On the family of affine threefolds $x^m y = F(x, z, t)$.

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24.11.2021, 08.12.2021

This talk is based on the following paper by Neena Gupta :

On the family of affine threefolds $x^m y = F(x, z, t)$,
Compositio Math. **150** (2014) 979–998.
(NG)

Investigations on the threefold $x^m y = F(x, z, t)$

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Notation and Definition

Throughout my talk k is a field.
For integral domains $R \subset B$,
$B = R^{[n]}$ denotes: $B = R[t_1, \ldots, t_n]$ for elements $t_1, \ldots, t_n \in B$ algebraically independent over R.

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Def 1: A polynomial $F \in k[Z, T]$ is said to be non-trivial line if

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Def 3: An affine k-algebra A is said to be geometrically factorial if $A \otimes \bar{k}$ is a factorial (UFD) domain, where $\bar{k}$ is an algebraic closure of k.

Zariski Cancellation Problem

ZCP. Is $k^{[n]}$ $(= k[X_1, \ldots, X_n])$ cancellative?, i. e.,

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Three well-known results ([AEH72], 2.6, 2.8, 4.8)

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Theorem AEH3 : Let R be a UFD and D be an R-algebra such that $R \subset D \subset R[X_1, \ldots, X_n]$, $\operatorname{tr.deg}_R D = 1$ and D is factorially closed in $R[X_1, \ldots, X_n]$. Then $D = R^{[1]}$.

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Remark : R UFD, A an R-algebra. Then
$A^{[m]} \cong R^{[m+1]} \implies A = R^{[1]}$.

Zariski Cancellation Problem

n $=$ **2**: **YES** ch $k = 0$ (Fujita 1979, Miyanishi-Sugie 1980)

YES k perfect (Russell 1981)

YES ch $k \geq 0$, k any field (Bhatwadekar-Gupta (2015))

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n ≥ 3: **NO** ch $k > 0$ (Gupta 2014[NG])

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n $\geq$ **3**: **NO** ch $k > 0$ (Gupta 2014[NG])

n $\geq$ **3**: **OPEN** ch $k = 0$

Epimorphism Problem for Hyperplanes

Q 1. $\dfrac{\mathbf{k[X_1,\ldots,X_n]}}{\mathbf{(G)}} = \mathbf{k^{[n-1]}} \implies \mathbf{k[X_1,\ldots,X_n]} = \mathbf{k[G]^{[n-1]}}$?

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$n = 2$: **YES** ch $k = 0$ (Abhyankar-Moh; Suzuki 1975)

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Example : (Segre-Nagata)
Let ch $k = p > 0$, $s(\geqslant 2) \in \mathbb{N}$ where $p \nmid s$,

$$\mathbf{F} = \mathbf{Z}^{\mathbf{p}^2} + \mathbf{T} + \mathbf{T}^{\mathbf{sp}} \in \mathbf{k[Z, T]}$$

Then f is a non-trivial line in $k[Z, T]$, i.e.,

$$\frac{\mathbf{k[Z, T]}}{\mathbf{(F)}} \cong \mathbf{k}^{[1]} \text{ but } \mathbf{k[Z, T]} \neq \mathbf{k[F]}^{[1]}.$$

Segre-Nagata example

A corollary of the Automorphism Theorem of $k[Z, T]$ (Jung 1942, van der Kulk 1953) :

Corollary 2: For an element $F \in k[Z, T]$, TFAE

- $k[Z, T] = k[F]^{[1]}$
- For each pair of variables Z_i, T_i of $k[Z, T]$, either $deg_{Z_i} F \mid deg_{T_i} F$ or $deg_{T_i} F \mid deg_{Z_i} F$.

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Define a k-algebra homomorphism $\phi : k[Z, T] \to k[U]$ given by
$\phi(Z) = -U - U^{sp}$ and $\phi(T) = U^{p^2}$. Since
$\phi(Z + (Z^p - T^s)^s) = -U$, the map is onto and
$\operatorname{Ker} \phi = (Z^{p^2} + T + T^{sp})$. So $\dfrac{k[Z, T]}{(F)} = k^{[1]}$ but by Corollary 2, $k[Z, T] \neq k[F]^{[1]}$.

