Lefschetz formulas for foliated flows and some analogies in number theory

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Finite-dimensional case: Euler characteristic

A finite-dimensional complex over $\mathbb{C}$:

$$0 \xrightarrow{d_{-1}} E_0 \xrightarrow{d_0} E_1 \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} E_n \xrightarrow{d_n} 0, \quad d_{p+1} \circ d_p = 0.$$

The cohomology of (E, d):

$$H^p(E, d) = \ker d_p / \operatorname{im} d_{p-1}, \quad p = 0, 1, \ldots, n.$$

Euler characteristic

$$\chi(E, d) := \sum_{p=0}^{n} (-1)^p \dim H^p(E, d) = \sum_{p=0}^{n} (-1)^p \dim E_p.$$

Finite-dimensional case: Lefschetz number

More generally, $T : (E, d) \to (E, d)$ an endomorphism of the complex:

$$T_p : E_p \to E_p, \quad d_p \circ T_p = T_{p+1} \circ d_p$$

$\Rightarrow$ the induced action on cohomology

$$H^p T : H^p(E, d) \to H^p(E, d)$$

Lefschetz number of T

$$L(T) := \sum_{p=0}^{n} (-1)^p \operatorname{Tr} H^p T = \sum_{p=0}^{n} (-1)^p \operatorname{Tr} T_p.$$

Finite-dimensional case: Hodge theory

Assume E_p are Hermitian vector spaces.
The Laplacian on E_p

$$\Delta_p = d_p^* d_p + d_{p-1} d_{p-1}^* : E_p \to E_p.$$

The Hodge theorems:

$$H^p(E,d) \cong \mathcal{H}_p = \operatorname{Ker} \Delta_p, \quad p = 0, 1, \dots, n.$$

MacKean-Singer formula

$$\chi(E,d) = \sum_{p=0}^{n} (-1)^p \dim \mathcal{H}_p = \sum_{p=0}^{n} (-1)^p \operatorname{Tr} e^{-t\Delta_p},$$

$T : (E,d) \to (E,d)$ an endomorphism of the complex,

$$L(T) = \sum_{p=0}^{n} (-1)^p \operatorname{Tr} T_p P_{\mathcal{H}_p} = \sum_{p=0}^{n} (-1)^p \operatorname{Tr} T_p e^{-t\Delta_p}, \quad t \geq 0.$$

De Rham complex

- M a compact smooth manifold, $\dim M = n$.
- $(\Omega(M), d)$ the de Rham complex of M:
  - $\Omega^p(M) = C^\infty(M, \Lambda^p T^*M)$ the space of smooth differential p-forms;
  - $d = d_p : \Omega^p(M) \to \Omega^{p+1}(M)$ the de Rham differential.
- In a coordinate chart with coordinates $(x_1, \ldots, x_n) \in \mathbb{R}^n$, a p-form $\omega \in \Omega^p(M)$ is written as

$$\omega = \sum_{\alpha_1<\alpha_2<\ldots<\alpha_p} a_\alpha(x) dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}$$

  and $d\omega \in \Omega^{p+1}(M)$ is given by

$$d\omega = \sum_{j=1}^{n} \sum_{\alpha_1<\alpha_2<\ldots<\alpha_p} \frac{\partial a_\alpha}{\partial x_j}(x) dx_j \wedge dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}.$$

De Rham cohomology

The de Rham complex $(\Omega(M), d)$

$$0 \xrightarrow{d_{-1}} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} 0, \quad d_{p+1} \circ d_p = 0.$$

The de Rham cohomology of M is the cohomology of $(\Omega(M), d)$:

$$H^p(M) = \ker d_p / \operatorname{im} d_{p-1}, \quad p = 0, 1, \dots, n.$$

Since the complex $(\Omega(M), d)$ is elliptic, it is Fredholm.
Therefore, $\operatorname{im} d_p$ is closed in C^∞-topology, and

$$\dim H^p(M) < \infty, \quad p = 0, 1, \dots, n.$$

The Lefschetz number

- $f : M \to M$ a smooth map.
- The induced action on differential forms:

$$f^* : \Omega^p(M) \to \Omega^p(M).$$

- $d \circ f^* = f^* \circ d \Longrightarrow$ the induced action on the cohomology:

$$f^* : H^p(M) \to H^p(M).$$

The Lefschetz number of f:

$$L(f) = \sum_{p=0}^{n} (-1)^p \mathrm{tr}\,(f^* : H^p(M) \to H^p(M)).$$

Fact. $L(f)$ a homotopy invariant of f.

