Holomorphic endomorphisms of $\mathbb{C}^n$ and countable subsets

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Main results

Let $n > 1$. We consider holomoprhic endomorphisms of $\mathbb{C}^n$.

Theorem

Let $\{p_j\}$ be a discrete sequence and $\{w_j\}$ be an arbitrary sequence in $\mathbb{C}^n$. There exists a holomorphic map $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\Phi(p_j) = w_j$ and $(J\Phi)(z) = 1$ for all $z \in \mathbb{C}^n$.

Theorem

If X and Y are countable dense subsets of $\mathbb{C}^n$, then there is $F \in \mathrm{Aut}(\mathbb{C}^n)$ so that $F(X) = Y$ and $JF \equiv 1$.

Theorem

Let $N = \{e_1, 2e_1, 3e_1, \dots\}$ be an arithmetic progression. For every permutation w_j of N there is $F \in \mathrm{Aut}(\mathbb{C}^n)$ such that $F(je_1) = w_j$.

Theorem

There exists countable $D \subset \mathbb{C}^n$ such that if any $F \in \mathrm{Aut}(\mathbb{C}^n)$ satisfies $D \subset F(D)$ then $F = id$.

Particularly, D is not conjugate to N.

Compact-open topology

We denote by $\mathrm{End}(\mathbb{C}^n)$ the vector space of holomorphic maps $\mathbb{C}^n \to \mathbb{C}^n$. The *compact-open* topology on $\mathrm{End}(\mathbb{C}^n)$ is given by the basis

$$V(K, U) = \{f \in \mathrm{End}\,\mathbb{C}^n \colon f(k) \subset U\},$$

where $K \subset \mathbb{C}^n$ is a compact and $U \subset \mathbb{C}^n$ is open. Alternatively, this topology is given by the series of norms

$$\|f\|_K = \sup_{z \in K} |f(z)|$$

for $f \in \mathrm{End}(\mathbb{C}^n)$ and compact $K \subset \mathbb{C}^n$.

Theorem

$\mathrm{End}(\mathbb{C}^n)$ *is complete in the compact-open topology. That is, the limit of holomorphic maps is always holomorphic.*

We denote by $\mathrm{Aut}(\mathbb{C}^n) \subset \mathrm{End}(\mathbb{C}^n)$ the subgroup of holomoprhic automorphisms. **It is not closed!** The sequence of automorphisms F_n converges to an automorphism if and only F_n and F_n^{-1} converge in $\mathrm{End}(\mathbb{C}^n)$.

Shears

Definition

Shear in the direction of e_j is a map $F \in \mathrm{Aut}(\mathbb{C}^n)$ given by

$$F(z_1, ..., z_n) = (z_1, ..., z_i + f, ..., z_n),$$

where $f = f(z_1, ...\widehat{z_i}, ..., z_n)$ is holomorphic.

Obviuously, shears are holomorphic automorphisms with $JF \equiv 1$.

Basic example

Let $f\colon \mathbb{C} \to \mathbb{C}$ be an entire function and $\Lambda\colon \mathbb{C}^n \to \mathbb{C}$ be a linear functional. For $u \in \ker \Lambda$ the map

$$\sigma(z) = z + f(\Lambda z)u$$

is a shear.

A lemma

Lemma

Let $\varepsilon > 0$ *and* $K \subset \mathbb{C}^n$ *is a convex compact. Let* $a_1, \ldots, a_n \in K$ *and* p, q *are points in the hyperplane* Π *which does not intersect* K.
Then there is a shear τ *such that* $\tau(p) = q$, $\tau(a_i) = a_i$ *for all* $1 \leq i \leq n$ *and*

$$|z - \tau(z)| < \varepsilon \text{ for all } z \in K.$$

Proof.

- Take linear $\Lambda\colon \mathbb{C}^n \to \mathbb{C}$ so that $\Lambda(p) = \Lambda(q)$ and $\Lambda(p) \notin \Lambda(K)$;
- There is a unit vector $u \in \ker \Lambda$ such that $q = p + cu$ for $c \in \mathbb{C}$;
- Take a polynomial $g\colon C \to C$ with

$$g(\Lambda p) = c, \;\; g(\Lambda a_i) = 0, \;\text{ and } |g(\Lambda z)| < \varepsilon \text{ for all } z \in K.$$

Define

$$\tau(z) = z + g(\Lambda z)u.$$

It has all the desired properties. □

Immersion-interpolation

Theorem

Let $\{p_j\}$ be a discrete sequence and $\{w_j\}$ be an arbitrary sequence in $\mathbb{C}^n$. There exists a holomorphic map $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ such that $\Phi(p_j) = w_j$ and $(J\Phi)(z) = 1$ for all $z \in \mathbb{C}^n$.

Proof.

We may assume that $0 < |p_1| < |p_2| < \dots$ and $w_j \notin \{z_1 = 0\}$. Let $E(z_1, \dots, z_n) = (e^{z_1}, z_2 e^{-z_1}, z_3, \cdots z_n)$. We will construct Φ as $\Phi = E \circ F$, where F is a limit of composition of shears.

