Arc of any Conic Hyperbola, by Means of Two Elliptic Arcs, with ſome other new and uſeful Theorems deduced therefrom. By* John Landen, *F.R.S.*

Redde, Mar. 23, 1775.

IN a paper, which the Society did me the honour to publiſh in the Philoſophical Tranſactions for the year 1771, I announced, that I had diſcovered a general theorem for finding the length of any arc of any conic hyperbola, by means of two elliptic arcs; and I promiſed to communicate the inveſtigation of ſuch theorem. I now purpoſe to perform my promiſe; and, being pleaſed with the diſcovery (by which we are enabled to bring out very elegant concluſions in many intereſting enquiries, as well mechanical as purely geometrical), I cannot but flatter myſelf, that what I am about to communicate will be acceptable to gentlemen who are curious in ſuch inquiries.

1. From the theorem taken notice of in Art. 1. of the

Adrien-Marie Legendre (1752−1833)

1792

MÉMOIRE

SUR

LES TRANSCENDANTES

ELLIPTIQUES,

Où l'on donne des méthodes faciles pour comparer et évaluer ces transcendantes, qui comprennent les arcs d'ellipse, et qui se rencontrent fréquemment dans les applications du calcul intégral.

Lu à la ci-devant Académie des Sciences en avril 1792.

PAR ADRIEN-MARIE LE GENDRE.

A PARIS,

Chez { le C. DU PONT, Imprimeur-Libraire, rue de la Loi, N°. 14.
le C. FIRMIN DIDOT, Libraire rue Thionville.

L'AN DEUXIÈME DE LA RÉPUBLIQUE.

MÉMOIRE

SUR

LES TRANSCENDANTES

ELLIPTIQUES.

PAR ADRIEN-MARIE LE GENDRE.

UN grand nombre d'intégrales peuvent se déterminer par le seul secours des arcs de cercle et des logarithmes, qui sont les plus simples des quantités transcendantes; mais pour étendre les applications du calcul intégral, il faut nécessairement avoir recours à des transcendantes plus composées. C'est en examinant avec soin la nature et les propriétés de ces transcendantes, au moins de celles dont l'usage est le plus fréquent, qu'on parviendra à simplifier considérablement les résultats de la théorie, et à en rendre l'usage commode pour la pratique.

Les arcs d'ellipse sont, après les arcs de cercle et les logarithmes, une des transcendantes les plus simples, et dont on pourroit faire en quelque sorte un nouvel instrument de calcul, si une fois on s'étoit familiarisé avec leurs propriétés, et que l'on eût des moyens faciles de les évaluer avec précision. C'est sur quoi nous avons proposé quelques vues nouvelles dans les Mémoires de l'Académie des Sciences de Paris, année 1786.

Mais si les arcs d'ellipse ajoutent beaucoup aux moyens de l'analyse, sur-tout depuis qu'on a remarqué que les arcs

A 2

Niels Henrik Abel (1802−1829)

1827

12.

Recherches sur les fonctions elliptiques.

(Par M. *N. H. Abel.*)

