Natural lacunae method and Schatten-von Neumann classes of the convergence exponent

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Abstract

The first our aim is to clarify the results obtained by Lidsky V.B. devoted to the decomposition on the root vector system of the non-selfadjoint operator. We use a technique of the entire function theory and introduce a so-called Schatten-von Neumann class of the convergent exponent. Considering strictly accretive operators satisfying special conditions formulated in terms of the norm, we construct a sequence of contours of the power type in the contrary to the results of Lidsky V.B., where a exponential type sequence of contours was used.

Keywords: Strictly accretive operator; Abel-Lidsky basis property; Schatten-von Neumann class; convergence exponent; counting function.

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1 Introduction

To write this paper, we were firstly motivated by the boundary value problems of the Sturm-Liouville type for fractional differential equations. Many authors devoted their attention to the topic, nevertheless this kind of problems are relevant for today. First of all, it is connected with the fact that they model various physical - chemical processes: filtration of liquid and gas in highly porous fractal medium; heat exchange processes in medium with fractal structure and memory; casual walks of a point particle that starts moving from the origin by self-similar fractal set; oscillator motion under the action of elastic forces which is characteristic for viscoelastic media, etc. In particular, we would like to study the eigenvalue problem for a differential operator with a fractional derivative in final terms, in this connection such operators as a Kipriyanov fractional differential operator, Riesz potential, difference operator are involved.

In the case corresponding to a selfadjoint senior term we can partially solve the problem having applied the results of the perturbation theory, within the framework of which the following papers are well-known [16], [23], [28], [29],[27], [38]. Generally, to apply the last paper results for a concrete operator L we must be able to represent it by a sum $L = T + A$, where the senior term T must be either a selfadjoint or normal operator. In other cases we can use methods of the papers [21],[20] which are relevant if we deal with non-selfadjoint operators and allow us to study spectral properties of operators whether we have the mentioned above representation or not. We should add that the results of the paper [27] can be also applied to study non-selfadjoin operators (see a detailed remark in [38]).

In many papers [5]-[7], [32] the eigenvalue problem was studied by methods of a theory of functions and it is remarkable that special properties of the fractional derivative were used in these papers, bellow we present a brief review. The singular number problem for the resolvent of a second order differential operator with the Riemann-Liouville fractional derivative in final terms was considered in the paper [5]. It was proved that the resolvent belongs to the Hilbert-Schmidt class. The problem of completeness of the root functions system was studied in the paper [6], also similar problems were considered in the paper [7].

However, we deal with a more general operator — a differential operator with a fractional integro-differential operator composition in final terms, which covers the operator mentioned above. Note that several types of compositions of fractional integro-differential operators were studied by such mathematicians as Prabhakar T.R. [35], Love E.R. [26], Erdelyi A. [12], McBride A. [30], Dimovski I.H., Kiryakova V.S. [11], Nakhushev A.M. [33].

The central idea of this paper is to formulate sufficient conditions of the basis property of the root functions system. We clarify the results obtained by V.B. Lidsky [25] devoted to the decomposition on the root vector system of the non-selfadjoint operator. We use a technique of the entire function theory and introduce a so-called Schatten-von Neumann class of the convergent exponent. Considering strictly accretive operators satisfying special conditions formulated in terms of the norm, we construct a sequence of contours of the power type in the contrary to the results of Lidsky V.B., where a exponential type sequence of contours was used. Finally, we produce applications to differential equations in the abstract Hilbert space.

2 Preliminaries

Let $C, C_i,\ i \in \mathbb{N}_0$ be real constants. We assume that a value of C is positive and can be different in various formulas but values of C_i are certain. Everywhere further, if the contrary is not stated, we consider linear densely defined operators acting on a separable complex Hilbert space $\mathfrak{H}$. Denote by $\mathcal{B}(\mathfrak{H})$ the set of linear bounded operators on $\mathfrak{H}$. Denote by $\tilde{L}$ the closure of an operator L. We establish the following agreement on using symbols $\tilde{L}^i := (\tilde{L})^i$, where i is an arbitrary symbol. Denote by $\mathrm{D}(L)$, $\mathrm{R}(L)$, $\mathrm{N}(L)$ the *domain of definition*, the *range*, and the *kernel* or *null space* of an operator L respectively. The deficiency (codimension) of $\mathrm{R}(L)$, dimension of $\mathrm{N}(L)$ are denoted by $\operatorname{def} T$, $\operatorname{nul} T$ respectively. Assume that L is a closed operator acting on $\mathfrak{H}$, $\mathrm{N}(L) = 0$, let us define a Hilbert space $\mathfrak{H}_L := \big\{f, g \in \mathrm{D}(L),\ (f,g)_{\mathfrak{H}_L} = (Lf, Lg)_{\mathfrak{H}}\big\}$. Consider a pair of complex Hilbert spaces $\mathfrak{H}, \mathfrak{H}_+$, the notation $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ means that $\mathfrak{H}_+$ is dense in $\mathfrak{H}$ as a set of elements and we have a bounded embedding provided by the inequality

$$\|f\|_{\mathfrak{H}} \leq C_0 \|f\|_{\mathfrak{H}_+},\ C_0 > 0,\ f \in \mathfrak{H}_+,$$

moreover any bounded set with respect to the norm $\mathfrak{H}_+$ is compact with respect to the norm $\mathfrak{H}$. Let L be a closed operator, for any closable operator S such that $\tilde{S} = L$, its domain $\mathrm{D}(S)$ will be called a core of L. Denote by $\mathrm{D}_0(L)$ a core of a closeable operator L. Let $\mathrm{P}(L)$ be the resolvent set of an operator L and $R_L(\zeta)$, $\zeta \in \mathrm{P}(L)$, $[R_L := R_L(0)]$ denotes the resolvent of an operator

L. Denote by $\lambda_i(L)$, $i \in \mathbb{N}$ the eigenvalues of an operator L. Suppose L is a compact operator and $N := (L^*L)^{1/2}$, $r(N) := \dim \mathrm{R}(N)$; then the eigenvalues of the operator N are called the *singular numbers* (*s-numbers*) of the operator L and are denoted by $s_i(L)$, $i = 1, 2, \ldots, r(N)$. If $r(N) < \infty$, then we put by definition $s_i = 0$, $i = r(N) + 1, 2, \ldots$. According to the terminology of the monograph [13] the dimension of the root vectors subspace corresponding to a certain eigenvalue λ_k is called the *algebraic multiplicity* of the eigenvalue λ_k. Let $\nu(L)$ denotes the sum of all algebraic multiplicities of an operator L. Let $\mathfrak{S}_p(\mathfrak{H})$, $0 < p < \infty$ be a Schatten-von Neumann class and $\mathfrak{S}_\infty(\mathfrak{H})$ be the set of compact operators. By definition, put

$$\mathfrak{S}_p(\mathfrak{H}) := \left\{ L : \mathfrak{H} \to \mathfrak{H}, \sum_{i=1}^{\infty} s_i^p(L) < \infty,\ 0 < p < \infty \right\}.$$

Suppose L is an operator with a compact resolvent and $s_n(R_L) \leq C\, n^{-\mu}$, $n \in \mathbb{N}$, $0 \leq \mu < \infty$; then we denote by $\mu(L)$ order of the operator L in accordance with the definition given in the paper [38]. Denote by $\mathfrak{Re} L := (L + L^*)/2$, $\mathfrak{Im} L := (L - L^*)/2i$ the real and imaginary components of an operator L respectively. In accordance with the terminology of the monograph [15] the set $\Theta(L) := \{z \in \mathbb{C} : z = (Lf, f)_{\mathfrak{H}},\ f \in \mathrm{D}(L),\ \|f\|_{\mathfrak{H}} = 1\}$ is called the *numerical range* of an operator L. An operator L is called *sectorial* if its numerical range belongs to a closed sector $\mathfrak{L}_\gamma(\theta) := \{\zeta : |\arg(\zeta - \gamma)| \leq \theta < \pi/2\}$, where γ is the vertex and θ is the semi-angle of the sector $\mathfrak{L}_\gamma(\theta)$. An operator L is called *bounded from below* if the following relation holds $\mathrm{Re}(Lf, f)_{\mathfrak{H}} \geq \gamma_L \|f\|^2_{\mathfrak{H}}$, $f \in \mathrm{D}(L)$, $\gamma_L \in \mathbb{R}$, where γ_L is called a lower bound of L. An operator L is called *accretive* if $\gamma_L = 0$. An operator L is called *strictly accretive* if $\gamma_L > 0$. An operator L is called *m-accretive* if the next relation holds $(A + \zeta)^{-1} \in \mathcal{B}(\mathfrak{H})$, $\|(A + \zeta)^{-1}\| \leq (\mathrm{Re}\zeta)^{-1}$, $\mathrm{Re}\zeta > 0$. An operator L is called *m-sectorial* if L is sectorial and $L + \beta$ is m-accretive for some constant β. An operator L is called *symmetric* if one is densely defined and the following equality holds $(Lf, g)_{\mathfrak{H}} = (f, Lg)_{\mathfrak{H}}$, $f, g \in \mathrm{D}(L)$.

Consider a sesquilinear form $t[\cdot,\cdot]$ (see [15]) defined on a linear manifold of the Hilbert space $\mathfrak{H}$. Denote by $t[\cdot]$ the quadratic form corresponding to the sesquilinear form $t[\cdot,\cdot]$. Let $\mathfrak{h} = (t + t^*)/2$, $\mathfrak{k} = (t - t^*)/2i$ be a real and imaginary component of the form t respectively, where $t^*[u, v] = \overline{t[v, u]}$, $\mathrm{D}(t^*) = \mathrm{D}(t)$. According to these definitions, we have $\mathfrak{h}[\cdot] = \mathrm{Re}\, t[\cdot]$, $\mathfrak{k}[\cdot] = \mathrm{Im}\, t[\cdot]$. Denote by $\tilde{t}$ the closure of a form t. The range of a quadratic form $t[f]$, $f \in \mathrm{D}(t)$, $\|f\|_{\mathfrak{H}} = 1$ is called *range* of the sesquilinear form t and is denoted by $\Theta(t)$. A form t is called *sectorial* if its range belongs to a sector having a vertex γ situated at the real axis and a semi-angle $0 \leq \theta < \pi/2$. Suppose t is a closed sectorial form; then a linear manifold $\mathrm{D}_0(t) \subset \mathrm{D}(t)$ is called *core* of t, if the restriction of t to $\mathrm{D}_0(t)$ has the closure t (see[15, p.166]). Due to Theorem 2.7 [15, p.323] there exist unique m-sectorial operators $T_t, T_{\mathfrak{h}}$ associated with the closed sectorial forms $t, \mathfrak{h}$ respectively. The operator $T_{\mathfrak{h}}$ is called a *real part* of the operator T_t and is denoted by $Re\, T_t$. Suppose L is a sectorial densely defined operator and $t[u, v] := (Lu, v)_{\mathfrak{H}}$, $\mathrm{D}(t) = \mathrm{D}(L)$; then due to Theorem 1.27 [15, p.318] the corresponding form t is closable, due to Theorem 2.7 [15, p.323] there exists a unique m-sectorial operator $T_{\tilde{t}}$ associated with the form $\tilde{t}$. In accordance with the definition [15, p.325] the operator $T_{\tilde{t}}$ is called a *Friedrichs extension* of the operator L. Everywhere further, unless otherwise stated, we use notations of the papers [13], [15], [17], [18], [37].

1. Some properties of non-selfadjoint operators.

In this section we explore a special operator class for which a number of spectral theory theorems can be applied. As an application of the obtained abstract results we study a basis property of the root vectors of the operator in terms of the order of the operator real part. By virtue of such an approach we express a convergence exponent of s-numbers through the order of the operator real part. Bellow, we give a slight generalization of the results presented in [20].