Using Lemma, we construct F_k by induction such that

1. $F_k(p_i) = F_{k-1}(p_i)$ for $1 \leq p_i \leq k-1$;
2. $F_k(p_k) = v_k$ such that $E(v_k) = w_k$ and $v_k \notin F_{k-1}(\bar{B}_{|p_k|})$;
3. $|F_k(z) - F_{k-1}(z)| < 2^{-k}$ for $z \in \bar{B}_{|p_k|}$.

Due to 3, F_k converges to $F \in \mathrm{End}(\mathbb{C}^n)$ with $JF \equiv 1$ and $\Phi = E \circ F$ has all desired properties. □

Continuous infinitie transitivity lemma

Lemma

Suppose $E, K, D \subset \mathbb{C}^n$, *$E$ is finite, K is compact, and D is dense. If* $a \in \mathbb{C}^n \setminus E$ *and* $\varepsilon > 0$, *then there is a shear* σ *so that*

1. $\sigma(p) = p$ *for* $p \in E$;
2. $\sigma(a) \in D$, *and*
3. $|\sigma(z) - z| + |\sigma^{-1}(z) - z| < \epsilon$ *for* $z \in K$.

Proof.

Assume that $a = 0$. Take a unit vector u such that $\langle p, u \rangle \neq 0$ for $p \in E$. Take sequences of unit vectors $u_i \in \mathbb{C}^n$ and $w_i \in D$ such that $w_i \to 0$, $u_i \to u$, and $\langle u_i, w_i \rangle = 0$. Consider

$$g_i(\lambda) = \prod_{p \in E} \left(1 - \frac{\lambda}{\langle p, u_i \rangle} \right).$$

The shear $\sigma_i(z) = z + g_i(\langle z, u_i \rangle) w_i$ works for big enough i. □

Dense transitivity

Theorem

If X and Y are countable dense subsets of $\mathbb{C}^n$, then there is $F \in \mathrm{Aut}(\mathbb{C}^n)$ so that $F(X) = Y$ and $JF \equiv 1$.

Proof.

- Enumerate $X = \{x_i\}$ and $Y = \{y_i\}$. By induction, we have $F_j \in \mathrm{Aut}(\mathbb{C}^n)$, which maps $p_1, \ldots, p_{2j}$ to $q_1, \ldots, q_{2j}$. We are going to construct F_{j+1}.
- Let p_{2j+1} be the first x_i in $X \setminus \{p_i\}$. Find a shear σ which leaves $q_1, \ldots, q_{2j}$ and maps $F_j(p_{2j+1})$ to Y. (The shear should be close enough to idenity.)
- Let q_{2j+2} be the first y_i not in $\{q_1, ..., q_{2j+1}\}$. Construct a shear τ which leaves $q_1, \ldots, q_{2j+1}$ and maps q_{2j+2} to $\sigma(F_j(X))$.

Define F_{j+1} as $\tau^{-1} \circ \sigma \circ F_j$. Then F is a limit of F_j. □

Permutations of an arithmetic progression

Lemma

If $\{\alpha_1\}$ and $\{\beta_1\}$ are discrete sequences in $\mathbb{C}$, then there are three sheers in $\mathbb{C}^n$ whose composition τ satisfies

$$\tau(\alpha_i e_1) = \beta_i e_1.$$

Proof.

Enough to prove for $\mathbb{C}^2$. Find $f, g\colon \mathbb{C} \to \mathbb{C}$ with $f(\alpha_i) = \beta_i - \alpha_i$ and $g(\beta_i) = -\alpha_i$. The three shears are

$$\sigma_1(z) = z + z_1 e_2,\ \sigma_2(z) = z + f(z_2)e_1,\ \sigma_3(z) = z + g(z_1)e_2.$$

The composition is $\tau = \sigma_3 \circ \sigma_2 \circ \sigma_1$. □

Theorem (Corollary)

Let $N = \{e_1, 2e_1, 3e_1, \dots\}$ be an arithmetic progression. For every permutation w_j of N there is $F \in \mathrm{Aut}(\mathbb{C}^n)$ such that $F(je_1) = w_j$.

Tame sets

Let $N = \{e_1, 2e_1, 3e_1, \dots\}$ be an arithmetic progression.

Definition

A set $E \subset \mathbb{C}^n$ is called *tame* if $E = F(N)$ for some $F \in \mathrm{Aut}(\mathbb{C}^n)$. It is called *very tame* if F can be chosen so that $JF \equiv 1$.

Let $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^m$ and π' be a projection on $\mathbb{C}^k$.

Theorem

Suppose $E \subset \mathbb{C}^n$ is infinite, $\pi'(E)$ is discrete and π' has finite fibers on E. Then E is very tame.

Corollary

- *Every discrete infinite set $E \subset \mathbb{C}^{n-1}$ is very tame in $\mathbb{C}^n$.*
- *The union of a finite set and a (very) tame set is (very) tame.*
- *Every discrete infinite series $E \subset \mathbb{C}^n$ is the union of two very tame subsets.*

Proof.