The Lefschetz number of the identity map

In the case $f = \mathrm{id}$:

$$L(\mathrm{id}) = \sum_{p=0}^{n} (-1)^p \mathrm{tr}\,(\mathrm{id} : H^p(M) \to H^p(M)) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M:

$$\chi(M) = \sum_{p=0}^{n} (-1)^p \dim H^p(M).$$

Hodge theorem

Fix a Riemannian metric g on M.
It induces inner products on $\Omega(M)$ and we can introduced the formal adjoint of d (the de Rham codifferential):

$$\delta_p = d_p^* : \Omega^p(M) \to \Omega^{p-1}(M).$$

The signature operator is a first order elliptic differential operator

$$D = d + \delta : \Omega(M) \to \Omega(M).$$

Its square is the Hodge Laplacian

$$\Delta_p = d_{p-1}\delta_{p-1} + \delta_p d_p : \Omega^p(M) \to \Omega^p(M).$$

The Hodge Theorem:

$$H^p(M) = \ker \Delta_p, \quad H^+(M) = \ker(d + \delta)$$

The Chern-Gauss-Bonnet theorem

By the Hodge Theorem, we have

$$\chi(M) = \operatorname{ind} D := \ker D - \ker D^*.$$

where

$$D = d + \delta : \Omega^{ev}(M) \to \Omega^{odd}(M),$$

Theorem (The Chern-Gauss-Bonnet theorem)

$$\chi(M) = \int_M \mathrm{Pf}\left(\frac{R_g}{2\pi}\right),$$

- *$n = \dim M$ is even.*
- *R_g the curvature of the Riemannian metric g.*
- *The Euler form*

$$\mathrm{Pf}\left(\frac{R_g}{2\pi}\right) \in \Omega^n(M).$$

Simple fixed points

Let $f : M \to M$ be a smooth map.

- $x \in M$ is a fixed point of f, $x \in \mathrm{Fix}(f)$, if

$$f(x) = x.$$

- A fixed point x of f is called simple, if

$$\det(\mathrm{id} - df_x : T_xM \to T_xM) \neq 0.$$

- For a simple fixed point x of f, put

$$\varepsilon_x := \mathrm{sign}\,\det(\mathrm{id} - df_x : T_xM \to T_xM).$$

Atiyah-Bott-Lefschetz formula

Theorem (Atiyah-Bott-Lefschetz formula)

Assume that all fixed points of a smooth map f are simple. Then

$$L(f) = \sum_{x \in \mathrm{Fix}(f)} \varepsilon_x.$$

$\varepsilon_x := \mathrm{sign}\ \det(\mathrm{id} - df_x : T_x M \to T_x M).$

Corollary

If $L(f) \neq 0$, then f has a fixed point.

About the proof

Formally

$$L(f)=\sum_{p=0}^{n}(-1)^p \mathrm{tr}\,(f^*:H^p(M)\to H^p(M))$$

$$=\sum_{p=0}^{n}(-1)^p \mathrm{tr}\,(f^*:\Omega^p(M)\to \Omega^p(M)).$$

For the operator $f^*:\Omega^p(M)\to\Omega^p(M)$, we have

$$f^*\omega(x)=\Lambda^p df^*_x[\omega(f(x))]=\int_M \Lambda^p df^*_x\delta(y-f(x))\omega(y)$$

$\Lambda^p df^*_x:\Lambda^p T^*_{f(x)}M\to\Lambda^p T^*_x M$ the induced action in the fibers.

About the proof

$f^* : \Omega^p(M) \to \Omega^p(M)$ is an integral operator with the kernel

$$K_{f^*}(x,y) = \Lambda^p df_x^* \delta(y - f(x)), \quad x, y \in M.$$

$$\operatorname{tr}(f^* : \Omega^p(M) \to \Omega^p(M)) = \int_M \operatorname{Tr} K_{f^*}(x,x) = \int_M \operatorname{Tr} \Lambda^p df_x^* \delta(x - f(x))$$
$$= \sum_{x \in \operatorname{Fix}(f)} \frac{\operatorname{Tr} \Lambda^p df_x^*}{|\det(\operatorname{id} - df_x)|}$$

$$L(f) = \sum_{x \in \operatorname{Fix}(f)} \frac{\sum_{p=0}^n (-1)^p \operatorname{Tr} \Lambda^p df_x^*}{|\det(\operatorname{id} - df_x)|} = \sum_{x \in \operatorname{Fix}(f)} \frac{\det(\operatorname{id} - df_x)}{|\det(\operatorname{id} - df_x)|}$$
$$= \sum_{x \in \operatorname{Fix}(f)} \operatorname{sign} \det(\operatorname{id} - df_x).$$

About the proof

Fix a Riemannian metric g on M:

$$L(f^*) = \sum_{p=0}^{n} (-1)^p \operatorname{Tr} f^* e^{-t\Delta_p}, \quad t > 0.$$

$f^* e^{-t\Delta_p}$ is an integral operator with smooth kernel

$$f^* e^{-t\Delta_p}\omega(x) = \int_M H_{t,f}(x,y) dv_g(y).$$

Its trace is well-defined

$$\operatorname{tr} f^* e^{-t\Delta_p} = \int_M \operatorname{Tr}[H_{t,f}(x,x)]\omega(y)\, dv_g(x).$$

$$\operatorname{tr} f^* e^{-t\Delta_p} \to \sum_{x \in \operatorname{Fix}(f)} \varepsilon_x, \quad t \to 0.$$

Lefschetz function for discrete dynamical systems

- $f : M \to M$ a diffeomorphism.
- $T = \{f^k : k \in \mathbb{Z}\}$ a discrete dynamical system on M.
- The Lefschetz function

$$L(T) : \mathbb{Z} \to \mathbb{Z}, \quad k \mapsto L(f^k).$$

- For $k = 0$, $L(f^0) = L(\mathrm{id}) = \chi(M)$ is given by the index theorem.
- For $k \neq 0$, $L(f^k)$ is given by the Lefschetz formula under the assumption that the fixed points of f^k are simple.
- For any x, the trajectory (the orbit) of x

$$\mathcal{O}_x = \{f^k(x) : k \in \mathbb{Z}\}.$$

$f^k(x) = x \Leftrightarrow \mathcal{O}_x$ is periodic (closed) with period k.