Theorem 1. *Assume that L is a non-sefadjoint operator acting in $\mathfrak{H}$, the following conditions hold*

*(*H1*) There exists a Hilbert space $\mathfrak{H}_+ \subset\subset \mathfrak{H}$ and a linear manifold $\mathfrak{M}$ that is dense in $\mathfrak{H}_+$. The operator L is defined on $\mathfrak{M}$.*

(H2) $|(Lf,g)_{\mathfrak{H}}| \le C_1\|f\|_{\mathfrak{H}_+}\|g\|_{\mathfrak{H}_+},\ \mathrm{Re}(Lf,f)_{\mathfrak{H}} \ge C_2\|f\|^2_{\mathfrak{H}_+},\ f,g\in\mathfrak{M},\ C_1,C_2>0.$

Let W be a restriction of the operator L on the set $\mathfrak{M}$. Then the following propositions are true.

*(**A**) We have the following classification*

$$R_{\tilde{W}} \in \mathfrak{S}_p,\ p = \begin{cases} l,\ l>2/\mu,\ \mu\le 1, \\ \qquad\quad 1,\ \mu>1 \end{cases},$$

where μ is order of $H := Re\,\tilde{W}$. Moreover under the assumptions $\lambda_n(R_H) \ge C\,n^{-\mu},\ n\in\mathbb{N}$, we have the following implication

$$\{R_{\tilde{W}} \in \mathfrak{S}_p,\ 1\le p<\infty\} \ \Rightarrow\ \mu p>1.$$

*(**B**) The following relation holds*

$$\sum_{i=1}^{n}|\lambda_i(R_{\tilde{W}})|^p \le C\sum_{i=1}^{n}\lambda_i^p(R_H),\ 1\le p<\infty,\ \ (n=1,2,...,\ \nu(R_{\tilde{W}})),$$

moreover if $\nu(R_{\tilde{W}})=\infty$ and $\mu\ne 0$, then the following asymptotic formula holds

$$|\lambda_i(R_{\tilde{W}})| = o\left(i^{-\mu+\varepsilon}\right),\ i\to\infty,\ \forall\varepsilon>0.$$

*(**C**) Assume that $\theta<\pi\mu/2$, where θ is the semi-angle of the sector $\mathfrak{L}_0(\theta)\supset\Theta(\tilde{W})$. Then the system of root vectors of $R_{\tilde{W}}$ is complete in $\mathfrak{H}$.*

Proof. Note that due to the first condition H2, by virtue of Theorem 3.4 [15, p.268] the operator W is closable. Let us show that $\tilde{W}$ is sectorial. By virtue of condition H2, we get

$$\mathrm{Re}(\tilde{W}f,f)_{\mathfrak{H}} \ge C_2\|f\|^2_{\mathfrak{H}_+} \ge C_2\varepsilon\|f\|^2_{\mathfrak{H}_+} + \frac{C_2(1-\varepsilon)}{C_0}\|f\|^2_{\mathfrak{H}};$$

$$\mathrm{Re}(\tilde{W}f,f)_{\mathfrak{H}} - k|\mathrm{Im}(\tilde{W}f,f)_{\mathfrak{H}}| \ge (C_2\varepsilon - kC_1)\|f\|^2_{\mathfrak{H}_+} + \frac{C_2(1-\varepsilon)}{C_0}\|f\|^2_{\mathfrak{H}} = \frac{C_2(1-\varepsilon)}{C_0}\|f\|^2_{\mathfrak{H}},$$

where $k=\varepsilon C_2/C_1$. Hence $\Theta(\tilde{W})\subset\mathfrak{L}_\gamma(\theta),\ \gamma=C_2(1-\varepsilon)/C_0$. Thus, the claim of Lemma 1 [20] is true regarding the operator $\tilde{W}$. Using this fact, we conclude that the claim of Lemma 2 [20] is true regarding the operator $\tilde{W}$ i.e. $\tilde{W}$ is m-accretive.

Using the first representation theorem (Theorem 2.1 [15, p.322]) we have a one-to-one correspondence between m-sectorial operators and closed sectorial sesquilinear forms i.e. $\tilde{W} = T_t$ by symbol, where t is a sesquilinear form corresponding to the operator $\tilde{W}$. Hence $H := Re\,\tilde{W}$ is defined (see [15, p.337]). In accordance with Theorem 2.6 [15, p.323] the operator H is selfadjoint, strictly accretive.

A compact embedding provided by the relation $\mathfrak{h}[f] \geq C_2\|f\|_{\mathfrak{H}_+} \geq C_2/C_0\|f\|_{\mathfrak{H}}$, $f \in \mathrm{D}(\mathfrak{h})$ proves that R_H is compact (see proof of Theorem 4 [20]) and as a result of the application of Theorem 3.3 [15, p.337], we get $R_{\tilde{W}}$ is compact. Thus the claim of Theorem 4 [20] remains true regarding the operators R_H, $R_{\tilde{W}}$.

In accordance with Theorem 2.5 [15, p.323] , we get $W^* = T_{t^*}$ (since $W^* = \tilde{W}^*$). Now if we denote $t_1 := t^*$, then it is easy to calculate $\mathfrak{k} = -\mathfrak{k}_1$. Since t is sectorial, than $|\mathfrak{k}_1| \leq \tan\theta \cdot \mathfrak{h}$. Hence, in accordance with Lemma 3.1 [15, p.336], we get $\mathfrak{k}[u,v] = (BH^{1/2}u, H^{1/2}v)$, $\mathfrak{k}_1[u,v] = -(BH^{1/2}u, H^{1/2}v)$, $u, v \in \mathrm{D}(H^{1/2})$, where $B \in \mathcal{B}(\mathfrak{H})$ is a symmetric operator. Let us prove that B is selfadjoint. Note that in accordance with Lemma 3.1 [15, p.336] $\mathrm{D}(B) = \mathrm{R}(H^{1/2})$, in accordance with Theorem 2.1 [15, p.322], we have $(Hf, f)_{\mathfrak{H}} \geq C_2/C_0\|f\|^2_{\mathfrak{H}}$, $f \in \mathrm{D}(H)$, using the reasonings of Theorem 5 [20], we conclude that $\mathrm{R}(H^{1/2}) = \mathfrak{H}$ i.e. $\mathrm{D}(B) = \mathfrak{H}$. Hence B is selfadjoint. Using Lemma 3.2 [15, p.337], we obtain a representation $\tilde{W} = H^{1/2}(I + iB)H^{1/2}$, $W^* = H^{1/2}(I - iB)H^{1/2}$. Noting the fact $\mathrm{D}(B) = \mathfrak{H}$, we can easily obtain $(I \pm iB)^* = I \mp iB$. Since B is selfadjoint, then $\mathrm{Re}([I \pm iB]f, f)_{\mathfrak{H}} = \|f\|^2_{\mathfrak{H}}$. Using this fact and applying Theorem 3.2 [15, p.268], we conclude that $\mathrm{R}(I \pm iB)$ is a closed set. Since $\mathrm{N}(I \pm iB) = 0$, then $\mathrm{R}(I \mp iB) = \mathfrak{H}$ (see (3.2) [15, p.267]). Thus, we obtain $(I \pm iB)^{-1} \in \mathcal{B}(\mathfrak{H})$. Taking into account the above facts, we get $R_{\tilde{W}} = H^{-1/2}(I + iB)^{-1}H^{-1/2}$, $R_{W^*} = H^{-1/2}(I - iB)^{-1}H^{-1/2}$. In accordance with the well-known theorem (see Theorem 5 [39, p.557]), we have $R^*_{\tilde{W}} = R_{W^*}$. Note that the relations $(I \pm iB) \in \mathcal{B}(\mathfrak{H})$, $(I \pm iB)^{-1} \in \mathcal{B}(\mathfrak{H})$, $H^{-1/2} \in \mathcal{B}(\mathfrak{H})$ allow as to obtain the following formula by direct calculations

$$\mathfrak{Re} R_{\tilde{W}} = \frac{1}{2} H^{-1/2}(I + B^2)^{-1}H^{-1/2}.$$

This formula is a crucial point of the matter, we can repeat the rest part of the proof of Theorem 5 [20] in terms $H := Re\,\tilde{W}$. By virtue of these facts Theorems 7-9 [20], can be reformulated in terms $H := Re\,\tilde{W}$, since they are based on Lemmas 1, 3, Theorems 4, 5 [20].

□

Remark 1. *Consider a condition $\mathfrak{M} \subset \mathrm{D}(W^*)$, in this case the operator $\mathcal{H} := \mathfrak{Re}\, W$ is defined on $\mathfrak{M}$, the fact is that $\tilde{\mathcal{H}}$ is selfadjoint, bounded from bellow (see Lemma 3 [20]). Hence a corresponding sesquilinear form (denote this form by h) is symmetric and bounded from bellow also (see Theorem 2.6 [15, p.323]). It can be easily shown that $h \subset \mathfrak{h}$, but using this fact we cannot claim in general that $\tilde{\mathcal{H}} \subset H$ (see [15, p.330]). We just have an inclusion $\tilde{\mathcal{H}}^{1/2} \subset H^{1/2}$ (see [15, p.332]). Note that the fact $\tilde{\mathcal{H}} \subset H$ follows from a condition $\mathrm{D}_0(\mathfrak{h}) \subset \mathrm{D}(h)$ (see Corollary 2.4 [15, p.323]). However, it is proved (see proof of Theorem 4 [20]) that relation H2 guaranties that $\tilde{\mathcal{H}} = H$. Note that the last relation is very useful in applications, since in most concrete cases we can find a concrete form of the operator $\mathcal{H}$.*

Some facts of the entire functions theory

Here we introduce some notions and facts of the entire functions theory, we follow the monograph [24] In this subsection we will use the following notations

$$G(z,p):=(1-z)e^{z+\frac{z^2}{2}+\ldots+\frac{z^p}{p}},\, G(z,0):=(1-z).$$

Consider such an entire function that its zeros satisfy the following relation for some $\lambda>0$

$$\sum_{n=1}^{\infty}\frac{1}{|a_n|^{\lambda}}<\infty. \tag{1}$$

In this case we denote by p the smallest integer number for which the following condition holds

$$\sum_{n=1}^{\infty}\frac{1}{|a_n|^{p+1}}<\infty. \tag{2}$$

It is clear that $0\leq p<\lambda$. It is proved that under the assumption (1) the infinite product

$$\prod_{n=1}^{\infty}G\left(\frac{z}{a_n},p\right) \tag{3}$$

is uniformly convergent, we will call it a canonical product and call p the genus of the canonical product. By the *convergence exponent* of the sequence

$$\{a_n\}_1^{\infty}\subset\mathbb{C},\, a_n\neq 0,\, a_n\to\infty$$

we mean the greatest lower bound for numbers λ for which series (1) converges. Note that if λ equals to a convergent exponent then series (1) may or may not be convergent. For instance, the sequences $a_n=1/n^{\lambda}$ and $1/(n\ln^2 n)^{\lambda}$ have the same convergent exponent $\lambda=1$, but in the first case the series (1) is divergent when $\lambda=1$ while in the second one it is convergent. In this paper we have a special interest regarding the first case. Consider the following obvious relation between the convergence exponent ρ_1 and the genus p of the corresponding canonical product $p\leq\rho_1\leq p+1$. It is clear that if ρ_1 is integer, then $p=\rho_1$, when the series (1) diverges for $\lambda=\rho_1$, while $\rho_1=p+1$ means that the series converges (in accordance with the definition of p). In the monograph [24] it is considered a more precise characteristic of the density of the sequence $\{a_n\}_1^{\infty}$ than the convergence exponent. Thus, there is defined the so called growth of the function $n(r)$ equals to a number of points of the sequence in the circle $|z|<r$. By upper density of the sequence we call a number

$$\Delta=\overline{\lim_{r\to\infty}}\, n(t)/t^{\rho_1},$$

if a limit exists in the ordinary sense (not in the sense of the upper limit) then Δ is called the density. Note that it is proved in Lemma 1 [24] that

$$\lim_{r\to\infty} r(t)/t^{\rho_1+\varepsilon}\to 0,\, \varepsilon>0.$$

We need the following fact (see [24] Lemma 3)

Lemma 1. *If the series* (2) *converges, then the corresponding infinite product* (3) *satisfies the following inequality in the entire complex plane*

$$\ln\left|\prod_{n=1}^{\infty}(z)\right|\leq Cr^p\left(\int\limits_0^r\frac{n(t)}{t^{p+1}}dt+r\int\limits_r^{\infty}\frac{n(t)}{t^{p+2}}dt\right),\, r:=|z|.$$

Using this result it is not hard to prove a relevant fact mentioned in the monograph [24]. Since it has a principal role in the further narrative, we formulate it as a theorem in terms of the upper density.