Epimorphism problem : $n = 3$

Some special cases :

- $\mathbf{G} = \mathbf{a}(\mathbf{X}, \mathbf{Z})\mathbf{Y} - \mathbf{b}(\mathbf{X}, \mathbf{Z})$

  YES ch $k = 0$ (Sathaye (1976)),

  YES ch $k \geq 0$ (Russell (1976))

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- $\mathbf{G} = \mathbf{a}(\mathbf{X}, \mathbf{Z})\mathbf{Y}^{\mathbf{n}} - \mathbf{b}(\mathbf{X}, \mathbf{Z})$, $n \geq 2$

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  YES any field k (Das, Dutta (2011)).

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  YES any field k (Das, Dutta (2011)).
- $\dfrac{\mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}]}{(\mathbf{G} - \lambda)} = \mathbf{k}^{[2]}$ for almost every $\lambda \in k$
  YES $k = \mathbb{C}$ (Kaliman (2002)),
  YES ch $k = 0$ (Daigle, Kaliman (2009)).

Epimorphism problem : $n = 4$

Some special cases :

$\mathbf{F} = \mathbf{a}(\mathbf{X}, \mathbf{Y})\mathbf{W} - \mathbf{b}(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ and $\mathbf{B} := \dfrac{\mathbb{C}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}]}{(\mathbf{F})} = \mathbb{C}^{[3]}$.

Then $\mathbb{C}[X, Y, Z, W] = \mathbb{C}[F]^{[3]}$ in the following cases (Kaliman, Vénéreau and Zaidenberg (2004))

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Then $\mathbb{C}[X, Y, Z, W] = \mathbb{C}[F]^{[3]}$ in the following cases (Kaliman, Vénéreau and Zaidenberg (2004))

- $a \in \mathbb{C}[X]$;
- $deg_Z \; b \leqslant 1$;
- b is of the form $b_0(X, Y) + b_2(X, Y, Z)Z^2$;
- $ht(a_1 B \cap \mathbb{C}[X, Y]) \leqslant 1$ for every irreducible factor a_1 of a;
- a is square-free. In fact, it is enough to assume that for every irreducible factor c of a such that $ht(cB \cap \mathbb{C}[X, Y]) = 1$, we have $c^2 \nmid a$.

Over any field k of arbitrary characteristic, one has the following special case of the previous result.

Theorem A1 : (N. Gupta (2014)[Theorem 3.11][NG])
Let k be any field and A an integral domain defined by

$$A = \frac{k[X, Y, Z, W]}{(X^m W - b(X, Y, Z))}, \text{ where } m > 1.$$

Set $f(Y, Z) := b(0, Y, Z)$ and $F := X^m W - b(X, Y, Z)$.
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Epimorphism problem : Gupta's Example

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- $f(Y, Z)$ is a variable in $k[Y, Z]$.
- $A = k[x]^{[2]}$, where x denotes the image of X in A.
- $A = k^{[3]}$.
- F is a variable in $k[X, Y, Z, W]$.
- F is a variable in $k[X, Y, Z, W]$ along with X.

Formulation of $\mathbb{G}_a$-action as ring homomorphism

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Remark : ϕ_U is always an A-algebra homomorphism. When $A = k[V]$ for an affine variety V over a field k, the exponential maps on A correspond to the algebraic actions of $(k, +)$ on $\mathrm{Spec}(A)$ (van den Essen (2000)[9.5][Es]).

Example of invariants of $\mathbb{G}_a$-actions

For an exponential map ϕ on A, the ring of ϕ-invariants is denoted by

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Example : For the exponential map ϕ on $k[X, Y]$ (ch. $k = p > 0$) defined by $\phi_U(X) = X$ and $\phi_U(Y) = Y + U + XU^p$, the ring of invariants $k[X, Y]^{\phi} = k[X]$.

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In general it is difficult to classify all the exponential map of a ring.

Locally finite iterative higher derivations

A a k-alg. For $\phi \in \mathrm{EXP}(A)$, $a \in A$ and $n \in \mathbb{N} \cup \{0\}$ define $D^n(a) :=$ coefficient of U^n in $\phi(a)$ and $D := \{D^0, D^1, \ldots\}$.
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- Since $\phi(a) \in A^{[1]}$, the sequence $\{D^i(a)\}$ has **finitely many non-zero elts.**
- $D^n : A \to A$ is a k-**linear map**.
- $D^n(ab) = \sum_{i+j=n} D^i(a)D^j(b)$, for all $a, b \in A$ (**Leibniz rule**).