Lefschetz formulas for flows

- M a compact smooth manifold, $\dim M = n$.
- $\phi = \{\phi^t : M \to M, t \in \mathbb{R}\}$ a smooth flow on M
  a one-parameter group of diffeomorphisms of M:
  $\phi^{t+s} = \phi^t \circ \phi^s, \phi^0 = \mathrm{id}$.

Problems:

- To define a Lefschetz number of ϕ:

$$L(\phi) = \sum_{j=0}^{n-1} (-1)^j \mathrm{Tr}\,(\phi^* : H^j \to H^j)$$

  H^j is some cohomology theory, Tr is some trace.
- To prove the corresponding Lefschetz trace formula:

$$L(\phi) = \text{a contribution of closed orbits and fixed points of the flow.}$$

Foliated flows

- M a closed manifold, $\dim M = n$;
- $\mathcal{F}$ a codimension one foliation on M;
  In a foliated chart with coordinates $(x_1, \ldots, x_{n-1}, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$, the leaves of $\mathcal{F}$ are given by $y = c \in \mathbb{R}$.
- $\phi = \{\phi^t : M \to M, t \in \mathbb{R}\}$ is a foliated flow (i.e., each ϕ^t takes an arbitrary leaf to a leaf).

- $\mathrm{Fix}(\phi)$ the fixed point set of ϕ (closed in M):

$$x \in \mathrm{Fix}(\phi) \Leftrightarrow \phi^t(x) = x \ \forall t \in \mathbb{R}.$$

- M^0 the $\mathcal{F}$-saturation of $\mathrm{Fix}(\phi)$ (the union of leaves with fixed points). Observe that M^0 is ϕ-invariant.
- $M^1 = M \setminus M^0$ the transitive point set.

Simple flows

The foliated flow ϕ is simple, if:

- Any closed orbit c of period l (not necessarily minimal) is simple:

$$\det(\mathrm{id} - \phi^l_* : T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0, \quad x \in c, \quad \phi^l(x) = x.$$

- any fixed point x of ϕ is simple:

$$\det(\mathrm{id} - \phi^t_* : T_xM \to T_xM) \neq 0, \quad t \neq 0.$$

- its orbits in M^1 are transverse to the leaves:

$$T_xM = \mathbb{R}\, Z(x) \oplus T_x\mathcal{F}, \quad x \in M^1,$$

  Z is the infinitesimal generator of ϕ.

Remark

If ϕ is simple, then M^0 is a finite union of compact leaves.

Guiilemin-Sternberg formula

If the flow ϕ is simple, there is a canonical expression for the right-hand side of the Lefschetz trace formula, which follows from the Guiilemin-Sternberg formula:
$L(\phi)$ is a distribution on $\mathbb{R}$ and, in $\mathcal{D}'(\mathbb{R}^+)$,

$$L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c)\delta_{kl(c)} + \sum_x \varepsilon_x |1 - e^{\varkappa_x t}|^{-1},$$

c runs over all closed orbits and x over all fixed points of ϕ:

- $l(c)$ the minimal period of c,
- $\varepsilon_l(c) := \text{sign}\ \det\left(\text{id} - \phi^l_* : T_x\mathcal{F} \to T_x\mathcal{F}\right), x \in c.$
- $\varepsilon_x := \text{sign}\ \det\left(\text{id} - \phi^t_* : T_x\mathcal{F} \to T_x\mathcal{F}\right), t > 0.$
- $\varkappa_x \neq 0$ is a real number such that

$$\bar{\phi}^t_* : T_xM/T_x\mathcal{F} \to T_xM/T_x\mathcal{F}, \quad v \mapsto e^{\varkappa_x t} v.$$

Leafwise de Rham complex

$(\Omega(\mathcal{F}), d_{\mathcal{F}})$ the leafwise de Rham complex of $\mathcal{F}$:

- $\Omega^{\cdot}(\mathcal{F}) = C^\infty(M, \Lambda^{\cdot} T^*\mathcal{F})$ smooth leafwise differential forms;
- $d_{\mathcal{F}} : \Omega^p(\mathcal{F}) \to \Omega^{p+1}(\mathcal{F})$ the leafwise de Rham differential.