Lemma 2. *Assume that the following series is convergent for some values of $\lambda > 0$ i.e.*

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda}} < \infty,$$

then the following relation holds

$$\left| \prod_{n=1}^{\infty} (z) \right| \leq e^{\beta(r) r^{\rho_1}}, \; \beta(r) = r^{p-\rho_1} \left(\int_0^r \frac{n(t)}{t^{p+1}} dt + r \int_r^{\infty} \frac{n(t)}{t^{p+2}} dt \right), \tag{4}$$

where ρ_1 is a convergent exponent of the sequence $\{a_n\}_1^{\infty}$. Moreover $\beta(r) \to 0$, if the convergent exponent ρ_1 is non-integer and such that $\rho_1 < \lambda$ and the density equals zero, or the convergent exponent is such that $\rho_1 = \lambda$. In addition the last equality guaranties that the density equals zero.

Proof. Applying Lemma 1, we establish relation (4). Taking into account the fact that the density equals zero, using the L'Hôpital's rule, in the case when $\rho_1 < \lambda$ is non-integer, we easily obtain

$$r^{p-\rho_1} \int_0^r \frac{n(t)}{t^{p+1}} dt \to 0; \; r^{p+1-\rho_1} \int_r^{\infty} \frac{n(t)}{t^{p+2}} dt \to 0. \tag{5}$$

Therefore $\beta(r) \to 0$. Consider the case when $\rho_1 = \lambda$, then let us write the series (1) in the form of the Stiltes integral

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda}} = \int_0^{\infty} \frac{dn(t)}{t^{\rho_1}}.$$

Using integration by parts formulae, we get

$$\int_0^r \frac{dn(t)}{t^{\rho_1}} = \frac{n(r)}{r^{\rho_1}} - \frac{n(\gamma)}{\gamma^{\rho_1}} + \rho_1 \int_0^r \frac{n(t)}{t^{\rho_1+1}} dt,$$

where γ denotes a positive constant, we should note that there exists a neighborhood of the point zero in which $n(t) = 0$. The latter representation shows us that the following integral converges i.e.

$$\int_0^{\infty} \frac{n(t)}{t^{\rho_1+1}} dt < \infty.$$

In its own turn, it follows that

$$\frac{n(r)}{r^{\rho_1}} = n(r) \rho_1 \int_r^{\infty} \frac{1}{t^{\rho_1+1}} dt < \rho_1 \int_r^{\infty} \frac{n(t)}{t^{\rho_1+1}} dt \to 0, \; r \to \infty.$$

Thus, in the way used above, we conclude that (5) holds if ρ_1 is non-integer. If $\rho_1 = \lambda$ is integer then it is clear that we have $\rho_1 = p + 1$, here we should recall that it is not possible to assume $\rho_1 = p$ due to the definition of p. In the case $\rho_1 = p + 1$, in the way analogous with (5), we get

$$r^{-1}\int\limits_0^r \frac{n(t)}{t^{p+1}}dt \to 0; \ \int\limits_r^\infty \frac{n(t)}{t^{p+2}}dt \to 0,$$

from what follows the fact that $\beta(r) \to 0$. □

Further, if we consider s - numbers of the operator A then we will use a notation β_A for a corresponding function β defined by formula (4).

Example 1. *There exists a sequence $\{a_n\}_1^\infty$ such that the corresponding series* (1) *is divergent when $\lambda = \rho_1$ while density equals zero and $\beta(r)\ln r \to 0$.*

We can construct the required sequence supposing $n(r) \sim r^\rho(\ln r \cdot \ln\ln r)^{-1}$, $\rho > 0$, Further, we will show that $\rho_1 = \rho$. It follows from the latter relation directly that the density equals zero. It is clear that we can represent partial sums of series (1) due to the Stiltes integral

$$\sum_{n=1}^k \frac{1}{|a_n|^\lambda} = \int\limits_0^{r(k)} \frac{dn(t)}{t^\lambda}.$$

Thus the sequence $\{a_n\}_1^\infty$ is defined by the function $n(r)$. Applying the integration by parts formulae, we get

$$\int\limits_0^r \frac{dn(t)}{t^\lambda} = \frac{n(r)}{r^\lambda} - a_1^\lambda + \lambda\int\limits_0^r \frac{n(t)}{t^{\lambda+1}}dt.$$

Using the latter relation, by direct calculation, we can easily establish the fact that the density equals zero while the last integral is divergent when $\lambda = \rho,\ r \to \infty$, we have

$$\int\limits_0^r \frac{n(t)}{t^{\rho+1}}dt = \int\limits_0^r \frac{dt}{t\ln t \cdot \ln\ln t} = \ln\ln\ln r - C.$$

On the other hand, we have for values $\lambda > \rho$

$$\int\limits_0^\infty \frac{n(t)}{t^{\lambda+1}}dt = \int\limits_0^\infty \frac{dt}{t^{1+\lambda-\rho}\ln t \cdot \ln\ln t} < \infty.$$

Thus we get that the series (1) is divergent if $\lambda = \rho$ and convergent if $\lambda = \rho + \varepsilon$, $\varepsilon > 0$. The fact that ρ is a converges exponent is proved. Assume that ρ_1 is not integer, then to prove the fact $\beta(r)\ln r \to 0$, $r \to \infty$, we should use representation (4), we have

$$r^{p-\rho_1}\left(\int\limits_0^r \frac{n(t)}{t^{p+1}}dt + r\int\limits_r^\infty \frac{n(t)}{t^{p+2}}dt\right)\ln r \to I,\ r \to \infty.$$

Denoting

$$a(r) := \ln r \int\limits_0^r \frac{n(t)}{t^{p+1}} dt, \; b(r) := r^{\rho_1 - p}, \; a_1(r) := \ln r \int\limits_r^\infty \frac{n(t)}{t^{p+2}} dt, \; b_1(r) := r^{\rho_1 - p - 1},$$

using the L'Hôpital's rule it is not hard to prove that $I = 0$, what gives us the desired result. More precisely

$$a'(r) := \frac{1}{r} \int\limits_0^r \frac{n(t)}{t^{p+1}} dt + \ln r \frac{n(r)}{r^{p+1}}; \; a_1'(r) := \frac{1}{r} \int\limits_r^\infty \frac{n(t)}{t^{p+2}} dt + \ln r \frac{n(r)}{r^{p+2}}.$$

In general, we have

$$\frac{1}{rb'(r)} \int\limits_0^r \frac{n(t)}{t^{p+1}} dt \to 0, \; \frac{1}{rb_1'(r)} \int\limits_r^\infty \frac{n(t)}{t^{p+2}} dt \to 0, \; r \to \infty.$$

Substituting $r^{\rho_1} (\ln r \cdot \ln \ln r)^{-1}$ instead of $n(r)$, we get

$$\ln r \frac{n(r)}{b'(r) r^{p+1}} \sim \ln r \frac{r^{\rho_1}}{r^{p+1} r^{\rho_1 - p - 1} \ln r \cdot \ln \ln r} \to 0,$$

$$\ln r \frac{n(r)}{b_1'(r) r^{p+2}} \sim \ln r \frac{r^{\rho_1}}{r^{p+2} r^{\rho_1 - p - 2} \ln r \cdot \ln \ln r} \to 0, \; r \to \infty.$$

Note that this example does not cover the case $\rho_1 = p$, since in this case $\beta(r) \nrightarrow 0$, what can be verified by direct calculations.

Remark 2. *Having taken into account the above reasonings, we see that the following implication holds*

$$\ln r \frac{n(r)}{r^{\rho_1}} \to 0, \Rightarrow \beta(r) \ln r \to 0.$$

Schatten-von Neumann class and the particular case corresponding to the normal resolvent

Let $\mathfrak{S}_p(\mathfrak{H})$, $0 < p < \infty$ be a Schatten-von Neumann class and $\mathfrak{S}_\infty(\mathfrak{H})$ be the set of compact operators. By definition, put

$$\mathfrak{S}_p(\mathfrak{H}) := \left\{ L : \mathfrak{H} \to \mathfrak{H}, \sum_{i=1}^{\infty} s_i^p(L) < \infty, \; 0 < p < \infty \right\}.$$

Denote by $\mathfrak{T}_\rho(\mathfrak{H})$ the class of the operators such that

$$A \in \mathfrak{T}_\rho(\mathfrak{H}) \Rightarrow \{ A \in \mathfrak{S}_{\rho+\varepsilon}, \, A \overline{\in} \mathfrak{S}_{\rho-\varepsilon}, \, \forall \varepsilon > 0 \}.$$

This operator class we will call a *Schatten-von Neumann class of the convergence exponent.* Note that there exists a one to one correspondence between selfadjoint compact operators and monotonically decreasing sequences. In this regard, if we consider example 1 then we see that the

made definition becomes relevant. Thus, in these terms the claim (**A**) of the Theorem 1 can be reformulated in particular as follows.
(**AI**) *We have the following classification*

$$R_{\tilde{W}} \in \mathfrak{S}_p,\ p = 2/\mu,\ \mu \leq 1,$$

where μ is order of $H := Re\,\tilde{W}$. Moreover under the assumptions $\lambda_n(R_H) \geq C\,n^{-\mu},\ n \in \mathbb{N}$, we have the following implication

$$\{R_{\tilde{W}} \in \mathfrak{S}_p,\ 1 \leq p < \infty\} \ \Rightarrow\ \mu = 1/p.$$

Lemma 3. *Assume that $(\ln^{\mu+1} x)'_{\lambda_i(H)} = o(i^{-\mu}),\ 0 < \mu \leq 1$, then in the general case, we get $R_{\tilde{W}} \in \mathfrak{S}_p,\ 1/\mu \leq p \leq 2/\mu,\ n(\lambda) = o(\lambda^{2/\mu}/\ln\lambda)$. Moreover, under the made assumptions if $\mu \neq 1$, $\tilde{W}$ is normal, then $R_{\tilde{W}} \in \mathfrak{S}_{1/\mu},\ n(\lambda) = o(\lambda^{1/\mu}/\ln\lambda)$, where $n(\lambda)$ is the counting function of the sequence $\{s_i^{-1}(R_{\tilde{W}})\}_1^\infty$.*