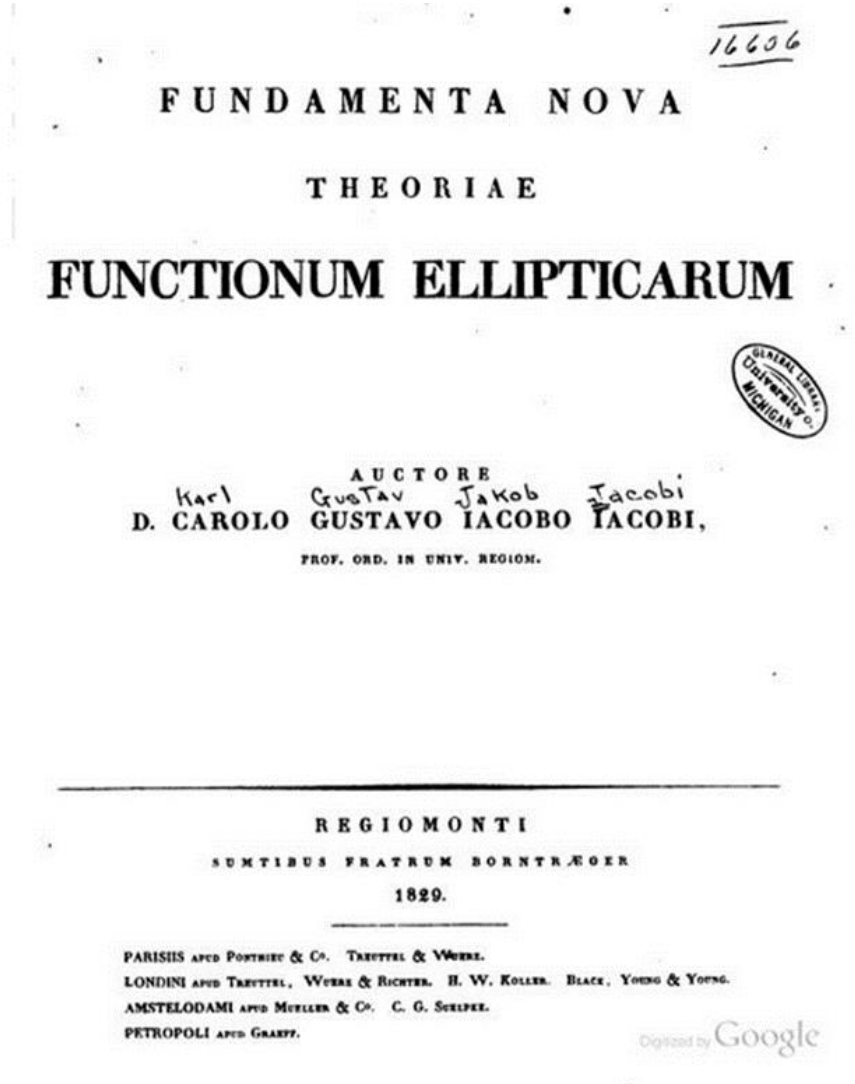
Depuis longtems les fonctions logarithmiques, et les fonctions exponentielles et circulaires ont été les seules fonctions transcendantes, qui ont attiré l'attention des géomètres. Ce n'est que dans les derniers tems, qu'on a commencé à en considérer quelques autres. Parmi celles-ci il faut distinguer les fonctions, nommées elliptiques, tant pour leurs belles propriétés analytiques, que pour leur application dans les diverses branches des mathématiques. La première idée de ces fonctions a été donnée par l'immortel Euler, en démontrant, que l'équation séparée

$$1. \quad \frac{\partial x}{\sqrt{(\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4)}}+\frac{\partial y}{\sqrt{(\alpha+\beta y+\gamma y^2+\delta y^3+\varepsilon y^4)}}=0$$

est intégrable algébriquement. Après Euler, Lagrange y a ajouté quelque chose, en donnant son élégante théorie de la transformation de l'intégrale $\int\frac{R.\partial x}{\sqrt{[(1-p^2x^2)(1-q^2x^2)]}}$, ou R est une fonction rationelle de x. Mais le premier et, si je ne me trompe, le seul, qui ait approfondi la nature de ces fonctions, est *M.* Legendre, qui, d'abord dans un mémoire sur les fonctions elliptiques, et ensuite dans ses excellents exercices de mathématiques, a développé nombre de poropriétés élégantes de ces fonctions, et a montré leur application. Lors de la publication de cet ouvrage, rien n'a été ajouté à la théorie de M. Legendre. Je crois, qu'on ne verra pas ici sans plaisir des recherches ultérieures sur ces fonctions.

Carl Gustav Jacob Jacobi (1804−1851)

1829


16606

FUNDAMENTA NOVA

THEORIAE

FUNCTIONUM ELLIPTICARUM

AUCTORE

Karl Gustav Jakob Jacobi

D. CAROLO GUSTAVO IACOBO IACOBI,

PROF. ORD. IN UNIV. REGIOM.