In a foliated chart with coordinates $(x_1, \ldots, x_{n-1}, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ such that leaves are given by $y = c$, a p-form $\omega \in \Omega^p(\mathcal{F})$ is written as

$$\omega = \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_p} a_\alpha(x, y) dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}$$

and $d_{\mathcal{F}}\omega \in \Omega^{p+1}(\mathcal{F})$ is given by

$$d_{\mathcal{F}}\omega = \sum_{j=1}^{n-1} \sum_{\alpha_1 < \alpha_2 < \ldots < \alpha_p} \frac{\partial a_\alpha}{\partial x_j}(x, y) dx_j \wedge dx_{\alpha_1} \wedge \ldots \wedge dx_{\alpha_p}$$

Leafwise de Rham cohomology

- The leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ is not elliptic and non-Fredholm.
- The reduced leafwise de Rham cohomology of $\mathcal{F}$:

$$\overline{H}(\mathcal{F}) = \ker d_{\mathcal{F}} / \overline{\operatorname{im} d_{\mathcal{F}}},$$

  the closure is in C^∞-topology.
- ϕ is a foliated flow $\Longrightarrow d_{\mathcal{F}} \circ \phi^t = \phi^t \circ d_{\mathcal{F}}$.
  The induced action:

$$\phi^{t*} : \overline{H}(\mathcal{F}) \to \overline{H}(\mathcal{F}).$$

Question

The trace of $\phi^{t*} : \overline{H}(\mathcal{F}) \to \overline{H}(\mathcal{F})$?

The case of a nonsingular flow

- Assume that there are no fixed points of ϕ
- The orbits of ϕ are transverse to the leaves:

$$T_xM = \mathbb{R}\, Z(x) \oplus T_x\mathcal{F}, \quad x \in M.$$

- Any closed orbit c of period l of ϕ is simple:

$$\det(\mathrm{id} - \phi^l_* : T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0, \quad x \in c.$$

- $\varepsilon_l(c) := \mathrm{sign}\ \det\left(\mathrm{id} - \phi^l_* : T_x\mathcal{F} \to T_x\mathcal{F}\right).$

$L(\phi)$ is a distribution on $\mathbb{R}$ and, in $\mathcal{D}'(\mathbb{R}^+)$,

$$L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c)\delta_{kl(c)},$$

c runs over all closed orbits; $l(c)$ the minimal period of c.

The leafwise Hodge decomposition

- g the Riemannian metric on M such that the infinitesimal generator Z of the flow ϕ satisfies:

$$|Z| = 1, \quad Z \perp T\mathcal{F},$$

  a bundle-like metric ($\mathcal{F}$ is a Riemannian foliation).
- $\Delta_{\mathcal{F}} = d_{\mathcal{F}}\delta_{\mathcal{F}} + \delta_{\mathcal{F}}d_{\mathcal{F}}$ the leafwise Laplacian on $\Omega(\mathcal{F})$ (a second order tangentially elliptic differential operator on M).
- $\mathcal{H}(\mathcal{F})$ the space of leafwise harmonic forms on M:

$$\mathcal{H}(\mathcal{F}) = \{\omega \in \Omega(\mathcal{F}) : \Delta_{\mathcal{F}}\omega = 0\}.$$

Theorem (Alvarez Lopez - Yu. K)

The Hodge isomorphism

$$\overline{H}(\mathcal{F}) \cong \mathcal{H}(\mathcal{F}).$$

The index theory for transversally elliptic operators

- Atiyah-Singer, 1973-1974, for compact Lie groups;
- Singer-Hörmander, 1974, definition of the index for noncompact Lie groups.

Transverse ellipticity

The leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ of $\mathcal{F}$ as well as the leafwise Laplacian $\Delta_{\mathcal{F}}$ are transversally elliptic relative to the action of the group $\mathbb{R}$, given by the flow ϕ

- The principal symbol $\sigma(\Delta_{\mathcal{F}})$ of $\Delta_{\mathcal{F}}$ is a section of the vector bundle $\mathrm{Hom}(\pi^*\Lambda^{\cdot} T^*\mathcal{F})$ over T^*M (actually, a scalar function). Here $\pi : T^*M \to M$ is a natural projection.
- Transverse ellipticity means that $\sigma(\Delta_{\mathcal{F}})(x, \xi)$ is invertible for any $(x, \xi) \in T^*M \setminus 0$ orthogonal to the orbits of ϕ.

For any $f \in C_c^\infty(\mathbb{R})$, define

$$A_f = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\, dt \circ \Pi : L^2\Omega(\mathcal{F}) \to L^2\Omega(\mathcal{F}),$$

where Π is the orthogonal projection in $L^2\Omega(\mathcal{F})$ to the subspace of leafwise harmonic L^2-forms on M.

A_f is a smoothing operator:

For any $f \in C_c^\infty(\mathbb{R})$, the Schwartz kernel $K_{A_f} = K_{A_f}(x, y)|dy|$ is smooth:

$$A_f u(x) = \int_M K_{A_f}(x, y) u(y) |dy|.$$

In particular, A_f is of trace class and

$$\operatorname{Tr} A_f = \int_M \operatorname{tr} K_{A_f}(x, x) |dx|.$$

The Lefschetz formula

The Lefschetz distribution $L(\phi) \in \mathcal{D}'(\mathbb{R})$:

$$< L(\phi), f >= \operatorname{Tr}^s A_f := \sum_{j=1}^{n-1} (-1)^j \operatorname{Tr} A_f^{(i)}, \quad f \in C_c^\infty(\mathbb{R}),$$

where $A_f^{(i)}$ is the restriction of A_f to $\Omega^i(\mathcal{F})$.

Theorem (Alvarez Lopez - Y.K.)

On $\mathbb{R} \setminus \{0\}$

$$L(\phi) = \sum_c l(c) \sum_{k \neq 0} \varepsilon_{kl(c)}(c) \delta_{kl(c)},$$

when c *runs over all closed orbits of* ϕ *and* $l(c)$ *denotes the minimal period of* c.