Proof. Note that the fact $R_{\tilde{W}} \in \mathfrak{S}_p,\ 0 < p \leq 2/\mu$ follows directly from the claim (**AI**). In accordance with relation (54) [20], we have

$$(|R_{\tilde{W}}|^2 f, f)_{\mathfrak{H}} = \|R_{\tilde{W}} f\|^2_{\mathfrak{H}} \leq C \cdot \mathrm{Re}(R_{\tilde{W}} f, f)_{\mathfrak{H}} = C(Vf, f)_{\mathfrak{H}},$$

where $V := (R_{\tilde{W}} + R^*_{\tilde{W}})/2$. In accordance with the Theorem 5 [20], we have $\lambda_i(V) \asymp \lambda_i(R_H)$, thus we have $s_i(R_{\tilde{W}}) \leq C\lambda_i^{1/2}(R_H)$; $s_i^{-1}(R_{\tilde{W}}) \geq C\lambda_i^{1/2}(H)$, the detailed proof of the latter fact see in the Theorem 7 [20]. Using the monotonous property of the functions, we have

$$\frac{\ln^t s_i^{-1}(R_{\tilde{W}})}{s_i^{-2}(R_{\tilde{W}})} \leq C \cdot \frac{\ln^t \lambda_i(H)}{\lambda_i(H)} \leq C \cdot \frac{\alpha_i}{i^t}; \quad \frac{\ln^{t/2} s_i^{-1}(R_{\tilde{W}})}{s_i^{-1}(R_{\tilde{W}})} \leq C \cdot \frac{\alpha_i}{i^{t/2}},$$

where $\alpha_i \to 0$. Hence

$$\frac{i \ln s_i^{-1}(R_{\tilde{W}})}{s_i^{-2/t}(R_{\tilde{W}})} \leq C \cdot \alpha_i.$$

Taking into account the facts $n(s_i^{-1}) = i$; $n(\lambda) = n(s_i^{-1}),\ s_i^{-1} < \lambda < s_{i+1}^{-1}$, the monotonous property of the functions, we get

$$\frac{n(\lambda)\ln\lambda}{\lambda^{2/t}} < C \cdot \alpha_i,\ s_i^{-1} < \lambda < s_{i+1}^{-1}.$$

The proof corresponding to the general case is complete. Assume in additional that the operator $R_{\tilde{W}}$ is normal. Let us show that the operator $V := (R_{\tilde{W}} + R^*_{\tilde{W}})/2$ has a complete orthonormal system of the eigenvectors. Using formula (53) [20], we get

$$V^{-1} = 2H^{\frac{1}{2}}(I + B^2)H^{\frac{1}{2}}.$$

Note that in accordance with relation (67) [20], we have

$$(V^{-1} f, f)_{\mathfrak{H}} = 2(SH^{\frac{1}{2}} f, H^{\frac{1}{2}} f)_{\mathfrak{H}} \geq 2\|H^{\frac{1}{2}} f\|^2_{\mathfrak{H}} = 2(Hf, f)_{\mathfrak{H}},\ f \in \mathrm{D}(V^{-1}), \tag{6}$$

where $S = I + B^2$. Since V is selfadjoint, then due to Theorem 3 [4, p.136] the operator V^{-1} is selfadjoint also. Combining (6) with Lemma 3 [20], we get that V^{-1} is strictly accretive. Using these facts we can write

$$\|f\|_{V^{-1}} \geq C\|f\|_H,\ f \in \mathfrak{H}_{V^{-1}}.$$

Since the operator H has a discrete spectrum (see Theorem 5.3 [19]), then any set bounded with respect to the norm $\mathfrak{H}_H$ is a compact set with respect to the norm $\mathfrak{H}$ (see Theorem 4 [31, p.220]). Combining this fact with (6), Theorem 3 [31, p.216], we get that the operator V^{-1} has a discrete spectrum, i.e. it has the infinite set of the eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_i \leq ...,\ \lambda_i \to \infty,\ i \to \infty$ and the complete orthonormal system of the eigenvectors. Now note that the operators V, V^{-1} have the same eigenvectors. Therefore the operator V has the complete orthonormal system of the eigenvectors. Recall that any complete orthonormal system is a basis in separable Hilbert space. Hence the complete orthonormal system of the eigenvectors of the operator V is a basis in the space $\mathfrak{H}$. Since the operator $R_{\tilde{W}}$ is compact and normal (the last fact follows from the fact that $\tilde{W}$ is normal), then in accordance with the well-known theorem we have a fact that there exists an orthonormal system of the eigenvectors $\{\psi_i\}_1^\infty$ of the operator $R_{\tilde{W}}$. The system is complete in $\overline{\mathrm{R}(R_{\tilde{W}})}$ in the following sense

$$f = \sum_{i=1}^{\infty} (f, \psi_i)\psi_i,\ f \in \overline{\mathrm{R}(R_{\tilde{W}})}.$$

The corresponding system of eigenvalues is such that

$$R_{\tilde{W}}\psi_i = \lambda_i\psi_i,\ R^*_{\tilde{W}}\psi_i = \overline{\lambda_i}\psi_i,\ i \in \mathbb{N}.$$

The latter facts give us $R^*_{\tilde{W}}R_{\tilde{W}}\psi_i = |\lambda_i|^2\psi_i$. Since the operator $R^*_{\tilde{W}}R_{\tilde{W}}$ is selfadjoint and compact, then it is not hard to prove that $s_i(R_{\tilde{W}}) = |\lambda_i(R_{\tilde{W}})|$. Thus, we get

$$s_i(R_{\tilde{W}}) = |(R_{\tilde{W}}\psi_i, \psi_i)| = \left(1 + \tan\theta_i^2\right) |\mathrm{Re}(R_{\tilde{W}}\psi_i, \psi_i)| = \tag{7}$$
$$= \left(1 + \tan\theta_i^2\right) |(V\psi_i, \psi_i)| = \left(1 + \tan\theta_i^2\right) \lambda_i(V),$$

where the sequence $\{\tan\theta_i^2\}_1^\infty$ is bounded by virtue of the sectorial property of the operator. Note that the fact $\overline{\mathrm{R}(R_{\tilde{W}})}$ indicates that $\{\psi_i\}_1^\infty$ is complete in $\mathfrak{H}$. It follows that the operators V and $R_{\tilde{W}}$ have the same eigenvectors (since the system of the eigenvectors of the operator V is complete) and as a result we can claim that all eigenvalues of the operator V are involved in the right-hand side of relation (7). Taking into account the fact $\lambda_i(V) \asymp \lambda_i(R_H)$, we obtain the following relation

$$C_1 \sum_{i=1}^{\infty} |\lambda_i(R_H)|^p \leq \sum_{i=1}^{\infty} |s_i(R_{\tilde{W}})|^p \leq C_2 \sum_{i=1}^{\infty} |\lambda_i(R_H)|^p,\ p > 1.$$

Using the latter relation, we have $R_{\tilde{W}} \in \mathfrak{S}_p,\ p = 1/\mu$. At the same time applying the above reasonings, we get

$$\frac{\ln^t s_i^{-1}(R_{\tilde{W}})}{s_i^{-1}(R_{\tilde{W}})} \leq C \cdot \frac{\ln^t \lambda_i(H)}{\lambda_i(H)} \leq C \cdot \frac{\alpha_i}{i^t}.$$

Hence, in an analogous way, we get

$$\frac{i \ln s_i^{-1}(R_{\tilde{W}})}{s_i^{-1/t}(R_{\tilde{W}})} \leq C \cdot \alpha_i;\quad \frac{n(\lambda)\ln\lambda}{\lambda^{1/t}} \to 0,\ \lambda \to \infty.$$

□

Consider the following example.

Example 2. *Here we would like to produce an example of the sequence* $\{\lambda_i\}_1^\infty$ *that satisfies the condition*

$$(\ln^{\mu+1} x)'_{\lambda_i} = o(i^{-\mu}),\ (0 < \mu < 1),$$

$$\sum_{n=1}^{\infty} \frac{1}{|\lambda_n|^{1/\mu}} = \infty.$$

Consider a sequence $\lambda_i = i^\mu \ln^\mu i \cdot \ln^\mu \ln i$, then using the integral test for convergence we can easily see that the previous series is divergent. At the same time substituting, we get

$$\frac{\ln^\mu \lambda_i}{\lambda_i} \leq \frac{C \ln^\mu i}{i^\mu \ln^\mu i \cdot \ln^\mu \ln i} = \frac{C}{i^\mu \cdot \ln^\mu \ln i},$$

what gives us the fulfilment of the first condition.

Bellow, we produce an auxiliary technique to study the central problem of the paper. The estimates for the Fredholm Determinant were studied by Lidsky in the paper [25] and gave a main tool in questions related to contour integrals, their estimation. We have slightly improved results by Lidsky having involved the function β and obtaining in this way more accurate results.

Estimates for the Fredholm Determinant

In this section we produce an adopted version of the propositions given in the paper [25], we consider a case when a compact operator A belongs to the class $\mathfrak{I}_\rho$, where ρ is not integer. Having taken into account the facts considered in the previous subsection we can reformulate Lemma 2 [25] in the refined form.

Lemma 4. *Assume that a compact operator* A *satisfies the condition* $A \in \mathfrak{I}_\rho$, *where* ρ *is non-integer, then for arbitrary numbers* R, δ *such that* $R > 0,\ 0 < \delta < 1$, *there exists a circle* $|\lambda| = \tilde{R}$, $(1-\delta)R < \tilde{R} < R$, *so that the following estimate holds*

$$\|(I - \lambda A)^{-1}\| \leq e^{\gamma_m(|\lambda|)|\lambda|^\rho} |\lambda|^m,\ m = [\rho],\ |\lambda| = \tilde{R},$$

where

$$\gamma_m(|\lambda|) = \beta_m(|\lambda|^{m+1}) + C\beta_m(|C\lambda|^{m+1}),\ \beta_m(r) = r^{-\frac{\rho}{m+1}} \left(\int\limits_0^r \frac{n_{A^{m+1}}(t)dt}{t} + r \int\limits_r^\infty \frac{n_{A^{m+1}}(t)dt}{t^2} \right).$$

Proof. In accordance with the definition, we have $A \in \mathfrak{S}_{\rho+\varepsilon}$, $\varepsilon > 0$. By direct calculation we get

$$(I - \lambda^{m+1} A^{m+1})^{-1} (I + \lambda A + \lambda^2 A^2 + ... + \lambda^m A^m) = (I - \lambda A)^{-1}. \tag{8}$$

Note that in accordance with Lemma 3 [25], for sufficiently small $\varepsilon > 0$, we have

$$\sum_{i=1}^{\infty} \lambda_i^{\frac{\rho+\varepsilon}{m+1}}(\tilde{A}) \leq \sum_{i=1}^{\infty} \lambda_i^{\rho+\varepsilon}(A) < \infty,$$

where $\tilde{A} := (A^{*m+1}A^{m+1})^{1/2}$. By virtue of the fact $\rho/(m+1) < 1$, we can apply inequality (1.27) [25, p.10], using Lemma 1, we get

$$\|\Delta_{A^{m+1}}(\lambda^{m+1})(I-\lambda^{m+1}A^{m+1})^{-1}\| \leq C\prod_{i=1}^{\infty}\{1+|\lambda^{m+1}s_i(A^{m+1})|\} \leq Ce^{\beta_m(r^{m+1})r^{\rho}},$$

where $\Delta_{A^{m+1}}(\lambda^{m+1})$ is a Fredholm determinant of the operator A^{m+1} (see [25, p.8]). In accordance with Theorem 11 [24, p.33], we have

$$\Delta_{A^{m+1}}(\lambda^{m+1}) \geq e^{-(2+\ln\{12e/\delta\})\ln\xi_m},\ \xi_m = \max_{\psi\in[0,2\pi/(m+1)]}\{\Delta_{A^{m+1}}([2e\tilde{R}e^{i\psi}]^{m+1})\},$$

where R,δ arbitrary numbers such that $R>0$, $0<\delta<1$, the values of λ belong to the circle $|\lambda|=\tilde{R}$, which radius (it can be different in the above relations) is defined by R,δ and satisfy the condition $(1-\delta)R<\tilde{R}<R$. Note that in accordance with estimate (1.21) [25, p.10], we have

$$\Delta_{A^{m+1}}(\lambda) \leq C\prod_{i=1}^{\infty}\{1+|\lambda s_i(A^{m+1})|\},$$

thus applying Lemmas 1, 2, we get $\xi_m \leq e^{\beta_m([2e\tilde{R}]^{m+1})(2e\tilde{R})^{\rho}}$. Consider relation (8), we have the following estimate

$$\|(I-\lambda A)^{-1}\| \leq \|(I-\lambda^{m+1}A^{m+1})^{-1}\|\cdot\|(I+\lambda A+\lambda^2A^2+...+\lambda^mA^m)\| \leq$$

$$\leq \|(I-\lambda^{m+1}A^{m+1})^{-1}\|\cdot\frac{|\lambda|^{m+1}\|A\|^{m+1}-1}{|\lambda|\cdot\|A\|-1}.$$