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- $D^i D^j = \binom{i+j}{j} D^{i+j}$ (criteria (ii) for exp map).

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- $D^iD^j = \binom{i+j}{j} D^{i+j}$ (criteria (ii) for exp map).
- When ch. k=0, $D^1 \in \mathrm{LND}(A)$ and $\phi := exp(UD^1) = \sum_i \frac{1}{i!}(UD^1)^i \in \mathrm{EXP}(A)$ with $A^\phi = \mathrm{Ker}\ D^1$. (not possible in ch. p).

Some properties of exponential maps

We summarise some useful properties of exponential maps (c.f. Makar-Limanov, Crachiola (2005), Crachiola (2005)[Cr05], Gupta (2014)[NG])

Lemma 3 : Let A be an affine k-domain and $\phi \in EXP(A)$ be non-trivial. Then

(i) A^{ϕ} is factorially closed in A.

(ii) A^{ϕ} is alg. closed in A.

(iii) $\operatorname{tr.deg}_k(A^{\phi}) = \operatorname{tr.deg}_k(A) - 1$.

(iv) There exists $c \in A^{\phi}$, such that $A[\frac{1}{c}] = A^{\phi}[\frac{1}{c}]^{[1]}$.

(v) $\operatorname{tr.deg}_k(A) = 1 \Rightarrow A = \bar{k}^{[1]}$ and $A^{\phi} = \bar{k}$, where $\bar{k} =$ alg. closure of k in A.

(vi) For $S \subseteq A^{\phi} \setminus \{0\}$ mult. closed, ϕ extends to a non-trivial exp map $S^{-1}\phi$ on $S^{-1}A$ with ring of invariants $S^{-1}(A^{\phi})$.

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Corollary: Let A be an affine k-algebra s.t. $\mathrm{tr.deg}_k A > 1$. If $\mathrm{DK}(A) \subsetneq A$, then A is not a polynomial ring over k.

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$\mathbf{A = k[X, Y, Z, T]/(X^m Y - F(X, Z, T))}$, where $\mathbf{m > 1}$.
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Consequences

k: a field of ANY characteristic, $\mathbf{m > 1}$.

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- A is a **non-trivial** $\mathbb{A}^2$-fibration over $k[x]$ if and only if $F(0, Z, T)$ is a **non-trivial line**.

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Thus, over a field k of positive characteristic, there is an infinite family of non-isomorphic rings which are stably isomorphic to $k^{[3]}$.

Lemma I

Let k be a field and A an integral domain defined by

$$A = k[X, Y, Z, T]/(X^m Y - F(X, Z, T)), \quad \text{where} \quad m \geq 1,$$

$f(Z, T) := F(0, Z, T)$; x is the image of X in A. TFAE:

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(ii) x is prime in A or x is a unit in A.

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Proof: (i) $\Rightarrow$ (ii): ETS that either x is an irreducible element in A or x is a unit in A. Let z, t denote, respectively, the images of Z, T in A. Suppose that x is **not irreducible** in A. Then, since x is irreducible in $k[x, z, t]$, there exist $a, b \in A$ such that $x = ab$ and $a \notin k[x, z, t]$. Since $A \subseteq A[x^{-1}] = k[x, x^{-1}, z, t]$, we have $a = \alpha/x^i$ and $b = \beta/x^j$ for some $\alpha, \beta \in k[x, z, t]$ and some integers $i, j \geq 0$. Therefore, $x^{i+j+1} = \alpha\beta$ in $k[x, z, t]$.

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Since x is prime in $k[x, z, t]$, we have $\alpha = \lambda x^r$ and $\beta = \lambda^{-1} x^s$ for some $\lambda \in k^*$ and $r, s \geq 0$ satisfying $r + s = i + j + 1$. Thus $a = \lambda x^{r-i}$ and $= \lambda x^{s-j}$. Since $a \notin k[x, z, t]$, we have $r - i < 0$ and hence $x^{-1} \in A$.

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(ii) $\Rightarrow$ (i): $A[\frac{1}{x}] = k[x, \frac{1}{x}]^{[2]}$ is a UFD. Therefore, if x is prime in A then, by Nagata's well known criterion, A is a UFD. If x is a unit in A, then clearly $A = A[x^{-1}]$ is a UFD.