About the proof

The proof is based on the heat equation approach to the index theorem:

- For $u > 0$,

$$P_{u,f} = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ e^{-uD^2_{\mathcal{F}^c}}.$$

- $\frac{d}{du} \operatorname{Tr}^s P_{u,f} = 0$, which means that $\operatorname{Tr}^s P_{u,f}$ is independent of u.
- As $u \to +\infty$,

$$\operatorname{Tr}^s P_{u,f} \to \operatorname{Tr}^s \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \Pi = \langle L(\phi), f \rangle.$$

- As $u \to 0$, $\operatorname{Tr}^s P_{u,f}$ can be computed, using heat kernel approximations (fantastic cancellations).

Contribution of zero

Theorem (Alvarez Lopez - Y.K.)

In some neighborhood of 0 *in* $\mathbb{R}$*:*

$$L(\phi) = \chi_\Lambda(\mathcal{F}) \cdot \delta_0.$$

- δ_0 the delta-function at 0.
- Λ holonomy invariant transverse measure of $\mathcal{F}$ determined by dt;
- $\chi_\Lambda(\mathcal{F})$ the Λ-Euler characteristic of $\mathcal{F}$ given by the holonomy invariant transverse measure Λ (Connes, 1979):

$$\chi_\Lambda(\mathcal{F}) = \sum_i (-1)^i \mathrm{Tr}_{\Lambda} P_i,$$

- The foliation Gauss-Bonnet theorem (Connes, 1979):

$$\chi_\Lambda(\mathcal{F}) = \int_M \mathrm{Pf}\left(\frac{R_{\mathcal{F}}}{2\pi}\right) \Lambda.$$

Example

- $F : X \to X$ a diffeomorphism of a compact manifold X;
- $M = \mathbb{R} \times X / \sim$, $(s, x) \sim (s+1, F(x))$ mapping torus;
- $\pi : M \to S^1 = \mathbb{R}/\mathbb{Z}, [(s,x)] \mapsto s \mod \mathbb{Z}$ a fibration;
- $\mathcal{F}$ is formed by the fibers of π;
- The flow $\phi^t : M \to M$, $\phi^t([(s,x)]) = [(s+t, x)]$.
- Leafwise differential forms $\omega \in \Omega(\mathcal{F})$ are smooth families $\{\omega(s) \in \Omega(X), s \in \mathbb{R}\}$ such that $F^*[\omega(s+1)] = \omega(s)$.
- A leafwise Riemannian metric $g_{\mathcal{F}}$ is a smooth family $\{g_s, s \in \mathbb{R}\}$ of Riemannian metrics on X such that $F^*[g_{s+1}] = g_s$
- Leafwise harmonic forms $\omega \in \mathcal{H}(\mathcal{F})$ are smooth families $\{\omega(s) \in \mathcal{H}(X, g_s), s \in \mathbb{R}\}$ such that $F^*[\omega(s+1)] = \omega(s)$.

$$L(\phi) = \sum_{n \in \mathbb{Z}} L(F^{-n}) \delta_n,$$

The Riemann hypothesis

- The Riemann zeta-function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

- The function $\zeta(s)$ has a holomorphic continuation to $\mathbb{C}\setminus\{1\}$ with a simple pole at $s=1$.
- The completed zeta-function

$$\hat{\zeta}(s)=(2\pi)^{-s}\Gamma(s)\zeta(s)$$

  is holomorphic continuation in $\mathbb{C}\setminus\{1\}$ with simple poles at $s=0,1$.
- The zeroes of $\hat{\zeta}(s)$ are the so-called non-trivial zeros of $\zeta(s)$, i.e. those in the critical strip $0<\mathrm{Re}\, s<1$.
- The Riemann hypothesis: all the non-trivial zeros of $\zeta(s)$ lie on the line $\mathrm{Re}\, s=1/2$

The explicit formulas in the analytic number theory

For a test function $\varphi \in C_c^\infty(\mathbb{R})$, an entire function

$$\Phi(s) = \int_{\mathbb{R}} \varphi(t) e^{ts}\, dt.$$

Then

$$\Phi(0) - \sum_\rho \Phi(\rho) + \Phi(1)$$

$$= \sum_p \log p \left(\sum_{k\geq 1} \varphi(k \log p) + \sum_{k \leq -1} p^k \varphi(k \log p) \right) + W(\varphi).$$

- ρ runs over the non-trivial zeroes of $\zeta(s)$ and p over the primes.
- W a distribution on $\mathbb{R}$:

$$W(\varphi) = \int_0^{+\infty} \frac{\varphi(t)}{1 - e^{-2t}} dt, \text{ if } \operatorname{supp} \varphi \in \mathbb{R}_+.$$

Comparison with the trace formula

The explicit formula

$$1-\sum_{\rho} e^{t\rho}+e^{t}=\sum_{p}\log p\left(\sum_{k\geq 1}\delta_{k\log p}+\sum_{k\leq -1}p^{k}\delta_{k\log p}\right)+W(t).$$