We can easily see, it follows from the latter relation, that to obtain the desired estimate it suffices to estimate the term $\|(I-\lambda^{m+1}A^{m+1})^{-1}\|$, using the obtained estimates, we have

$$\|(I-\lambda^{m+1}A^{m+1})^{-1}\| \leq e^{\gamma_m(|\lambda|)|\lambda|^{\rho}},\ |\lambda|=\tilde{R},$$

where $\gamma_m(|\lambda|)=\beta_m(|\lambda|^{m+1})+(2+\ln\{12e/\delta\})\beta_m(|2e\lambda|^{m+1})(2e)^{\rho}$. Thus we get

$$\|(I-\lambda A)^{-1}\| \leq Ce^{\gamma_m(|\lambda|)|\lambda|^{\rho}}|\lambda|^m,\ |\lambda|=\tilde{R}.$$

□

Abel-Lidsky summarizing the series

In this subsection we improve results obtained by Lidsky [25] considering the class $\mathfrak{T}_\alpha$ under the additional assumption $\beta(r)=o(\ln^{-1}r)$. As an application we consider differential equations in the Hilbert space. We should stress that a significant refinement takes place in comparison with the reasonings by Lidsky [25]. However, let us begin our narrative. In accordance with the Hilbert theorem (see [13, p.32]) the spectrum of an arbitrary compact operator A consists of the so called normal eigenvalues it gives us the opportunity to consider a decomposition

$$\mathfrak{H}=\mathfrak{N}_q\dot{+}\mathfrak{M}_q, \tag{9}$$

where the first summand is an invariant subspace regarding the operator A a finite dimensional root subspace corresponding to the eigenvalue μ_q. Let n_q is a dimension of $\mathfrak{N}_q$ and let A_q is the operator induced in $\mathfrak{N}_q$. We can chose a basis (Jordan basis) in $\mathfrak{N}_q$ that consists of Jordan chains of eigenvectors and root vectors of the operator A_q. Each chain $e_{q_\xi}, e_{q_\xi+1}, ..., e_{q_\xi+k}$, where e_{q_ξ}, $\xi \in \mathbb{N}$ are the eigenvectors corresponding to the eigenvalue μ_q and other terms are root vectors, can be transformed by the operator A according with the following formulas

$$Ae_{q_\xi} = \mu_q e_{q_\xi},\ Ae_{q_\xi+1} = \mu_q e_{q_\xi+1} + e_{q_\xi}, ..., Ae_{q_\xi+k} = \mu_q e_{q_\xi+k} + e_{q_\xi+k-1}. \tag{10}$$

Considering the sequence $\{\mu_i\}_1^\infty$ of the eigenvalues of the operator A and choosing a Jordan basis in each corresponding space $\mathfrak{N}_i$ we can arrange a system of vectors $\{e_k\}_1^\infty$ which we will call a system of the root vectors or following Lidsky V.B. a system of the major vectors of the operator A. Let $e_1, e_2, ..., e_{n_i}$ be the Jordan basis in the subspace $\mathfrak{N}_i$, then in accordance with Lidsky V.B. there exists a corresponding biorthogonal basis $g_1, g_2, ..., g_{n_i}$ in the space $\mathfrak{M}_i^\perp$ (see [25, p.14]), note that in accordance with our clarification $\mathfrak{M}_i^\perp = \mathfrak{N}_i$. Moreover the set $\{g_k\}_1^{n_i}$ consists of the Jordan chains of the operator A^* which correspond to the Jordan chains (10) due to the following formula

$$A^* g_{q_\xi+k} = \overline{\mu_q} g_{q_\xi+k},\ A^* g_{q_\xi+k-1} = \overline{\mu_q} g_{q_\xi+k-1} + g_{q_\xi+k}, ..., A^* g_{q_\xi} = \overline{\mu_q} g_{q_\xi} + g_{q_\xi+1}.$$

Let us show that $\mathfrak{N}_i \subset \mathfrak{M}_j,\ i \neq j$ for this purpose note that in accordance with the representation $P_{\mu_i}\mathfrak{H} = \mathfrak{N}_i$ and the property $P_{\mu_i}P_{\mu_j} = 0,\ i \neq j$, where P_{μ_i} is a Riesz projector (integral) corresponding to the eigenvalue μ_i (see [13] Chapter I §1.3), we have an orthogonal decomposition $\mathfrak{H} = \mathfrak{N}_i \dot{+} \mathfrak{N}_j \dot{+} \mathfrak{M}_{ij}$, where $\mathfrak{M}_{ij} = (I - P_{\mathfrak{N}_i \dot{+} \mathfrak{N}_j})\mathfrak{H}$. On the other hand in accordance with [13] Chapter I §2.1 we can claim that the following orthogonal decomposition is unique

$$\mathfrak{H} = \mathfrak{N}_j \dot{+} \mathfrak{M}_j,$$

hence we have an orthogonal sum $\mathfrak{M}_j = \mathfrak{N}_i \dot{+} \mathfrak{M}_{ij}$, what proves the desired result. Taking into account relation (9), we conclude that the set $g_1, g_2, ..., g_{n_i},\ i \neq j$ is orthogonal to the set $e_1, e_2, ..., e_{n_j}$. Gathering the sets $g_1, g_2, ..., g_{n_i},\ i = 1, 2, ...$, we can obviously create a biorthogonal system $\{g_i\}_1^\infty$ with respect to the system of the major vectors of the operator A. It is rather reasonable to call it as a system of the major vectors of the operator A^*. Note that if an element $f \in \mathfrak{H}$ allows a decomposition in the strong sense

$$f = \sum_{n=1}^{\infty} e_n c_n,\ c_n \in \mathbb{C},$$

then by virtue of the biorthogonal system existing we can claim that such a representation is unique. Further, let us come to the previously made agrement that the vectors in each Jourdan chain are arranged in the same order as in (10) i.e. at the first place there stands an eigenvector. It is clear that under such an assumption we have

$$c_{q_\xi+i} = \frac{(f, g_{q_\xi+k-i})}{(e_{q_\xi+i}, g_{q_\xi+k-i})},\ 0 \leq i \leq k(q_\xi),$$

where $k(q_\xi)+1$ is a number of elements in the q_ξ-th Jourdan chain. In particular, if the vector e_{q_ξ} is included to the major system solo, there does not exist a root vector corresponding to the same eigenvalue, then

$$c_{q_\xi} = \frac{(f, g_{q_\xi})}{(e_{q_\xi}, g_{q_\xi})}.$$

Note that in accordance with the property of the biorthogonal sequences we can expect that the denominators equal to one in the previous two relations. Consider a formal series corresponding to a decomposition on the major vectors of the operator A

$$f \sim \sum_{n=1}^{\infty} e_n c_n,$$

where each number n corresponds to a number $q_\xi + i$ (thus, the coefficients c_n are defined in accordance with the above and numerated in a simplest way). Consider a set of the polynomials with respect to a real parameter t

$$P_m^\alpha(\zeta^{-1}, t) = \frac{e^{t\zeta^{-\alpha}}}{m!} \frac{d^m}{d\zeta^m} e^{-t\zeta^{-\alpha}}, \ \alpha > 0, \ m = 1, 2, ..., .$$

Consider a series

$$\sum_{n=1}^{\infty} c_n(t) e_n, \tag{11}$$

where the coefficients $c_n(t)$ are defined in accordance with the correspondence between the indexes n and $q_\xi + i$ in the following way

$$c_{q_\xi + i}(t) = e^{-\lambda_q^\alpha t} \sum_{m=0}^{k-i} P_m^\alpha(\lambda_{q_\xi}, t) c_{q_\xi + i + m}, \ i = 0, 1, 2, ..., k,$$

here $\lambda_q = 1/\mu_q$ is a characteristic number corresponding to e_{q_ξ}. It is clear that in any case, we have $c_n(t) \to c_n$, $t \to 0$ (it can be established by direct calculations). In accordance with the definition given in [25, p.17] we will say that series (11) converges to the element f in the sense (A, λ, α), if there exists a sequence of the natural numbers $\{N_j\}_1^\infty$ such that

$$f = \lim_{t \to +0} \lim_{j \to \infty} \sum_{n=1}^{N_j} c_n(t) e_n.$$

Note that sums of the latter relation forms a subsequence of the partial sums of the series (11).

To prove the main theorem we need the following lemmas by Lidsky, note that in spite of the fact that we have rewritten the lemmas in the refined form the proof has not been changed and can be found in the paper [25].

Lemma 5. *Let* $\Theta(A) \subset \mathfrak{L}_0(\pi/2v)$, $v > 1/2$, *then on each ray* ζ *containing the point zero and not belonging to the sector* $\mathfrak{L}_0(\pi/2v)$ *as well as real axis, we have*

$$\|(I - \lambda A)^{-1}\| \leq \frac{1}{\sin \varphi}, \ \lambda \in \zeta,$$

where $\varphi = \min\{|\mathrm{arg}\zeta - \pi/2v|, |\mathrm{arg}\zeta + \pi/2v|\}$.

Lemma 6. *Assume that* $f \in \mathrm{R}(A)$, *then*

$$\lim_{t \to +0} \int\limits_{\gamma} e^{-\lambda^\alpha t} A(I - \lambda A)^{-1} f d\lambda = f,$$

where

$$\gamma:=\{\lambda:\ |\lambda|=r>0,\ |\mathrm{arg}\lambda|\leq\pi/2\upsilon+\varepsilon\}\cup\{\lambda:\ |\lambda|>r,\ |\mathrm{arg}\lambda|=\pi/2\upsilon+\varepsilon\}\,,\ \upsilon>1/2,$$

the number r is chosen so that the function $\|(I-\lambda A)^{-1}\|$ is regular within the corresponding circle.

Lemma 7. *We claim that, in the pole λ_q of the operator $(I-\lambda A)^{-1}$, the residue of the function $e^{-\lambda^\alpha t}A(I-\lambda A)^{-1}f$, $(f\in\mathfrak{H})$ equals*

$$-\sum_{\xi=1}^{m(q)}\sum_{i=0}^{k(q_\xi)}e_{q_\xi+i}c_{q_\xi+i}(t),$$

where $m(q)$ is a geometrical multiplicity of the q-th eigenvalue, $k(q_\xi)+1$ is a number of elements in the q_ξ-th Jourdan chain.

3 Main results

In this section we consider the operator classes under the point of view made in the latter section. Firstly, we consider a general statement with the made refinement related to the involved notion of the convergence exponent. Secondly, having formulated conditions in terms of the operator order, we produce an example establishing the fact in accordance with which the contours may be chosen in a concrete way, under the assumption $\rho=\alpha$, what provides a peculiar validity of the statement. Finally, we consider applications to the differential equations in the Hilbert space. The structure of the proof of the following theorem completely belongs to Lidsky. However we produce the proof since we make a refinement corresponding to consideration of the case when a convergent exponent does not equals the index of the Schatten-von Neumann class.