Only the last assertion needs a proof. Take $E_1 = \{z' + z'' \in E, |z'| \geq |z''|\}$ and $E_2 = \{z' + z'' \in E, |z'| < |z''|\}$. □

More about tame sets

Theorem

Suppose that $E \subset \mathbb{C}^n$ is an infinite discrete sequence and all coordinates z_i of every $z = (z_1, \ldots, z_n) \in E$ satisfy $|z_i| \geq 1$. Then E is very tame.

Proof.

Let $P(z) = z_1 z_2 \ldots z_n$ and $\{\lambda_1, \lambda_2, \lambda_3, \ldots\}$ be a set $P(E)$. Put

$$E_t = \{z \in E \colon P(z) = \lambda_t\} \ (t = 1, 2, 3, \ldots).$$

Observe that $\{\lambda_t\}$ is discrete in $\mathbb{C}$, and each E_t is finite.
Choose a holomorphic $f \colon \mathbb{C} \to \mathbb{C}$ with $f(\lambda_t) = t$. Define $w = \Phi(z)$ by

$$w_1 = z_1 e^{f(P(z))}, \ w_2 = z_2 e^{-f(P(z))}, \ w_i = z_i.$$

Then $\Phi(E)$ satisfies the hypothesis of the projection theorem. □

(Un)avoidable subsets

Proposition

If $E \subset \mathbb{C}^n$ is tame then there is a biholomorphic map $F\colon \mathbb{C}^n \to \mathbb{C}^n$ so that $F(\mathbb{C}^n)$ does not intersect E. If E is very tame we can choose F with $JF \equiv 1$.

Proof.

Basic fact: there exists a biholomorphic $G\colon \mathbb{C}^n \to \mathbb{C}^n$ with $JG \equiv 1$ and $G(\mathbb{C}^n) \neq \mathbb{C}^n$. Since $\Omega = G(\mathbb{C}^n)$ is homeomorphic to $\mathbb{C}^n$, $\mathbb{C}^n \setminus \Omega$ is unbounded and we can find a very tame $E_0 \subset \mathbb{C}^n$. □

Theorem

There is a discrete set $D \subset \mathbb{C}^n$ which is unavoidable by nondegenerate holomorphic maps from $\mathbb{C}^n$ into $\mathbb{C}^n$.

Tame subsets of Stein manifolds

Definition (J. Winkelmann (2017))

Let X be a complex manifold. An infinite discrete subset D is called *weakly tame* if for every exhaustion function $\rho : X \to \mathbb{R}^+$ and every map $\zeta : D \to \mathbb{R}^+$ there exists an automorphism ϕ of X such that $\rho(\phi(x)) \geq \zeta(x)$ for all $x \in D$.

Definition (R. Andrist and R. Ugolini (2018))

Let X be a complex manifold. An infinite discrete subset D is called *strongly tame* if for every injective map $f : D \to D$ there exists an automorphism ϕ of X such that $\phi(x) = f(x)$ for all $x \in D$.

Strongly tame implies weakly tame. For $X \cong \mathbb{C}^n$ and $X \cong SL_n(\mathbb{C})$ both notions coincide. For $\mathbb{C}^n$ it is the same as defined by Rosay and Rudin.

Results of J.Winkelmann

1. Any two tame discrete subsets in $SL_n(\mathbb{C})$ are equivalent.
2. Certain discrete subgroups may be verified to be tame discrete subsets. In particular, $SL_2(\mathbb{Z}[i])$ is a tame discrete subset, and also every discrete subgroup of a one-dimensional Lie subgroup of $SL_n(\mathbb{C})$ and every discrete subgroup fo a maximal torus.
3. Every discrete subset of $SL_n(\mathbb{C})$ is the union of n tame discrete subsets.
4. Every semisimple complex Lie group admits a non-tame discrete subset.
5. Every injective self-map of a tame discrete subset of $SL_n(\mathbb{C})$ extends to a biholomorphic self map of $SL_n(\mathbb{C})$.
6. For every semisimple complex Lie group S there exists a "threshold sequence", i.e., there exists a sequence of numbers $R_k > 0$ and an exhaustion function τ such that every sequence g_k with $\tau(g_k) > R_k$ defines a tame discrete subset.

Counterexamples by J. Winkelmann

1. On $\mathbb{C}^n \setminus \{(0,\ldots,0)\}$ ($n \geq 2$) there do exist discrete subsets which may not be realized as a finite union of tame discrete subsets.
2. On $\Delta \times \mathbb{C}$ there are (weakly) tame discrete sets which are not strongly tame. There are permutations of tame discrete sets which do not extend to biholomorphic self-maps of the ambient manifold.
3. On $\Delta \times \mathbb{C}$ there exist inequivalent tame discrete subsets.
4. On $\mathbb{C}^n \setminus \{(0,\ldots,0)\}$ there is no "threshold sequence".