The Lefschetz formula

$$\sum_{i=1}^{n-1}(-1)^{i}\operatorname{tr}\left(\phi_t^{*}:\overline{H}^{i}(\mathcal{F})\to\overline{H}^{i}(\mathcal{F})\right)$$

$$=\sum_{c} l(c)\sum_{k\neq 0}\varepsilon_{kl(c)}(c)\delta_{kl(c)}+\sum_{x}\varepsilon_{x}|1-e^{\varkappa_x t}|^{-1}.$$

Analogies: Deninger's program

- $\operatorname{spec}\mathbb{Z}\cup\infty \leftrightarrow$ 3-dimensional manifold M with a codimension one foliation $\mathcal{F}$ and a flow ϕ_t satisfying above conditions.
- an appropriate cohomology theory — the reduced leafwise de Rham cohomology $\bar{H}^{\cdot}(\mathcal{F})$:
  - $\operatorname{tr}(\phi_t^* : \overline{H}^0(\mathcal{F}) \to \overline{H}^0(\mathcal{F})) = 1$.
  - $\operatorname{tr}(\phi_t^* : \overline{H}^1(\mathcal{F}) \to \overline{H}^1(\mathcal{F})) = \sum_\rho e^{t\rho}$.
  - $\operatorname{tr}(\phi_t^* : \overline{H}^2(\mathcal{F}) \to \overline{H}^2(\mathcal{F})) = e^t$.
- a prime $p \leftrightarrow$ closed orbit c_p such that
  - $l(c_p) = \log p$;
  - $\varepsilon_{k\log p}(c_p) = 1$ for all $k \geq 1$.
- an infinite place $\leftrightarrow$ fixed point x_∞.
- $\chi_\Lambda(\mathcal{F}) \leftrightarrow$ Euler characteristic in Arakelov geometry.

Problems in Deninger's program

Problem 1

W a distribution on $\mathbb{R}$:

$$W(\varphi) = \int_0^{+\infty} \frac{\varphi(t)}{1-e^{-2t}} dt, \text{ if } \operatorname{supp} \varphi \in \mathbb{R}_+.$$

$$W(\varphi) = \int_{-\infty}^{0} \frac{\varphi(t)}{1-e^{2t}} e^t\, dt, \text{ if } \operatorname{supp} \varphi \in \mathbb{R}_-.$$

Solution: one should consider singular flows (with fixed points).

Problem 2

By Poincare duality $\overline{H}^0(\mathcal{F}) = \overline{H}^2(\mathcal{F})$, but $1 \neq e^t$.
Solution: M is a solenoid, $\mathcal{F}$ is a lamination.

The case of a singular flow

- M a closed manifold, $\dim M = n$.
- $\mathcal{F}$ a codimension one foliation on M.
- $\phi^t : M \to M, t \in \mathbb{R}$ a foliated flow.

- $\mathrm{Fix}(\phi)$ the fixed point set of ϕ (closed in M).
- M^0 the $\mathcal{F}$-saturation of $\mathrm{Fix}(\phi)$ (the union of leaves with fixed points).
  Observe that M^0 is ϕ-invariant.
- $M^1 = M \setminus M^0$ the transitive point set.

Simple flows

The foliated flow ϕ is simple, i.e.:

- any closed orbit c of period l of ϕ is simple:

$$\det(\mathrm{id} - \phi_*^l : T_x\mathcal{F} \to T_x\mathcal{F}) \neq 0, \quad x \in c.$$

- any fixed point x of ϕ is simple:

$$\det(\mathrm{id} - \phi_*^t : T_x M \to T_x M) \neq 0, \quad t \neq 0.$$

- its orbits in M^1 are transverse to the leaves:

$$T_x M = \mathbb{R}\, Z(x) \oplus T_x\mathcal{F}, \quad x \in M^1,$$

  Z is the infinitesimal generator of ϕ.

Remark

If ϕ is simple, then M^0 is a finite union of compact leaves.

Difficulties

- The leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ of $\mathcal{F}$ as well as the leafwise Laplacian $\Delta_{\mathcal{F}}$ are transversally elliptic only on the transitive point set M^1, not on M^0.
- As a consequence, the operator

$$A_f = \int_{\mathbb{R}} \phi^{t*} \cdot f(t)\, dt \circ \Pi : L^2\Omega(\mathcal{F}) \to L^2\Omega(\mathcal{F})$$

  is not a smoothing operator. Its Schwartz kernel is smooth on $M^1 \times M^1$ and singular near $M^0 \times M^0$. So its trace is not well-defined.
- We don't discuss the leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ and related cohomology theory and start with the Hodge theoretic setting.
- Two ideas: a special choice of Riemannian metric and renormalization of the trace.

Riemannian metric of bounded geometry

We use a very concrete choice of a Riemannian metric g^1 on the transitive point set M^1, which is singular at M^0: near each leaf L in M^0

$$g^1 = g_{\mathcal{F}} + \frac{dx^2}{x^2},$$

where $g_{\mathcal{F}}$ is a leafwise Riemanian metric and x is a defining function of L, i.e. $L = \{x = 0\}$, $dx \neq 0$ on L.

