GLOBAL STABILISATION OF DAMPED-DRIVEN CONSERVATION LAWS

JOINT WORK WITH A. DJURDJEVAC (BERLIN)

1. Introduction

$D \subset \mathbb{R}^d$ bounded domain, $\partial D \in C^\infty$

(1) $$\partial_t u - \nu \Delta u + \operatorname{div}_x A(u) = h(t,x), \quad x \in D$$

(2) $$u|_{\partial D} = 0.$$

$\nu > 0$, $A \in C^2(\mathbb{R}, \mathbb{R}^d)$, h external force

$$\|h\|_\gamma := \sup_{\substack{t \geq 0 \\ x \in D}} |h(t,x)| + \sup_{\substack{t_1, t_2 \geq 0 \\ x_1, x_2 \in D}} \frac{|h(t_1,x_1) - h(t_2,x_2)|}{|t_1-t_2|^{\gamma/2} + |x_1-x_2|^\gamma} < \infty, \quad \gamma \in (0,1)$$

(3) $$u(0,x) = u_0(x), \quad u_0 \in C^{2+\gamma}(D)$$

$\forall u_0\ \forall h$ satisfying the usual compatibility condition $\exists!$ solution for (1)-(3) in the space

$$X^\gamma = \{ u : \mathbb{R} \times D \to \mathbb{R} \ / \ u, \partial_t u, \nabla_x u, \nabla^2 u \text{ have } \|\cdot\|_\gamma \text{ norm finite} \}$$

(CP) $$u_0|_{\partial D} = 0, \quad -\nu \Delta u_0 + \operatorname{div}_x A(u_0) - h(0,x) = 0 \quad \text{on } \partial D$$

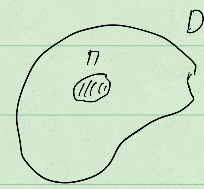
<u>Question</u> $\hat{u} \in X^\gamma$ solution of (1)-(3)

$\underline{u_0 \in C^{2+\gamma}(D)} \to u(t,x)$, $\|u(t) - \hat{u}(t)\| \not\to 0$

$u_0 \neq \hat{u}(0)$

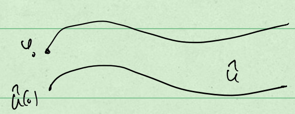
Is it possible to add a control to stabilise $\hat{u}$ as $t \to +\infty$.

(4) $$\partial_t u - \nu \Delta u + \operatorname{div}_x A(u) = h(t,x) + \eta(t,x)$$



(5) $$\|u(t) - \hat{u}(t)\| \leq C e^{-\alpha t}, \quad t \geq 0.$$

$\forall t \geq 0 \quad \operatorname{supp}_x \eta(t) \subset \Pi$

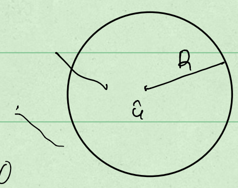


<u>Example</u> (Fursikov - Imanuvilov 95, Diaz 96)

$$\partial_t u + u \partial_x u - \nu \partial_x^2 u = \eta(t,x), \quad x \in (0,1), \quad \operatorname{supp}_x \eta \subset [a,b] \subset (0,1)$$

$\forall T_0 > 0 \quad \forall R > 0 \quad \exists \hat{u} \in L^2(0,1) \quad \forall u_0 \in L^2(0,1) \quad \forall \eta \in L^2_{loc}(\mathbb{R}, L^2) \quad \forall T \geq T_0$

$$\| u(T) - \hat{u} \| \geq R$$



$$\eta(t,x) = \underline{\xi(t)}\, \varphi(x), \quad \varphi \in C_0^\infty(\Omega), \quad \varphi \geq 0, \; \varphi \not\equiv 0$$

$\xi(t)$ control

2. Main result

$$\partial_t u - \nu \Delta u + \operatorname{div}_x A(u) = h(t,x) + \xi(t)\varphi(x)$$