Theorem 2. *Assume that $\Theta(A)\subset\mathfrak{L}_0(\pi/2\upsilon)$, $\upsilon>\max\{\alpha,1/2\}$, $\alpha\geq\rho$, then there exists such a sequence of natural numbers $\{M_j\}_1^\infty$ that*

$$\frac{1}{2\pi i}\int\limits_{\gamma}e^{-\lambda^\alpha t}A(I-\lambda A)^{-1}fd\lambda=\lim_{j\to\infty}\sum_{q=1}^{M_j}\sum_{\xi=1}^{m(q)}\sum_{i=0}^{k(q_\xi)}e_{q_\xi+i}c_{q_\xi+i}(t),$$

moreover

$$\sum_{\nu=0}^{\infty}\left\|\sum_{q=N_\nu+1}^{N_{\nu+1}}\sum_{\xi=1}^{m(q)}\sum_{i=0}^{k(q_\xi)}e_{q_\xi+i}c_{q_\xi+i}(t)\right\|<\infty. \tag{12}$$

Proof. Having fixed $R>0,0<\delta<1$, consider a monotonically increasing sequence $\{R_\nu\}_0^\infty$, $R_\nu=R(1-\delta)^{-\nu+1}$, then using Lemma 4, we get

$$\|(I-\lambda A)^{-1}\|\leq e^{\gamma_m(|\lambda|)|\lambda|^\rho}|\lambda|^m,\ m=[\rho],\ |\lambda|=\tilde{R}_\nu,\ R_\nu<\tilde{R}_\nu<R_{\nu+1}.$$

Denote by γ_ν a bound of the intersection of the ring $\tilde{R}_\nu<|\lambda|<\tilde{R}_{\nu+1}$ with the interior of the contour γ, thus we get $\gamma_\nu:=\{\lambda:\ |\lambda|=\tilde{R}_\nu,\ |\lambda|=\tilde{R}_{\nu+1},\ |\mathrm{arg}\lambda|<\pi/2\upsilon+\varepsilon\}\cup\{\lambda:\ \tilde{R}_\nu<$

$|\lambda| < \tilde{R}_{\nu+1}$, $|\arg\lambda| = \pi/2\upsilon + \varepsilon\}$. Denote by N_ν a number of poles being contained in the set $G_\nu := \{\lambda : \, r < |\lambda| < \tilde{R}_\nu, \, |\arg\lambda| < \pi/2\upsilon + \varepsilon\}$, where r is defined in Lemma 7, in accordance with which, we get

$$\frac{1}{2\pi i}\int\limits_{\gamma_\nu} e^{-\lambda^\alpha t} A(I-\lambda A)^{-1} f d\lambda = \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}(t).$$

Let us estimate the above integral for this purpose split the contour γ_ν on for terms $\gamma_{\tilde{\nu}} := \{\lambda : |\lambda| = \tilde{R}_\nu, \, |\arg\lambda| < \pi/2\upsilon + \varepsilon\}$, $\gamma_{\nu_+} := \{\lambda : \, \tilde{R}_\nu < |\lambda| < \tilde{R}_{\nu+1}, \, \arg\lambda = \pi/2\upsilon + \varepsilon\}$, $\gamma_{\nu_-} := \{\lambda : \, \tilde{R}_\nu < |\lambda| < \tilde{R}_{\nu+1}, \, \arg\lambda = -\pi/2\upsilon - \varepsilon\}$. In accordance with the above, we have

$$I_\nu := \left\| \int\limits_{\gamma_{\tilde{\nu}}} e^{-\lambda^\alpha t} A(I-\lambda A)^{-1} f d\lambda \right\| \le |\lambda| \cdot \|A(I-\lambda A)^{-1} f\| \int\limits_{-\pi/2\upsilon-\varepsilon}^{\pi/2\upsilon+\varepsilon} e^{-t\mathrm{Re}\lambda^\alpha} d \arg\lambda.$$

Note that in accordance with the imposed conditions $\upsilon > \alpha$, we get $|\arg\lambda| < \pi/2\alpha - \varepsilon$. It follows that

$$\mathrm{Re}\lambda^\alpha \ge |\lambda|^\alpha \cos\left[(\pi/2\alpha - \varepsilon)\alpha\right] = |\lambda|^\alpha \sin \alpha\, \varepsilon.$$

Thus, we get

$$I_\nu \le |\lambda| \cdot \|A(I-\lambda A)^{-1} f\| e^{-t|\lambda|^\alpha \sin\alpha\,\varepsilon}.$$

Taking into account Lemma 4, we get

$$I_\nu \le e^{\gamma_m(|\lambda|)|\lambda|^\rho - t|\lambda|^\alpha \sin\alpha\,\varepsilon} |\lambda|^{m+1} =$$

$$= e^{|\lambda|^\rho[\gamma_m(|\lambda|) - t|\lambda|^{\alpha-\rho}\sin\alpha\varepsilon]} |\lambda|^{m+1}, \; m = [\rho], \; |\lambda| = \tilde{R}_\nu.$$

It is clear that for a fixed t and a sufficiently large $|\lambda|$, we have $|\lambda|^\rho\{\gamma_m(|\lambda|) - t|\lambda|^{\alpha-\rho}\sin\alpha\,\varepsilon\}\ln|\lambda| < 0$, since we have the facts $\alpha \ge \rho$, in accordance with Lemma 2 $\gamma_m(|\lambda|) \to 0$, $|\lambda| \to \infty$. Therefore, the following series is convergent

$$\sum_{\nu=0}^{\infty} I_\nu < \infty.$$

Analogously, using Lemma 5, we get

$$J_\nu := \left\| \int\limits_{\gamma_{\nu_+}} e^{-\lambda^\alpha t} A(I-\lambda A)^{-1} f d\lambda \right\| \le \frac{\|f\|}{\sin\varphi} \int\limits_{R_\nu}^{R_{\nu+1}} |e^{-t\lambda^\alpha}||d\lambda| \le e^{-tR_\nu^\alpha \sin\alpha\,\varepsilon} \int\limits_{R_\nu}^{R_{\nu+1}} |d\lambda| =$$

$$= e^{-tR_\nu^\alpha \sin\alpha\,\varepsilon}\{R_{\nu+1} - R_\nu\}.$$

Hence

$$\sum_{\nu=0}^{\infty} J_\nu < \infty.$$

Thus, we obtain relation (12), from what follows the rest part of the claim. □

Bellow, we produce an application of the above results, we study a concrete operator class for which it is possible to chose the contour sequence of the power type. The following theorem is formulated in terms of the asymptotics of the operator H.

Sequence of contours

Recall that in the paper [25] there is considered a sequence of the contours of the exponentional type the condition $\alpha > \rho$ is imposed. Bellow, we improve this result in the following sense we produce a sequence of the power type contours which gives us a solution of the problem in the case $\alpha = \rho$. Let us come to the agrement to use a short-hand notation $A := R_{\tilde{W}}$.

Theorem 3. *Assume that the operator $\tilde{W}$ is normal, the conditions $(\ln^{\mu+1} x)'_{\lambda_i(H)} = o(i^{-\mu})$, $0 < \mu < 1$, $\arctan\{C_2/C_1\} < \pi\mu/2$ hold. Then the following relation holds*

$$\frac{1}{2\pi i}\int\limits_{\gamma} e^{-\lambda^{\alpha}t}A(I-\lambda A)^{-1}f d\lambda = \sum_{\nu=0}^{\infty}\sum_{q=N_\nu+1}^{N_{\nu+1}}\sum_{\xi=1}^{m(q)}\sum_{i=0}^{k(q_\xi)} e_{q_\xi+i}c_{q_\xi+i}(t),$$

where $\alpha = 1/\mu$, a sequence of contours $\{R_\nu\}_0^\infty$ is chosen so that such root vectors are united, in the partial sums corresponding to $q = N_\nu+1,\ N_\nu+2, ..., N_{\nu+1}$, for which

$$|\lambda_{N_\nu+1}| - |\lambda_{N_\nu}| \leq C|\lambda_{N_\nu+1}|^{1-1/(\mu-\varepsilon)},\ \forall\varepsilon > 0.$$

Proof. Note that the following relation follows from the conditions imposed on the operator W see introduction

$$|\lambda_i^{-1}| = o\left(i^{-\mu+\varepsilon}\right),\ i\to\infty,\ \forall\varepsilon>0,$$

thus $\lambda_i/i^{\mu-\varepsilon} \geq C$. Using this fact, we can prove that that there exists a subsequence $\{\lambda_{i_k}\}_{k=1}^\infty$, such that

$$|\lambda_{i_k+1}| - |\lambda_{i_k}| \geq K|\lambda_{i_k+1}|^{1-1/(\mu-\varepsilon)},\ K>0,$$

for this purpose it suffices to establish the following implication

$$\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n)/\lambda_n^{(p+1)/p} = 0,\ \Longrightarrow \lim_{n\to\infty}\lambda_n/n^p = 0,\ p>0.$$

Now, consider

$$|\lambda_{N_\nu+1}| - |\lambda_{N_\nu}| \geq C|\lambda_{N_\nu}|^{q_\varepsilon},\ q_\varepsilon := 1-1/(\mu-\varepsilon),$$

and let us find δ from the condition $R = K|\lambda_{N_\nu}|^{q_\varepsilon} + |\lambda_{N_\nu}|$, $R(1-\delta) = |\lambda_{N_\nu}|$, then $\delta^{-1} = K^{-1}|\lambda_{N_\nu}|^{1-q_\varepsilon}$. Note that in accordance with Lemma 4, reasonings of Theorem 2, there exists an arch $\gamma_{\tilde{\nu}} := \{\lambda :\ |\lambda| = \tilde{R}_\nu,\ |\mathrm{arg}\lambda| < \pi\mu/2\}$, in the ring $(1-\delta)R < |\lambda| < R$, on which the following estimate holds

$$I_\nu = \left\|\int\limits_{\gamma_{\tilde{\nu}}} e^{-\lambda^\alpha t}A(I-\lambda A)^{-1}f d\lambda\right\| \leq e^{|\lambda|^\rho[\gamma_m(|\lambda|)-t|\lambda|^{\alpha-\rho}\sin\alpha\varepsilon]}|\lambda|^{m+1},\ m=[\rho],\ |\lambda| = \tilde{R}_\nu,$$

where $\gamma_m(|\lambda|) = \beta_m(|\lambda|^{m+1}) + (2+\ln\{12e/\delta\})\beta_m(|2e\lambda|^{m+1})(2e)^\rho$. Substituting δ^{-1}, we have

$$\ln\{12e/\delta\} = \ln\{12eK^{-1}|\lambda_{N_\nu}|^{1-q_\varepsilon}\} = \ln\{|\lambda_{N_\nu}|^{1-q_\varepsilon}\} + C.$$

It is clear that to obtain the desired result we should prove that $\ln|\lambda_{N_\nu}|^{1-q_\varepsilon}\beta_m(|\lambda_{N_\nu}|^{m+1}) \to 0,\ \nu\to\infty$. Note that in accordance with Lemma 3, we get $n_A(\lambda)=o(\lambda^{1/\mu}/\ln\lambda)$, where $n_A(\lambda)$ is the counting function of the sequence $\{s_i^{-1}(R_{\tilde W})\}_1^\infty$. Applying the L'Hôpital's rule analogously to the technique using in Example 2, we come to the problem

$$\ln r\frac{n_{A^{m+1}}(r^{m+1})}{r^{1/\mu}}\to 0\,? \tag{13}$$

We need establish some facts, note that the following operators have the same eigenfunctions i.e.

$$(A^*A)^{1/2}f_n=\mu_n f_n \Longleftrightarrow (A^{m*}A^m)^{1/2}f_n=\mu_n^m f_n,\ m=[\rho]+1. \tag{14}$$