- M_l^1, $l = 1, \ldots, r$, the connected components of $M^1 = M \setminus M^0$:

$$(M^1, \mathcal{F}^1) = \bigsqcup_l (M_l^1, \mathcal{F}_l^1).$$

- M_l^1 equipped with $g_l := g^1|_{M_l^1}$ is a manifold of bounded geometry;
- g^1 is bundle-like for $\mathcal{F}^1$ and $\mathcal{F}_l^1$ a Riemannian foliation of bounded geometry;
- ϕ_l^t a flow of bounded geometry.

Operators on the transitive point set

- M^l is the closure of M_l^1: $M^l = \overline{M_l^1}$.
  Thus, M_l is a connected compact manifold with boundary, endowed with a smooth foliation $\mathcal{F}_l$ tangent to the boundary.
- $d_{\mathring{\mathcal{F}}_l}$ the leafwise de Rham differential on $\Omega(\mathring{\mathcal{F}}_l)$.
- $\delta_{\mathring{\mathcal{F}}_l}$ the leafwise de Rham codifferential on $\Omega(\mathring{\mathcal{F}}_l)$.
- $D_{\mathring{\mathcal{F}}_l} = d_{\mathring{\mathcal{F}}_l} + \delta_{\mathring{\mathcal{F}}_l}$.

Heat equation approach

For any $f \in C_c^\infty(\mathbb{R})$, we consider the operator

$$P_{l,f} = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\,dt \circ e^{-D^2_{\mathring{\mathcal{F}}_l}}.$$

Theorem (Alvarez Lopez, Yu.K., Leichtnam)

The operator $P_{l,f} = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ e^{-D^2_{\mathring{\mathcal{F}}_l}}$ belongs to the class $\Psi_{\mathrm{b}}^{-\infty}(M_l; \bigwedge T\mathcal{F}_l^*)$ in pseudodifferential b-calculus of R. Melrose.

- The Schwartz kernel $K_{P_{l,f}}$ is smooth in the interior $\mathring{M}_l \times \mathring{M}_l$.
- $K_{P_{l,f}}$ has a C^∞ extension to $M_l \times M_l \setminus \partial M_l \times \partial M_l$ that vanishes to all orders at $(\partial M_l \times M_l) \cup (M_l \times \partial M_l)$.
- Consider a tubular neighborhood of $L \subset \pi_0(\partial M_l)$ with coordinates $(\rho, y), \rho \in (0, \infty), y \in L$.
  Then $K_{P_{l,f}} = K_{P_{l,f}}(\rho, y, \rho', y')u(\rho', y')|d\rho'||dy'|$ has the form

$$K_{P_{l,f}}(\rho, y, \rho', y') = \frac{1}{\rho'}\kappa_{P_{l,f}}(\rho, y, \frac{\rho'}{\rho}, y'),$$

  where $\kappa_{P_{l,f}}(\rho, y, s, y')$ is smooth up to L (that is, up to $\rho = 0$).

The renormalized trace

In a tubular neighborhood of L with coordinates $\rho \in (0, \epsilon_0), y \in L$,

$$P_{l,f}u(\rho, y) = \int K_{P_{l,f}}(\rho, y, \rho', y')u(\rho', y')|d\rho'||dy'|,$$

$$K_{P_{l,f}}(\rho, y, \rho', y') = \frac{1}{\rho'}\kappa_{P_{l,f}}(\rho, y, \frac{\rho'}{\rho}, y'),$$

and $\kappa_{P_{l,f}}(\rho, y, s, y')$ is smooth up to L (that is, up to $\rho = 0$).

The b-trace:

$$^{\mathrm{b}}\mathrm{Tr}\,(P_{l,f}) = \lim_{\epsilon\to 0}\left(\int_{\rho>\epsilon} K_{P_{l,f}}(\rho, y, \rho, y)|d\rho||dy| + \ln\epsilon\int \kappa_{P_{l,f}}(0, y, 1, y)|dy|\right).$$

Key observation

The functional $^{\mathrm{b}}\mathrm{Tr}$ doesn't have trace propertry, but $^{\mathrm{b}}\mathrm{Tr}\,[P, P']$ is expressed in terms of traces of some explicit integral operators on ∂M_l.

Now we put together:

$M^{\mathrm{c}} = \bigsqcup_l M_l$ is a manifold with boundary, $\mathcal{F}^{\mathrm{c}} = \bigsqcup_l \mathcal{F}_l$.
We get the operator

$$P_f \equiv \bigoplus_l P_{l,f} = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ e^{-D^2_{\mathcal{F}^{\mathrm{c}}}}$$

We can define the Lefschetz distribution as the b-supertrace of P_f:

$$^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_f) = \sum_{j=1}^{n-1} (-1)^j\, {}^{\mathrm{b}}\mathrm{Tr}\,(P_f^{(j)}),$$

where $P_f^{(j)}$ is the restriction to j-forms.

Heat equation approach to the index theorem

- For $u > 0$,

$$P_{u,f} = \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ e^{-u^2 D^2_{\mathcal{F}^c}}.$$

- $\frac{d}{du} \operatorname{Tr}^s P_{u,f} = 0$, which means that $\operatorname{Tr}^s P_{u,f}$ is independent of u.
- As $u \to +\infty$,

$$\operatorname{Tr}^s P_{u,f} \to \operatorname{Tr}^s \int_{-\infty}^{\infty} \phi^{t*} \cdot f(t)\, dt \circ \Pi = \langle L(\phi), f \rangle.$$

- As $u \to 0$, $\operatorname{Tr}^s P_{u,f}$ can be computed, using heat kernel approximations (fantastic cancellations).