REGIOMONTI

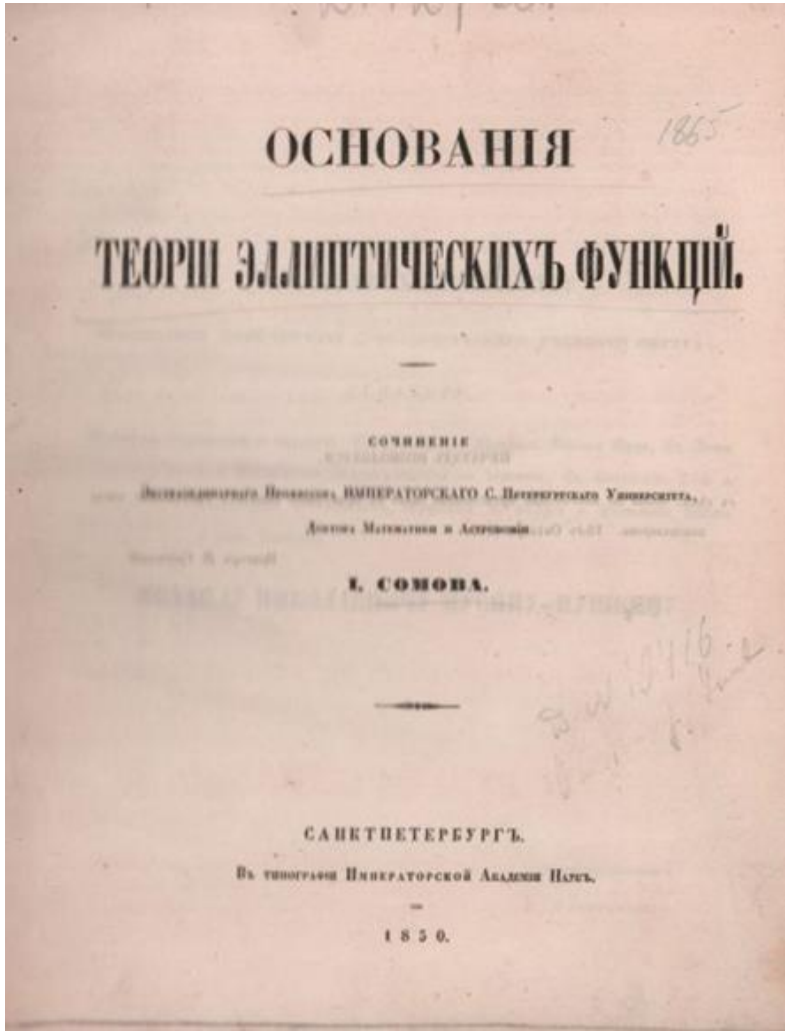
SUMTIBUS FRATRUM BORNTRÆGER

1829.

PARISIIS APUD PONTHIEU & C°. TREUTTEL & WÜRTZ.
LONDINI APUD TREUTTEL, WÜRTZ & RICHTER. H. W. KOLLER. BLACK, YOUNG & YOUNG.
AMSTELODAMI APUD MUELLER & C°. C. G. SULPKE.
PETROPOLI APUD GRAEFF.

Osip I. Somov (1815-1876)

1850



ОСНОВАНІЯ

ТЕОРІИ ЭЛЛИПТИЧЕСКИХЪ ФУНКЦІЙ.

СОЧИНЕНІЕ

[illegible] ИМПЕРАТОРСКАГО С. Петербургскаго Университета,

Доктора Математики и Астрономіи

І. СОМОВА.

САНКТПЕТЕРБУРГЪ.

Въ типографіи Императорской Академіи Наукъ.

1850.

Karl Theodor Wilhelm Weierstraß (1815−1897)

1862

$$\wp(z) = \wp(z;\, 2\omega_1,\, 2\omega_3) =$$

$$= \frac{1}{z^2} + \sum\nolimits'^{+\infty}_{m_1,\, m_3 = -\infty} \left[\frac{1}{(z - 2\Omega_{m_1, m_3})^2} - \frac{1}{(2\Omega_{m_1, m_3})^2} \right] =$$

$$= \frac{1}{z^2} + c_2 z^2 + c_4 z^4 + \ldots,$$

Sofya V. Kovalevskaya (1850–1891)

1888

SUR LE PROBLÈME DE LA ROTATION

D'UN CORPS SOLIDE AUTOUR D'UN POINT FIXE[1]

PAR

SOPHIE KOWALEVSKI

À STOCKHOLM.