To prove this fact, firstly let us show that $A^{*m}=A^{m*}$, it follows easily from the inclusion $A^{*m}\subset A^{m*}$ and the fact $\mathrm{D}(A^{*m})=\mathfrak{H}$. Thus, for a normal operator we have $(A^*A)^m=A^{*m}A^m$. Let us involve a notion of a spectral function of a selfadjoint non-negative operator, in accordance with a standard definition (see [22] Chapter 3), we have

$$(A^*A)^\tau=\int\limits_0^{\|A^*A\|}\lambda^\tau dP_\lambda,\ \tau>0,$$

where the latter integral is understood in the Riemann sense as a limit of the partial sums

$$\sum_{i=0}^n \xi_i^\tau P_{\Delta\lambda_i}\xrightarrow{\mathfrak{H}}\int\limits_0^{\|A^*A\|}\lambda^\tau dP_\lambda,\ \omega\to 0,$$

where $(0=\lambda_0<\lambda_1<...<\lambda_n=\|A^*A\|)$ is an arbitrary splitting of the segment $[0,\|A^*A\|]$, $\omega:=\max\limits_i(\lambda_{i+1}-\lambda_i)$, ξ_i is an arbitrary point belonging to $[\lambda_i,\lambda_{i+1}]$, the operators $P_{\Delta\lambda_i}$ is projectors corresponding to the selfadjoint operator. It follows easily from the well-known facts that if in additional A^*A is a compact operator, then the above formula reduces to

$$(A^*A)^\tau f=\sum_{n=1}^\infty \lambda_n^\tau(f,\varphi_n)\varphi_n,$$

where $\{\varphi_n\}_1^\infty$ is a set of eigenvectors of the operator A^*A. Taking into account the latter representation, an obvious fact that $(A^*A)^\tau$ is selfadjoint, it is not hard to obtain a relation

$$(A^*A)^{\frac{1}{2}\cdot m}=(A^*A)^{m\cdot\frac{1}{2}}.$$

Thus, using the property $A^*A=AA^*$, we get

$$(A^*A)^{\frac{1}{2}\cdot m}=(A^{m*}A^m)^{\frac{1}{2}},$$

from what follows the implication from the left-hand side to the right-hand side in formula (14). To obtain the other implication we should establish the fact that the eigenvectors of the operator and its positive powers are the same. To prove it, let us notice that

$$T^\tau e_i=\lambda_i^\tau\varphi_i,\ i\in\mathbb{N},$$

where $T := A^*A$. It follows that

$$T^\tau f = \sum_{n=1}^{\infty} \lambda_n^\tau (f, \varphi_n)\varphi_n = \sum_{n=1}^{\infty} (f, T^\tau \varphi_n)\varphi_n = \sum_{n=1}^{\infty} (T^\tau f, \varphi_n)\varphi_n.$$

Hence we have a fact

$$g = \sum_{n=1}^{\infty} (g, \varphi_n)\varphi_n, \; g \in \mathrm{R}(T^\tau).$$

Now, let us assume that there exists an eigenfunction h of the operator T^τ that differs from $\varphi_i, \; i \in \mathbb{N}$. Using the fact proved above, we get

$$T^\tau h = \sum_{n=1}^{\infty} \lambda_n^\tau (h, \varphi_n)\varphi_n = \zeta \sum_{n=1}^{\infty} (h, \varphi_n)\varphi_n,$$

where ζ is a corresponding eigenvalue. Multiplying (in the sense of the inner product) both sides of the latter relation on φ_k, we get $\lambda_k^\tau = \zeta$, hence $h = \varphi_k$, this contradiction proves the desired result. Thus, we complete the proof of formula (14). To complete the proof of relation (13) we need to mention the fact $n_A(\lambda) = n_{A^m}(\lambda^m)$ which follows easily from relation (14). Thus making a substitution and using the theorem condition, we claim that relation (13) holds. Finally, we should note that the integrals along the contours $\gamma_{\nu_+} := \{\lambda : (1-\delta)R < |\lambda| < R, \arg\lambda = \pi\mu/2 + \varepsilon\}$, $\gamma_{\nu_-} := \{\lambda : (1-\delta)R < |\lambda| < R, \arg\lambda = -\pi\mu/2 - \varepsilon\}$ converges uniformly i.e. analogously to Theorem 2, we have

$$J_\nu := \left\| \int\limits_{\gamma_{\nu_+}} e^{-\lambda^\alpha t} A(I - \lambda A)^{-1} f d\lambda \right\| \leq \frac{\|f\|}{\sin\varphi} \int\limits_{(1-\delta)R}^{R} |e^{-t\lambda^\alpha}||d\lambda| \leq e^{-t(1-\delta)^\alpha R^\alpha \sin\alpha\varepsilon} \int\limits_{(1-\delta)R}^{R} |d\lambda| =$$

$$= e^{-t(1-\delta)^\alpha R^\alpha \sin\alpha\varepsilon} \delta R.$$

From what follows the desired result. □

The following consequence follows immediately from Lemma 6.

Consequence 1. *Under Theorem 3 assumptions, we get*

$$f = \lim_{t\to+0} \sum_{\nu=0}^{\infty} \sum_{q=N_\nu+1}^{N_{\nu+1}} \sum_{\xi=1}^{m(q)} \sum_{i=0}^{k(q_\xi)} e_{q_\xi+i} c_{q_\xi+i}(t), \; f \in \mathrm{D}(\tilde{W}).$$

Differential equations in the Hilbert space

In this section we consider an operator $(E - \lambda A)^{-1}$, where $A := R_{\tilde{W}}$. Note that absolutely analogously to the proof of Lemma 2 [20] we can prove that $\mathrm{P}(A) \in \mathbb{C} \setminus \mathfrak{L}_0$, where $\mathfrak{L}_0$ is a sector containing the numerical range of the operator A. It follows easily from this fact that the operator $(E - \lambda A)^{-1}$ is defined on the Hilbert space $\mathfrak{H}$ when $\lambda \in \mathbb{C} \setminus \mathfrak{L}_0$.

1. Let $u:=u(t),\, t>0$ be an element-function in the Hilbert space $u:\mathbb{R}_+\to\mathfrak{H}$. Consider the Cauchy problem for a differential equation

$$\frac{du}{dt}+\tilde{W}^n u=0,\; n=2,3,...\,, \tag{15}$$

(here we should note that the case $n=1$ was considered by Lidsky [25]) under the initial condition

$$u(0)=h\in \mathrm{D}(\tilde{W}), \tag{16}$$

in the case when in additional $\tilde{W}^n$ is accretive, we can assume that $h\in\mathfrak{H}$. Assume that the following conditions holds regarding the operator $\tilde{W}$. It is a normal operator, the conditions of Theorem 3 holds, then there exists a solution

$$u(t)=\frac{1}{2\pi i}\int\limits_{\gamma} e^{-\lambda^n t}A(E-\lambda A)^{-1}h d\lambda=\sum\limits_{\nu=0}^{\infty}\sum\limits_{q=N_\nu+1}^{N_{\nu+1}}\sum\limits_{\xi=1}^{m(q)}\sum\limits_{i=0}^{k(q_\xi)}e_{q_\xi+i}c_{q_\xi+i}(t),\; h\in\mathrm{D}(\tilde{W}).$$

of the Cauchy problem (15),(16). Note that since the operator A is sectorial, then we can consider a contour γ defined in Lemma 6. Using Lemma 5, it is not hard to prove that the following integral converges and as a consequence presents an element of the Hilbert space i.e.

$$\frac{1}{2\pi i}\int\limits_{\gamma} e^{-\lambda^n t}(E-\lambda A)^{-1}h d\lambda=g(t)\in\mathfrak{H}.$$

Since A is bounded, then the latter relation gives us the fact

$$\tilde{W}u(t)=\frac{1}{2\pi i}\int\limits_{\gamma} e^{-\lambda^n t}(E-\lambda A)^{-1}h d\lambda,$$

from what follows that $u(t)\in\mathrm{D}(\tilde{W})$. Analogously the above, using Lemma 5 we can show the the following derivative exists i.e.

$$\frac{du}{dt}=-\frac{1}{2\pi i}\int\limits_{\gamma} e^{-\lambda^n t}\lambda^n A(E-\lambda A)^{-1}h\, d\lambda\in\mathfrak{H}.$$

Notice that $\lambda^n A^n(E-\lambda A)^{-1}=(E-\lambda A)^{-1}-(E+\lambda A+...+\lambda^{n-1}A^{n-1})$, substituting this relation to the above formula, we obtain

$$A^{n-1}\frac{du}{dt}=-\frac{1}{2\pi i}\int\limits_{\gamma} e^{-\lambda^n t}(E-\lambda A)^{-1}h\, d\lambda+\frac{1}{2\pi i}\int\limits_{\gamma} e^{-\lambda^n t}\sum\limits_{k=0}^{n-1}\lambda^k A^k h,\, d\lambda=I_1+I_2.$$

The second integral equals zero by virtue of the fact that the function under the integral is analytical inside the domain G, here we denotes by G the interior of the contour γ. Thus, we have come to the relation

$$A^{n-1}\frac{du}{dt}=-\frac{1}{2\pi i}\int\limits_{\gamma} e^{-\lambda^n t}(E-\lambda A)^{-1}h\, d\lambda.$$

Since the left-hand side of the latter relation belongs to $\mathrm{D}(\tilde{W}^{n-1})$, then it is true for the right-hand side also. It follows that $u \in \mathrm{D}(W^n)$. Now, if we consider the expression for u, we get

$$A^{n-1}\frac{du}{dt} + \tilde{W}u = 0,$$

Multiplying both sides on $\tilde{W}^{n-1}$, we obtain the fact that $u(t)$ is a solution of the equation (15). Let us show that the initial condition holds in the sense

$$u(t) \xrightarrow{\mathfrak{H}} h,\, t \to +0.$$

It becomes clear in the case $h \in \mathrm{D}(\tilde{W})$, in this cases it suffices to apply Lemma 6, what gives us the desired result. Consider a case when h is an arbitrary element of the Hilbert space $\mathfrak{H}$. It follows from Lemma 5 that for a fixed t the operator

$$S_t h = \frac{1}{2\pi i}\int\limits_{\gamma} e^{-\lambda^n t} A(E-\lambda A)^{-1} h d\lambda = u(t),$$

is bounded. Let us show that

$$\|S_t\| \leq 1,\ t > 0.$$

Note that the prove of this fact cannot be implemented by direct estimating of the operator, but requires a special technique. Bellow, we produce an estimate obtained directly. Analogously, as it has been done previously consider a contour $\tilde{\gamma}$ splitted on terms $\tilde{\gamma}_R := \{\lambda :\ |\lambda| = R,\, |\mathrm{arg}\lambda| < \pi\mu/2 - \varepsilon\}$, $\tilde{\gamma}_+ := \{\lambda :\ 0 < |\lambda| < R,\, \mathrm{arg}\lambda = \pi\mu/2 - \varepsilon\}$, $\tilde{\gamma}_- := \{\lambda :\ 0 < |\lambda| < R,\, \mathrm{arg}\lambda = -\pi\mu/2 + \varepsilon\}$. In accordance with the above, we have

$$I_R = \left\| \int\limits_{\tilde{\gamma}_R} e^{-\lambda^\alpha t} A(I-\lambda A)^{-1} f d\lambda \right\| \leq |\lambda| \cdot \|A(I-\lambda A)^{-1} f\| \int\limits_{-\pi/2\upsilon-\varepsilon}^{\pi/2\upsilon+\varepsilon} e^{-t\mathrm{Re}\lambda^\alpha} d\,\mathrm{arg}\lambda.$$

Using the imposed conditions $|\mathrm{arg}\lambda| < \pi\mu/2 - \varepsilon$, we have

$$\mathrm{Re}\lambda^\alpha \geq |\lambda|^\alpha \cos\left[(\pi/2\alpha - \varepsilon)\alpha\right] = |\lambda|^\alpha \sin\alpha\,\varepsilon.$$