(5) $\quad \exists\, C, c > 0 \quad \forall u \in \mathbb{R} \quad \left(\sum_{j=1}^{d} A_j'(u) \right) \operatorname{sgn}(u) \geq c|u| - C \quad \forall u \in \mathbb{R}$

Example $\quad d = 1, \quad A(u) = \frac{1}{2}u^2$ (Burgers equation)

Theorem $\quad h, \; \|h\|_\gamma < \infty, \; \gamma \in (0,1), \; \varphi \in C_0^\infty(\Omega), \; \varphi \geq 0, \; \varphi \not\equiv 0.$

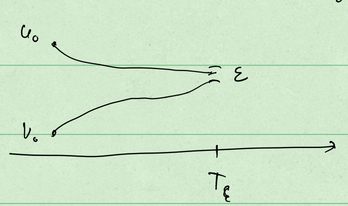
Then $\forall \hat{u} \in X^\gamma$ solution of (1), (2) $\forall u_0 \in C^{2+\gamma}(\Omega)$ satisfying (CP)

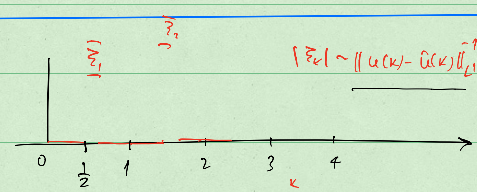
$\exists \xi : \mathbb{R}_+ \to \mathbb{R}$ piecewise constant s.t.

(6) $$\| u(t) - \hat{u}(t) \|_{C^{2+\sigma}} \leq C_\sigma e^{-\alpha_\sigma t}, \quad t \geq 0$$

where $\sigma \in (0, \gamma)$, and C_σ, α_σ are positive numbers depending only on $\|h\|_\gamma$, but not on $u_0, \hat{u}$.

OP1 $\quad$ Exact controllability





$|\xi_k| \sim \| u(k) - \hat{u}(k) \|_{L^1}^{-1}$

$\int_{k-1}^{k} \xi(t)\,dt \leq C \| u(k) - \hat{u}(k) \|_{L^1}$

$d = 1$: FI 95, Burgers equation

$d \geq 2 \quad |A'(u)| \leq C(1 + |u|) \quad$ Imanuvilov – Yamamoto (03)

OP2 $\quad$ Feedback control

- $\hat{u} = \hat{u}(x) \quad$ Raymond 07

- $h(t,x)$, η localized in the Fourier (Barbu-Rodrigues-AS 2011)

$$\eta(t,x) = K(u(t), \hat{u}(t))\,\varphi(x), \quad K \text{ continuous map}$$

3. Ideas of the proof

- Interpolation:

$$\|f\|_{C^{2+\sigma}} \le C \|f\|_{L^1}^{\theta} \|f\|_{C^{2+\gamma}}^{1-\theta}, \quad \theta = \frac{\gamma-\sigma}{d+\gamma+2}$$

decays exponentially — bounded

Besov-Il'in-Nikolski

Brezis-Mironescu

- L^1-contraction

$$\|u(t)-\hat{u}(t)\|_{L^1} \le \|u(0)-\hat{u}(0)\|_{L^1} \quad \forall t \ge 0 \quad (\text{if } \eta \equiv 0)$$

$$\|u(1)-\hat{u}(1)\|_{L^1} \le q\,\|u(0)-\hat{u}(0)\|_{L^1}, \quad q<1$$

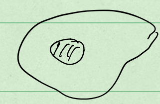
- Key observation $\quad u_0 \to v(t,x)$, $\eta \equiv 0$, $w = v - \hat{u}$

$$\partial_t w - \nu \Delta w + \operatorname{div}_x(a(t,x)w) = 0, \quad a(t,x) = \int_0^1 A'(\hat{u}+sw)\,ds$$