§ 1.

Le problème de la rotation d'un corps solide pesant autour d'un point fixe peut se ramener, comme on sait, à l'intégration du système d'équations différentielles suivant:

$$A\frac{dp}{dt} = (B - C)qr + Mg(y_0\gamma'' - z_0\gamma'), \qquad \frac{d\gamma}{dt} = r\gamma' - q\gamma'',$$

$$(1) \qquad B\frac{dq}{dt} = (C - A)rp + Mg(z_0\gamma - x_0\gamma''), \qquad \frac{d\gamma'}{dt} = p\gamma'' - r\gamma,$$

$$C\frac{dr}{dt} = (A - B)pq + Mg(x_0\gamma' - y_0\gamma), \qquad \frac{d\gamma''}{dt} = q\gamma - p\gamma'.$$

- **The history of the emergence of the concept of an elliptic function**

- **Mathematical preamble. The concept of an elliptic function. Reduction, comparison, and transformation of such functions**

- **Abel's theorem. Representation of elliptic functions in terms of theta functions**

- **Solution of the problem of the rotation of a rigid body about a fixed point in the case of an initial impact**

Mathematical preamble. The concept of an elliptic function. Reduction, comparison, and transformation of such functions

The concept of an elliptic function.

$$z = \int f(x, \sqrt{R})dx,$$

R is an entire function with respect to x

f is rational with respect to x and R

$$\int \frac{P dx}{\sqrt{R}} \sim a_0 + a_1 x + a_2 x^2 + \cdots + a_{m-3} x^{m-3}$$

$$P = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m$$

$$b_0 \int \frac{dx}{\sqrt{R}} + b_1 \int \frac{x dx}{\sqrt{R}} + b_2 \int \frac{x^2 dx}{\sqrt{R}} + \ldots + b_m \int \frac{x^m dx}{\sqrt{R}} = (a_0 + a_1 x + a_2 x^2 + \ldots + a_{m-3} x^{m-3}) \sqrt{R}.$$

m> 2

$m \geq 3$ - algebraic representation

$m < 3$ - transcendental with respect to the function x

$$\int \frac{dx}{\sqrt{R}}, \int \frac{xdx}{\sqrt{R}}, \int \frac{x^2dx}{\sqrt{R}}$$

$$\int \frac{Pdx}{\sqrt{R}}, \qquad P = \frac{A}{(x-\alpha)^m}$$

Where A and α are constants, and m is a positive integer

$$A \int \frac{dx}{(x-\alpha)^m \sqrt{R}} \qquad x - \alpha = \frac{1}{z}. \qquad \int \frac{x^m dx}{\sqrt{R}}$$

$$\int \frac{(x-\alpha)^m dx}{\sqrt{R}}$$

$$\int \frac{dx}{(x-\alpha)^m \sqrt{R}} = b_0 \int \frac{dx}{\sqrt{R}} + b_1 \int \frac{dx}{(x-\alpha)\sqrt{R}} + b_2 \int \frac{dx}{(x-\alpha)^2 \sqrt{R}} + \left(\frac{a_0}{(x-\alpha)^2} + \frac{a_1}{(x-\alpha)^3} + \ldots + \frac{a_{m-3}}{(x-\alpha)^{m-3}}\right)$$

$\int \frac{P dx}{\sqrt{R}}$, where P is some rational function

$$\int \frac{dx}{\sqrt{R}}, \int \frac{x\,dx}{\sqrt{R}}, \int \frac{x^2 dx}{\sqrt{R}}, \int \frac{dx}{(x-\alpha)\sqrt{R}}.$$

$$x = \frac{p+qy}{1+y}, R_1 = c(y^2 \pm a)(y^2 \pm b), \lambda = \frac{p-\alpha}{q-\alpha}$$

$$\int \frac{dy}{\sqrt{R_1}}, \int \frac{y^2 dy}{\sqrt{R_1}}, \int \frac{dy}{(y^2-\lambda^2)\sqrt{R_1}}$$

$$\int \frac{dy}{(y^2-\lambda^2)\sqrt{R_1}} \Longrightarrow \int \frac{f(\sin^2\varphi)\,d\varphi}{\sqrt{(1-k^2\sin^2\varphi)}}$$