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(ii) $\iff$ (iii) holds since
$A/xA = k[Y, Z, T]/(f) = (k[Z, T]/(f))^{[1]}$.

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Let k, A, f and x be as in previous Lemma. TFAE:

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By **Corollary AEH2**, $A/xA = k^{[2]} \implies k[Z,T]/(f) \cong k^{[1]}$.

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Let k, A, f and x be as in previous Lemma. TFAE:

(i) A is an $\mathbb{A}^2$-fibration over $k[x]$.

(ii) $A/xA = k^{[2]}$.

(iii) $f(Z,T)$ is a line in $k[Z,T]$, i.e., $k[Z,T]/(f) \cong k^{[1]}$.

Proof (i) $\Rightarrow$ (ii) follows from the definition of $\mathbb{A}^2$-fibration.

(ii) $\Rightarrow$ (iii):

$$A/xA = k[Y,Z,T]/(f) = (k[Z,T]/(f))^{[1]}.$$

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Let P be a prime ideal of $k[x]$ other than $xk[x]$.
Since $x \notin P$, $Pk[x, x^{-1}]$ is a prime ideal of $k[x, \frac{1}{x}]$.
Since $A[\frac{1}{x}] = k[x, \frac{1}{x}]^{[2]}$, we have $A \otimes_{k[x]} k(P) = k(P)^{[2]}$. Thus A is an $\mathbb{A}^2$-fibration over $k[x]$.

Admissible proper Z-filtration: Definition

A **proper** $\mathbb{Z}$**-filtration** of an affine domain A over a field k is a collection of k-linear subspaces $\{A_n\}_{n \in \mathbb{Z}}$ satisfying:

(i) $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{Z}$,

(ii) $A = \bigcup_{n \in \mathbb{Z}} A_n$,

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A proper $\mathbb{Z}$-filtration of A is called **admissible** if $\exists$ finite generating set Γ of A such that, for any $n \in \mathbb{Z}$ and $f \in A_n$, f can be written as a finite sum of monomials in elements of Γ and each of these monomials is an element of A_n.

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Example: The filtration induced by a graded structure of a finitely generated k-algebra is admissible.

Homogeneous Exponential map

Let A be a graded ring. Then it induces a (usual) graded structure to the polynomial ring $A[U]$ with U as a a homogeneous element.

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Ex 1: Let $A = k[X, Y]$ be a polynomial ring with the usual grading. Then the exponential map $\phi_1 : A \to A[U]$ defined by

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Ex 2: The exponential map $\phi_2 : A \to A[U]$ defined by

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Any proper $\mathbb{Z}$-filtration on A determines the following $\mathbb{Z}$-graded integral domain

$$\mathrm{gr}(A) := \bigoplus_i A_i/A_{i-1}, \quad \text{and a map}$$

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Some remarks :

• ρ is **not a ring homomorphism**. If $i < n$ and $a_1, \ldots, a_l \in A_n \setminus A_{n-1}$ such that $\sum_j a_j \in A_i \setminus A_{i-1} (\subseteq A_{n-1})$, then $\rho(\sum_j a_j) = \sum_j a_j + A_{i-1} \neq 0$ but $\sum_j \rho(a_j) = \sum_j a_j + A_{n-1} = 0$.

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• Suppose A has **proper** $\mathbb{Z}$-filtration and a finite generating set Γ which makes it **admissible**. Then $\mathrm{gr}(A)$ is generated by $\rho(\Gamma)$. Since if $a_1, \ldots, a_l$ and $a_1 + \cdots + a_l \in A_n \setminus A_{n-1}$, then $\rho(\sum_j a_j) = \sum_j \rho(a_j)$ and $\rho(ab) = \rho(a)\rho(b)$ for all $a, b \in A$.

Admissible proper Z-filtration : A Theorem

Remarks (cont'd) :

• Suppose A has a $\mathbb{Z}$-graded algebra structure, say, $A = \bigoplus_{i\in\mathbb{Z}} C_i$. Then there exists a proper $\mathbb{Z}$-filtration $\{A_n\}_{n\in\mathbb{Z}}$ on A defined by $A_n := \bigoplus_{i\leqslant n} C_i$. Moreover, $\mathrm{gr}(A) = \bigoplus_{n\in\mathbb{Z}} A_n/A_{n-1} \cong \bigoplus_{n\in\mathbb{Z}} C_n = A$ and, for any $a \in A$, the image of $\rho(a)$ under the isomorphism $\mathrm{gr}(A) \to A$ is the homogeneous component of a of maximum degree.