Proposition $\forall \Pi$ $\forall T>0$ $\forall \rho>0$ $\exists q<1$ $\exists \delta>0$ s.t. if $\|a\|_{C^1([0,1]\times\overline{D})} \le \rho$, then

either $\quad \|w(T)\|_{L^1} \le q\,\|w(0)\|_{L^1} \quad (7)$

or $\quad \inf_{x\in\Pi} |w(T,x)| \ge \delta\,\|w(0)\|_{L^1} \quad (8)$



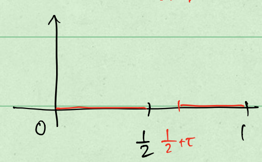
- $u_0, \hat{u}_0 \in B_{C^{2+\gamma}}(R)$ $\quad \overline{\Pi} \subset D$

$T = \frac{1}{2}$: $\quad \|u(\tfrac{1}{2}) - \hat{u}(\tfrac{1}{2})\|_{L^1} \le q\,\|u_0 - \hat{u}_0\|_{L^1} \qquad \zeta \equiv 0$

$$\|u(1) - \hat{u}(1)\|_{L^1} \le q\,\|u_0 - \hat{u}_0\|_{L^1}$$

In the opposite case, (8) holds.

$-\bar{\zeta}$



$$w(\tfrac{1}{2}, x) \ge \delta\|w(0)\|_{L^1} = \delta\|u(0)-\hat{u}(0)\|_{L^1}$$

$z = u - \hat{u}$

$$\partial_t z - \nu\Delta z + \operatorname{div}\big(A(\hat{u}+z) - A(\hat{u})\big) = \bar{\zeta}\,\varphi(x), \quad t \in [\tfrac{1}{2}, \tfrac{1}{2}+\tau]$$

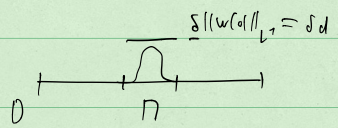
$$z(\tfrac{1}{2}) = \bar{z} = u(\tfrac{1}{2}) - \hat{u}(\tfrac{1}{2})$$

$$\frac{\partial_t Z = \bar{\xi}\varphi(x)}{Z(\tfrac{1}{2}) = \bar{z}}$$

<u>Lemma</u> $\quad \tau\bar{\xi} \le 1 \implies \|z(\tfrac{1}{2}+\tau) - Z(\tfrac{1}{2}+\tau)\|_{L^1} \le C_1\sqrt{\tau}$

$$\|u(\underbrace{\tfrac{1}{2}+\tau}_{T_\tau}) - \hat{u}(\tfrac{1}{2}+\tau)\|_{L^1} = \|z(T_\tau)\|_{L^1} \le \|z(T_\tau) - Z(T_\tau)\|_{L^1} + \underline{\|Z(T_\tau)\|_{L^1}}$$

$$\le \underline{C_1\sqrt{\tau}} + \|\bar{z} + \tau\bar{\xi}\varphi\|_{L^1}$$



$$\underline{\bar{\xi} = \pm(\tau M)^{-1}\delta d}$$

$$M = \|\varphi\|_{L^\infty}, \quad d = \|w(0)\|_{L^1}$$

$$\underline{\|\bar{z} + \tau\bar{\xi}\varphi\|}_{L^1} = \int_{\mathcal{D}} |\bar{z}(x) - \underbrace{M^{-1}\varphi}_{\le 1}\delta d|\,dx = \int_{\mathcal{D}\setminus\Pi} |\bar{z}| + \int_{\Pi} (\bar{z} - M^{-1}\varphi d\delta)\,dx$$

$$= \underbrace{\|\bar{z}\|_{L^1}}_{\le d} - M^{-1}\|\varphi\|_{L^1} d\delta = d\underbrace{(1 - M^{-1}\|\varphi\|_{L^1}\delta)}$$

$\partial_t Z = \bar{\xi}\varphi, \quad Z(\tfrac{1}{2}) = \bar{z}$

$Z(t) = \bar{z} + (t-\tfrac{1}{2})\bar{\xi}\varphi, \qquad t = \tfrac{1}{2}+\tau$