Substitution $$y^2 = \frac{A + B\sin^2\varphi}{C + D\sin^2\varphi}$$

$$\int \frac{y^2\,dy}{\sqrt{R_1}} = \int \frac{\alpha + \beta\sin^2\varphi}{\gamma + \delta\sin^2\varphi} \cdot \frac{d\varphi}{\sqrt{(1 - k^2\sin^2\varphi)}}$$

Notation $$\sqrt{(1 - k^2\sin^2\varphi)} = \Delta(k, \varphi)$$

amplitude

module

Modular function

$$\int \frac{y^2\,dy}{\sqrt{R_1}} = \int \frac{\alpha + \beta\sin^2\varphi}{\gamma + \delta\sin^2\varphi} \cdot \frac{d\varphi}{\Delta\varphi}$$

$$H(\varphi) = \int \frac{\alpha + \beta\sin^2\varphi}{\gamma + \delta\sin^2\varphi} \cdot \frac{d\varphi}{\Delta\varphi}$$

$$H(\varphi)$$

Some properties

$$H(-\varphi) = -H(\varphi)$$

$$H(\varphi) = H(n\pi \pm \psi) = 2nH(\tfrac{\pi}{2}) \pm H(\psi), 0<\psi<\frac{\pi}{2}.$$

Classification

$$F(\varphi) = \int_0^\varphi \frac{d\varphi}{\Delta\varphi} \text{ an elliptic function of the first kind,}$$

$$E(\varphi) = \int_0^\varphi \Delta\varphi * d\varphi \text{ an elliptic function of the second kind,}$$

$$\Pi(n,\varphi) = \int_0^\varphi \frac{d\varphi}{(1+n)sin^2\varphi\Delta\varphi} \text{ an elliptic function of the third kind.}$$

Hyperelliptic (ultra elliptic functions)

$$n = \frac{\delta}{\gamma} \text{ is an elliptic function parameter } \Pi(n, \varphi)$$

$$\sin \varphi = x.$$

$$F(\varphi), E(\varphi), \Pi(n, \varphi)$$

$$T_1(x) = \int_0^x \frac{dx}{\Delta x},$$

$$T_2(x) = \int_0^x \frac{x^2 dx}{\Delta x},$$

$$T_3(x) = \int_0^x \frac{dx}{(1+nx^2)\Delta x},$$

$$\Delta x = \sqrt{(1 - x^2)(1 - k^2 x^2)}$$

Jacobi notation for inverse functions

If α is the value of the function $F(\varphi)$, then to denote the inverse function we use the notation

$$\varphi = am(\alpha).$$

Trigonometric dependencies

$$\sin am(\alpha),\ \cos am(\alpha),\ \tan am(\alpha)$$
$$\Delta\varphi = \Delta am(\alpha)$$

General property of elliptic functions
(Fagnano, Euler)

If $\psi(x)$ is a transcendental function, and $\frac{d\psi(x)}{dx}$ is algebraic, then one can find such an algebraic relationship between the particular values $x = x_1, x_2, \dots x_n$, in which the sum $m_1\psi(x_1) + m_2\psi(x_2) + \cdots m_n\psi(x_n)$

CONST

$P(x_1, x_2, \dots x_n)$

$Log(x_1, x_2, \dots x_n)$

Transcendental Functions (Abel)

Giulio Carlo de' Toschi di Fagnano (1682-1766)

DI FAGNANO,
MARCHESE DE' TOSCHI,
E DI SANT' ONORIO
NOBILE ROMANO, E PATRIZIO SENOGAGLIESE
ALLA SANTITA' DI N. S.
BENEDETTO XIV.
PONTEFICE MASSIMO.
TOMO PRIMO.

DEO VERITATIS GLORIA

IN PESARO

L' ANNO DEL GIUBBILEO M. DCC. L.

The only portrait of Niels Henrik Abel

(made by the artist Herbitz in Paris in 1826.)