Remark

b-trace ${}^{\mathrm{b}}\mathrm{Tr}$ *doesn't satisfy the trace property. Therefore,* $\frac{d}{du} {}^{\mathrm{b}}\mathrm{Tr}^{\,s} P_u \neq 0$.

Derivative of the b-supertrace

For $u > 0$,

$$\frac{d}{du}{}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{u,f}) = \sum_{L\in\pi_0(M^0)}\sum_{\gamma\in\Gamma_L} a(D_{\tilde{L}}, u, \gamma, t_{L,\gamma})f(t_{L,\gamma}),$$

where $a(D_{\tilde{L}}, u, \gamma, t_{L,\gamma}) \in \mathbb{R}$ and $t_{L,\gamma} \in \mathbb{R}$, $\widetilde{L}$ is the holonomy covering of L associated with the holonomy group Γ_L of L, , $D_{\tilde{L}}$ is the lift of D_L to $\widetilde{L}$.

For $u, v > 0$,

$${}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{v,f}) - {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{u,f}) = \sum_{L\in\pi_0(M^0)}\sum_{\gamma\in\Gamma_L}\left(\int_u^v a(D_{\tilde{L}}, w, \gamma, t_{L,\gamma})dw\right) f(t_{L,\gamma}).$$

The last equality can be written as

$$ {}^{b}\mathrm{Tr}^{s}(P_{u,f}) = {}^{b}\mathrm{Tr}^{s}(P_{v,f}) - \sum_{L\in\pi_0(M^0)} \sum_{\gamma\in\Gamma_L} \left(\int_u^v a(D_{\tilde{L}}, w, \gamma, t_{L,\gamma}) dw \right) f(t_{L,\gamma}). $$

The Lefschetz distribution

$$ \begin{aligned} \langle L(\phi), f\rangle &= \lim_{u\to 0} {}^{b}\mathrm{Tr}^{s}(P_{u,f}) \\ &= {}^{b}\mathrm{Tr}^{s}(P_{v,f}) - \sum_{L\in\pi_0(M^0)} \sum_{\gamma\in\Gamma_L} \left(\int_0^v a(D_{\tilde{L}}, w, \gamma, t_{L,\gamma}) dw \right) f(t_{L,\gamma}). \end{aligned} $$

Here the right-hand side is independent of v.

Trace formula

Theorem (Alvarez Lopez, Yu.K., Leichtnam)

For $\operatorname{supp} f \subset \mathbb{R}_+$, *the limit of* ${}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{u,f})$ *as* $u \to 0$ *exists and is given by*

$$\lim_{u\to 0} {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{u,f}) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c) \cdot f(kl(c))$$

where c runs over all closed orbits of ϕ^t, $l(c)$ *denotes the minimal period of c, and x is an arbitrary point of c.*

Corollary

$L(\phi)$ is a well-defined distribution on $\mathbb{R}_+$ given by

$$L(\phi) = \sum_c l(c) \sum_{k=1}^{\infty} \varepsilon_{kl(c)}(c) \cdot \delta_{kl(c)}.$$

Concluding remarks

- We proved the Lefschetz formula for a singular foliated flow with correct contribution of closed orbits (as in the Guillemin-Sternberg formula).
- The next problems:
  to obtain contribution of fixed points as in the Guillemin-Sternberg formula.
  to give a cohomological interpretation of the limit as $v \to +\infty$ of

$$ {}^{\mathrm{b}}\mathrm{Tr}^{\,\mathrm{s}}(P_{v,f}) - \sum_{L\in\pi_0(M^0)} \sum_{\gamma\in\Gamma_L} \left(\int_0^v a(D_{\tilde{L}}, w, \gamma, t_{L,\gamma}) dw \right) f(t_{L,\gamma}). $$

Guiilemin-Patterson conjecture

- X a compact manifold of negative curvature.
- $M = S^*X$ the cosphere bundle.
- $\alpha_t : M \to M$ the geodesic flow.
- α_t is an Anosov flow $\Longrightarrow$ there are strongly stable and strongly unstable foliations $\mathcal{F}_{ss}$ and $\mathcal{F}_{su}$ invariant under the flow:

$$\alpha_t : \mathcal{F}_{ss} \to \mathcal{F}_{ss}, \quad \alpha_t : \mathcal{F}_{su} \to \mathcal{F}_{su}.$$

Conjecture

On $\mathbb{R} \setminus \{0\}$

$$\sum_i (-1)^i \operatorname{tr}\left(T_t^* : \overline{H}^i(\mathcal{F}_{su}) \to \overline{H}^i(\mathcal{F}_{su})\right) = \sum_c l(c) \sum_{k \neq 0} \varepsilon_{kl(c)}(c) \delta_{kl(c)},$$

when c runs over all primitive closed orbits of T_t and $l(c)$ denotes the length of c.

Guiilemin-Patterson conjecture

FORMALLY TRUE in the case when $X = \mathbb{H}/\Gamma$ is a Riemann surface of genus $g > 1$ or, more generally, when X is a locally symmetric space due to representation theory and the Selberg trace formula.

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