Thus, we get

$$I_R \leq |\lambda| \cdot \|A(I-\lambda A)^{-1} f\| \cdot e^{-t|\lambda|^\alpha \sin\alpha\,\varepsilon},\ |\lambda| = R,$$

from what follows that

$$I_R \to 0,\ R \to \infty.$$

Analogously, we get

$$J_- = \left\| \int\limits_{\tilde{\gamma}_-} e^{-\lambda^\alpha t} A(I-\lambda A)^{-1} f d\lambda \right\| \leq \frac{\|f\|}{\sin\varphi}\int\limits_0^R |e^{-t\lambda^\alpha}||d\lambda| \leq \frac{\|f\|}{\sin\varphi}\int\limits_0^R e^{-t|\lambda|^\alpha \sin\alpha\,\varepsilon}|d\lambda| \leq$$

$$\leq \frac{\|f\|}{\sin\varphi}\int\limits_0^R e^{-t|\lambda|\sin\alpha\,\varepsilon}|d\lambda| \leq Ct^{-1}\|f\|.$$

Thus, we obtain an estimate $\|S_t\| \leq Ct^{-1},\ t>0$ that is not uniform regarding t, it indicates that some difficulties are presented in this problem. To avoid this disadvantage consider the following reasonings. Firstly, assume that $h \in \mathrm{D}(\tilde{W})$, then as it was mentioned above, by virtue of Lemma 6, we get

$$u(t) \xrightarrow{\mathfrak{H}} h,\ t \to +0.$$

Thus we can claim the fact that $u(t)$ is continuous at the right-hand side of the point zero. Let us multiply the both sides of relation (15) on u in the sense of the inner product

$$\left(\frac{du}{dt}, u\right) + (\tilde{W}^n u, u) = 0.$$

Consider a real part of the latter relation, we have

$$\mathrm{Re}\left(\frac{du}{dt}, u\right) + \mathrm{Re}(\tilde{W}^n u, u) = \frac{1}{2}\left(\frac{du}{dt}, u\right) + \frac{1}{2}\left(u, \frac{du}{dt}\right) + \mathrm{Re}(\tilde{W}^n u, u) =$$

$$= \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 = -\mathrm{Re}(\tilde{W}^n u, u) \leq 0.$$

Hence

$$\|u(\tau)\|^2 - \|u(0)\|^2 = \int\limits_0^\tau \frac{d}{dt}\|u(t)\|^2 dt \leq 0.$$

The last relation can be rewritten in the form

$$\|S_t h\| \leq \|h\|,\ h \in \mathrm{D}(\tilde{W}),$$

since $\mathrm{D}(\tilde{W})$ is a dense set in $\mathfrak{H}$ then we obviously have a desired result i.e. $\|S_t\| \leq 1$. Now consider the following reasonings assuming that

$$h_n \xrightarrow{\mathfrak{H}} h,\ n \to \infty,\ \{h_n\} \subset \mathrm{D}(\tilde{W}),\ h \in \mathfrak{H},$$

we have

$$\|u(t) - h\| = \|S_t h - h\| = \|S_t h - S_t h_n + S_t h_n - h_n + h_n - h\| \leq \|S_t\| \cdot \|h - h_n\| + \|S_t h_n - h_n\| + \|h_n - h\|.$$

It is clear that if we chose n so that $\|h - h_n\| < \varepsilon/3$ and after chose t so that $\|S_t h_n - h_n\| < \varepsilon$, then we obtain $\forall \varepsilon > 0,\ \exists \delta(\varepsilon):\ \|u(t) - h\| < \varepsilon,\ t < \delta$. Thus the initial condition holds.

2. Let $\mathfrak{H}$ still be the abstract Hilbert space. Consider a fractional differential operator in the Riemann-Liouvile sense (see [37]) i.e. in the formal form, we have

$$\mathfrak{D}_-^{1/\alpha} f(t) := -\frac{1}{\Gamma(1-1/\alpha)}\frac{d}{dt}\int\limits_0^\infty f(t+x)x^{-1/\alpha}dx.$$

Let us study the Cauchy problem for a differential equation

$$\mathfrak{D}_-^{1/\alpha} u = \tilde{W} f,\ \alpha > 1, \tag{17}$$

where $u:=u(t),\, t>0$ is an element-function in the Hilbert space $u:\mathbb{R}_+\rightarrow\mathfrak{H}$, under the initial condition

$$u(0)=h\in \mathrm{D}(\tilde{W}). \tag{18}$$

In additional, in the case when the operator composition $\mathfrak{D}^{1-1/\alpha}_{-}\tilde{W}$ is an accretive operator we can weaken conditions assuming that $h\in\mathfrak{H}$. Suppose the operator $\tilde{W}$ is normal, the conditions of Theorem 3 holds, then there exists a solution

$$u(t)=\frac{1}{2\pi i}\int\limits_{\gamma}e^{-\lambda^{n}t}A(E-\lambda A)^{-1}h d\lambda=\sum\limits_{\nu=0}^{\infty}\sum\limits_{q=N_{\nu}+1}^{N_{\nu+1}}\sum\limits_{\xi=1}^{m(q)}\sum\limits_{i=0}^{k(q_{\xi})}e_{q_{\xi}+i}c_{q_{\xi}+i}(t),\; h\in \mathrm{D}(\tilde{W}). \tag{19}$$

of the Cauchy problem (17),(18). Analogously to the previous case let us find a solution in the form (18). In the same way we get $u\in \mathrm{D}(\tilde{W})$. Consider the following formula for the fractional derivative on the sufficiently smooth functions

$$-\frac{1}{\Gamma(1-1/\alpha)}\frac{d}{dt}\int\limits_{0}^{\infty}f(x+t)x^{-1/\alpha}dx=-\frac{1}{\Gamma(1-1/\alpha)}\int\limits_{0}^{\infty}f'(x+t)x^{-1/\alpha}dx.$$

On the one hand, using obvious reasonings, we have

$$\int\limits_{0}^{\infty}x^{-1/\alpha}dx\int\limits_{\gamma}e^{-\lambda^{\alpha}(t+x)}A(E-\lambda A)^{-1}hd\lambda=\int\limits_{\gamma}e^{-\lambda^{\alpha}t}A(E-\lambda A)^{-1}hd\lambda\int\limits_{0}^{\infty}x^{-1/\alpha}e^{-\lambda^{\alpha}x}dx=$$

$$=\Gamma(1-1/\alpha)\int\limits_{\gamma}\lambda^{1-\alpha}e^{-\lambda^{\alpha}t}A(E-\lambda A)^{-1}hd\lambda.$$

On the other hand, since the integrals

$$\int\limits_{\gamma}e^{-\lambda^{\alpha}t}A(E-\lambda A)^{-1}hd\lambda;\;\int\limits_{\gamma}e^{-\lambda^{\alpha}t}\lambda^{\alpha}A(E-\lambda A)^{-1}hd\lambda,$$

are uniformly convergent regarding t (to prove this fact it suffices to apply Lemma 5), then in accordance with the well-known theorem, we get

$$\frac{d}{dt}\int\limits_{\gamma}e^{-\lambda^{\alpha}t}A(E-\lambda A)^{-1}hd\lambda=\int\limits_{\gamma}\frac{d}{dt}e^{-\lambda^{\alpha}t}A(E-\lambda A)^{-1}hd\lambda=-\int\limits_{\gamma}e^{-\lambda^{\alpha}t}\lambda^{\alpha}A(E-\lambda A)^{-1}hd\lambda.$$

Thus, combining the above facts, we get

$$\mathfrak{D}^{1/\alpha}_{-}u=\frac{1}{2\pi i}\int\limits_{\gamma}e^{-\lambda^{\alpha}t}\lambda A(E-\lambda A)^{-1}hd\lambda.$$

Using the formula $\lambda^{n}A(E-\lambda A)^{-1}=(E-\lambda A)^{-1}-E$, we get

$$\mathfrak{D}^{1/\alpha}_{-}u=\frac{1}{2\pi i}\int\limits_{\gamma}e^{-\lambda^{n}t}(E-\lambda A)^{-1}h\,d\lambda-\frac{1}{2\pi i}\int\limits_{\gamma}e^{-\lambda^{n}t}h,\,d\lambda=I_1+I_2.$$

The second integral equals zero in accordance with the fact that the function under the integral is analytical inside the domain G. Now, if we consider the expression for u, we get

$$\mathfrak{D}_-^{1/\alpha} u = \tilde{W} u.$$

Let us show that the initial condition holds in the sense

$$u(t) \xrightarrow{\mathfrak{H}} h,\ t \to +0.$$

Consider a case when h is an arbitrary element of the Hilbert space $\mathfrak{H}$. It follows from Lemma 5 that for a fixed t the operator

$$S_t h = \frac{1}{2\pi i} \int\limits_{\gamma} e^{-\lambda^\alpha t} A(E - \lambda A)^{-1} h \, d\lambda = u(t),$$

is bounded. Let us show that

$$\|S_t\| \leq 1,\ t > 0.$$

Firstly, assume that $h \in \mathrm{D}(\tilde{W})$, then by virtue of Lemma 6, we get

$$u(t) \xrightarrow{\mathfrak{H}} h,\ t \to +0.$$

Thus we can claim the fact that $u(t)$ is continuous at the right-hand side of the point zero. Let us apply the operator $\mathfrak{D}_-^{1-1/\alpha}$ to the both sides of relation (17). Taking into account a relation

$$\mathfrak{D}_-^{1-1/\alpha} \mathfrak{D}_-^{1/\alpha} u = -\frac{du}{dt},$$

we get

$$\frac{du}{dt} + \mathfrak{D}_-^{1-1/\alpha} \tilde{W} u = 0.$$

Let us multiply the both sides of the latter relation on u, in the sense of the inner product, we get

$$\left(\frac{du}{dt}, u \right) + (\mathfrak{D}_-^{1-1/\alpha} \tilde{W} u, u) = 0.$$

Consider a real part of the latter relation, we have

$$\mathrm{Re} \left(\frac{du}{dt}, u \right) + \mathrm{Re} (\mathfrak{D}_-^{1-1/\alpha} \tilde{W} u, u) = \frac{1}{2} \left(\frac{du}{dt}, u \right) + \frac{1}{2} \left(u, \frac{du}{dt} \right) + \mathrm{Re} (\mathfrak{D}_-^{1-1/\alpha} \tilde{W} u, u) =$$

$$= \frac{1}{2} \frac{d}{dt} \|u(t)\|^2 = -\mathrm{Re} (\mathfrak{D}_-^{1-1/\alpha} \tilde{W} u, u) \leq 0.$$

Hence

$$\|u(\tau)\|^2 - \|u(0)\|^2 = \int\limits_0^\tau \frac{d}{dt} \|u(t)\|^2 dt \leq 0.$$

The last relation can be rewritten in the form

$$\|S_t h\| \leq \|h\|,\ h \in \mathrm{D}(\tilde{W}),$$

since $\mathrm{D}(\tilde{W})$ is a dense set in $\mathfrak{H}$ then we obviously have a desired result i.e. $\|S_t\| \leq 1$. Now consider the following reasonings assuming that

$$h_n \xrightarrow{\mathfrak{H}} h,\ n \to \infty,\ \{h_n\} \subset \mathrm{D}(\tilde{W}),\ h \in \mathfrak{H}.$$

We have

$$\|u(t)-h\| = \|S_t h-h\| = \|S_t h-S_t h_n+S_t h_n-h_n+h_n-h\| \leq \|S_t\|\cdot\|h-h_n\|+\|S_t h_n-h_n\|+\|h_n-h\|.$$

It is clear that if we chose n so that $\|h-h_n\| < \varepsilon/3$ and after that chose t so that $\|S_t h_n-h_n\| < \varepsilon$, then we obtain $\forall\varepsilon > 0,\ \exists\delta(\varepsilon) : \|u(t) - h\| < \varepsilon,\ t < \delta$. Thus the initial condition holds. The representation (19) follows immediately from Theorem 3.

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