Monument to Abel in the Royal Park in Oslo
(sculptor Gustav Vigeland. 1908)

Abel's theorem

" Précis d'une théorie des fonctions elliptiques "

This work of Abel was published posthumously in «Mémoires présentés par divers savants à l'Académie des sciences de l'Institut national de France et imprimés par son ordre T.VII, 1841."

Abel's theorem

Given :

$$T_3(x) = \int \frac{d\,x}{(1-\frac{x^2}{a^2})\Delta x}$$

$$\sqrt{(1-x^2)(1-k^2x^2)},\ k^2 > 1$$

$$\psi(x) = A(x^2 - x_1^2)(x^2 - x_2^2)(x^2 - x_3^2)\ldots(x^2 - x_\mu^2)$$

$\psi(x)$ → $\varphi(x)$ odd, $f(x)$ Ev en

$$\psi(x) = f(x)^2 - \varphi(x)^2(\Delta x)^2$$

entire functions with undefined coefficients

Prove :

$$T_3(x_1) + T_3(x_2) + \ldots T_3(x_\mu) = const - \frac{a}{2\Delta a} \cdot \log\left[\frac{f(a) + \varphi(a)\Delta a}{f(a) - \varphi(a)\Delta a}\right].$$

Proof :

$$x = x_1, x_2, \dots x_\mu \Rightarrow \quad \psi(x) = f(x)^2 - \varphi(x)^2(\Delta x)^2 = 0$$

$$\psi'(x)dx + \delta\,\psi(x) = 0$$

$$\delta\,\psi(x) = 2[f(x)\,\delta\,f(x) - \varphi(x)\delta\,\varphi(x)\,](\Delta x)^2$$

$$\psi(x) = f(x)^2 - \varphi(x)^2(\Delta x)^2 = 0 \;\Rightarrow f(x) = \pm\,\varphi(x)\Delta x, \varphi(x)(\Delta x)^2 = \pm\,f(x)(\Delta x)$$

$$\delta\,\psi(x) = -2[\varphi(x)\,\delta\,f(x) - f(x)\delta\,\varphi(x)\,]\Delta x = \; -\theta(x)\Delta x$$

$$\theta(x) = [\varphi(x)\,\delta\,f(x) - f(x)\delta\,\varphi(x)\,]$$

$$\psi'(x)dx = \theta(x)\Delta x \Rightarrow \frac{dx}{\Delta x} = \frac{\theta(x)}{\psi'(x)}$$

$$\Rightarrow T_3(x) = \int \frac{\theta(x)}{\left(1 - \frac{x^2}{a^2}\right)\psi'(x)}$$

$$x = x_1, x_2, \ldots x_\mu \Rightarrow \quad \sum T_3(x) = \int \sum \left[\frac{\theta(x)}{\left(1 - \frac{x^2}{a^2}\right)\psi'(x)}\right]$$

$$\sum T_3(x) = \frac{a}{2}\int \frac{\varphi(a)\delta f(a) - f(a)\delta\varphi(a)}{f^2(a) - \ \varphi^2(a)(\Delta a)^2}$$

$\frac{\varphi(a)\Delta a}{f(a)} = z.$

$$\sum T_3(x) = -\frac{a}{2\Delta a}\int \frac{\delta z}{1 - z^2} = Const. - \ \frac{a}{4\Delta a} log\left(\frac{1+z}{1-z}\right)^2$$

So finally,

$$\sum T_3(x) = Const. - \ \frac{a}{4\Delta a} log\left(\frac{f(a)+\varphi(a)\Delta a}{f(a)-\varphi(a)\Delta a}\right)^2$$

Representation of elliptic functions in terms of theta functions.

$$\Theta(x) = 2[\varphi(x)\delta f(x) - f(x)\delta\varphi(x)]$$

$$T_3(x) = \int \frac{\Theta(x)}{(1-\frac{x^2}{a^2})\psi'(x)}$$

$$T_3(x_1) + T_3(x_2) + \ldots T_3(x_\mu) = const - \frac{a}{2\Delta a} \cdot \log[\frac{f(a)+\varphi(a)\Delta a}{f(a)-\varphi(a)\Delta a}].$$

Carl Jacobi

$$T_3(x_1) + T_3(x_2) + \dots T_3(x_\mu) = const - \frac{a}{2\Delta a} \cdot \log[\frac{f(a) + \varphi(a)\Delta a}{f(a) - \varphi(a)\Delta a}].$$

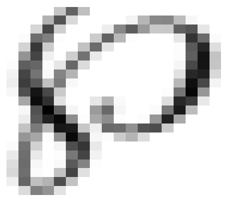
$$\theta_1(x)$$

$$\theta_2(x)$$

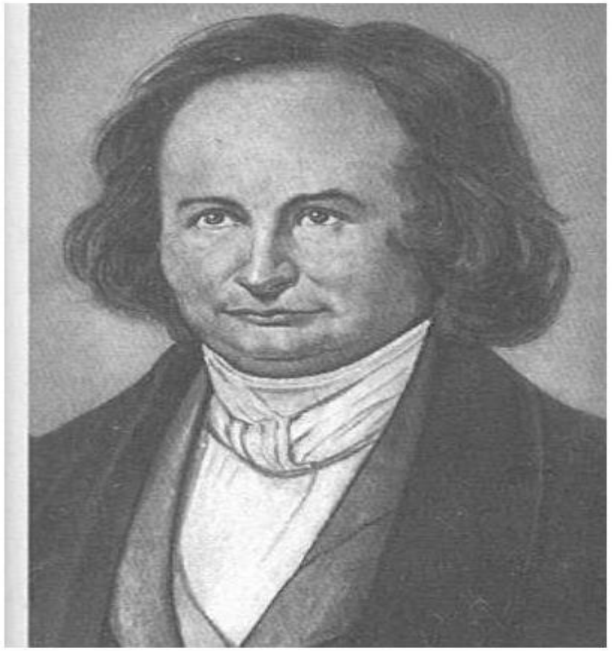
rapidly converging series

Karl Weierstrass

$$\begin{cases} \theta_1(x) \\ \theta_2(x) \end{cases}$$



Carl Gustav Jacob Jacobi (1804-1851)



(1815-1897)

Osip Ivanovich Somov 1815 - 1876

- Academician of the Imperial St. Petersburg Academy of Sciences
- Doctor of Mathematics and Astronomy
- Honored Professor of St. Petersburg University

ОСНОВАНІЯ
ТЕОРІИ ЭЛЛИПТИЧЕСКИХЪ ФУНКЦІЙ.

fundamental work "Foundations of the theory of elliptic functions" (1850)

СОЧИНЕНІЕ

Экстраординарнаго Профессора ИМПЕРАТОРСКАГО С. Петербургскаго Университета,

Доктора Математики и Астрономіи

I. СОМОВА.

САНКТПЕТЕРБУРГЪ.

Въ типографіи Императорской Академіи Наукъ.

1850.

The book contains seven chapters, and another additional chapter is devoted to applications of elliptic functions to some problems of geometry and mechanics.

ОГЛАВЛЕНІЕ.

The history of the analytical solution of the Euler problem on the rotation of a rigid body around a fixed point.

1. Leonhard Euler - 1758
2. Joseph Louis Lagrange - 1773
3. Osip Ivanovich Somov - 1850
4. Sofia Vasilievna Kovalevskaya - 1887

Dynamics problems (Leonhard Euler)

The first
Given: kinematic characteristics of motion
Search: power characteristics

The second
Given: power characteristics
Search: kinematics of motion

Free rigid body motion

- Free axes of rotation, main axes of inertia
- The movement of a free rigid body: translational movement with a center of inertia + rotation around this center
- A kinematic description of such a motion
- Drawing up dynamic equations

Equations of motion of a rigid body around a fixed point of Leonhard Euler

$$\begin{cases} P = A \cdot \dfrac{dP}{dt} + (C - B)qr, \\ Q = B \cdot \dfrac{dq}{dt} + (A - C)rp, \\ R = C \cdot \dfrac{dr}{dt} + (B - A)pq. \end{cases}$$

- "The sum of the whole Theory of Motion of Rigid Bodies is contained in these three fairly simple formulas“
- The use of a moving coordinate system invariably associated with the body - the system of the main axes of inertia of a rigid body
- Introduction of a movable trihedron, determination of the position of a movable coordinate system relative to a fixed one - Euler angles, 1748.

Joseph Louis Lagrange

- The concept of an instantaneous axis of rotation
- New derivation of the dynamic Euler equations, based on the differential equations of motion of the system ("Lagrange equations of the second kind")
- Formulation of the problem of the rotation of a rigid body: the fulcrum does not coincide with the center of gravity of the body (gravity "works")
- Introducing dynamic symmetry to the rotation problem

Solution of the problem of the rotation of a rigid body about a fixed point in the case of an initial impact

- the body has three degrees of freedom and begins to rotate from the initial impact. Further, there are no shock loads, only gravity acts.
- $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ are the moments of inertia about the axes parallel to the main axes of the rotating body. The position of these axes is determined by the Euler angles: $\boldsymbol{\psi}, \boldsymbol{\varphi}, \boldsymbol{\theta}$ (precession, proper rotation and nutation).
- $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}$ are the angular speeds of rotation relative to these axes.

The solution is based on differential equations of motion

$$\begin{cases} dt = \dfrac{A}{B-C} \cdot \dfrac{dp}{qr}, \\ dt = -\dfrac{B}{A-C} \cdot \dfrac{dq}{pr}, \\ dt = \dfrac{C}{A-B} \cdot \dfrac{dr}{pq}. \end{cases}$$

system of fundamental integrals

$A^2p^2 + B^2q^2 + C^2r^2 = l^2$ (l^2 - moment of initial impact),

$Ap^2 + Bq^2 + Cr^2 = h$ (h - kinetic energy).

$$d\psi = -\frac{Ap^2+Bq^2}{A^2p^2+B^2q^2} \cdot l \cdot dt.$$

$$\psi = -l \int \frac{Ap^2+Bq^2}{A^2p^2+B^2q^2} \cdot dt.$$

$$\psi = -\frac{l}{A}\int_{t_0}^{t} dt - \frac{l(A-B)}{A}\int_{t_0}^{t} \frac{Ap^2+Bq^2}{A^2p^2+B^2q^2} dt$$

$$
\begin{cases}
p = -\dfrac{l}{A}\sin\theta\sin\varphi = -\sqrt{\dfrac{l^2 - Ch}{A(A-C)}}\cos am(u)\,, \\[2ex]
q = -\dfrac{l}{B}\sin\theta\cos\varphi = \sqrt{\dfrac{l^2 - Ch}{B(B-C)}}\sin am(u)\,, \\[2ex]
r = \dfrac{l}{C}\cos\theta = \pm\sqrt{\dfrac{Ah - l^2}{C(A-C)}}\Delta am(u).
\end{cases}
$$

where $u = n(t - t_0), n = \sqrt{\frac{(B-C)(Ah-l^2)}{ABC}}$.

$$
\psi = -\frac{l}{An}\cdot u - \frac{l(A-B)(A-C)}{A^2(B-C)n}\int_0^u \frac{sin^2 am(u)du}{1 + \frac{C(A-B)}{A(B-C)}sin^2 am(u)}.
$$

The last integral is an elliptic function of the third kind with an imaginary parameter. Writing it down in terms of the theta function $\theta(x)$, we obtain an expression for the precession angle ψ:

$$\psi = -nu \pm \frac{i}{2} \cdot ln\frac{\theta(u-ai)}{u+ai}.$$

Conclusion

Newton defined the laws of dynamics, and Euler divided all problems of dynamics into two classes, and indicated a mathematical apparatus for solving problems of each class

Mechanical problem posed by Euler and developed in the works of Lagrange received its mathematical embodiment in the theory of elliptic functions.

The theory of elliptic functions, enriched by the discoveries of Abel and Jacobi, took an important place in mathematical analysis and theoretical mechanics in the period from the 18th to the 19th century.

The mathematical apparatus of elliptic functions created by Jacobi allowed O.I. Somov in 1850 brilliantly solved the problem of the rotation of a rigid body in the event of an initial impact.

Weierstrass replaced all particular forms of the θ-function with one $\wp$-function

Therefore, further investigation of the problem of the motion of the top was continued on the complex plane, the direction cosines during the rotation of the body were obtained in the form θ or $\wp$-functions.

Thank you for your attention