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Theorem DHM: (Derksen, Hadas and Makar-Limanov (2001))
Let A be an affine domain over a field k with an admissible proper $\mathbb{Z}$-filtration and $\operatorname{gr}(A)$ the induced $\mathbb{Z}$-graded domain. Let ϕ be a non-trivial exponential map on A. Then ϕ induces a non-trivial homogeneous exponential map $\bar{\phi}$ on $\operatorname{gr}(A)$ such that $\rho(A^{\phi}) \subseteq \operatorname{gr}(A)^{\bar{\phi}}$.

Applications of Russell-Sathaye criteria

We state two applications of Russell-Sathaye criteria (Russell-Sathaye (1979) [Theorems 2.4.2, 2.3.1][RS79]

Theorem RS1 : (NG (2014)

Let k be a field and $F \in k[X, Y]$ be such that $k[X, Y] \otimes_{k[F]} k(F) = k(F)^{[1]}$. Then $k[X, Y] = k[F]^{[1]}$.

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Theorem RS2 : (Bhatwadekar-Dutta (1994))
Let $R \subset D$ be domains such that D is a finitely generated R-algebra. Suppose there exists a prime element $\pi \in R$ such that π remains prime in D, $D[\frac{1}{\pi}] = R[\frac{1}{\pi}]^{[1]}$, $\pi D \cap R = \pi R$ and $R/\pi R$ is algebraically closed in $D/\pi D$. Then $D = R^{[1]}$.

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As a consequence, we have (NG (2014))

Theorem RS3 : Let k be a field and $F \in k[Z, T]$ be such that $k[F]$ is algebraically closed in $k[Z, T]$. Suppose that $k[Y, Z, T] \otimes_{k[Y,F]} k(Y, F) = k(Y, F)^{[1]}$ for an indeterminate Y over $k[Z, T]$. Then $k[Z, T] = k[F]^{[1]}$.

Some technical results : Lemma III

Lemma III: Let k, A, f and x be as in Lemma I. Let $B := \dfrac{k[X,Y,Z,T]}{(X^mY - f(Z,T))}$. Then there exists a proper $\mathbb{Z}$-filtration $\{A_n\}_{n\in\mathbb{Z}}$ on A with $x \in A_{-1} \setminus A_{-2}$ and $z, t \in A_0 \setminus A_{-1}$ such that $\mathrm{gr}(A) \cong B$.

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Proof (sketch) : • $A \hookrightarrow k[x, \frac{1}{x}, z, t] = \bigoplus_{i\in\mathbb{Z}} F_i$, where $F_i = k[z,t]x^i$ and consider the proper $\mathbb{Z}$-filtration $\{A_n\}_{n\in\mathbb{Z}}$ on A given by $A_n := A \cap \bigoplus_{i \geqslant -n} F_i$.

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• If $\bar{g}$ denotes image of g in $\mathrm{gr}(A)$, then $\bar{g} = g_i(\bar{z},\bar{t})\bar{x}^i$, for some $i \geqslant 0$ if $g \in k[x,z,t]$ and $\bar{g} = g_{ij}(\bar{z},\bar{t})\bar{x}^i\bar{y}^j$, (with $j > 0$ and $0 \leqslant i < m$) otherwise. It follows the filtration is admissible with $\Gamma = \{x, y, z, t\}$.

A major technical step : Proposition IV

Proposition IV: Let

$$\mathbf{B} = \mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}]/(\mathbf{X}^{\mathbf{m}}\mathbf{Y} - \mathbf{f}(\mathbf{Z}, \mathbf{T})), \quad \text{where} \quad m \geq 1.$$

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Then there exists $w \in B^{\phi}$ such that $k[z, t] = k[w]^{[1]}$.

Proof of Proposition IV

Case I : $B^{\phi} \subseteq k[y,z,t]$

• $D := B^{\phi} \cap k[z,t]$. Then $D \subsetneq k[z,t]$ (since $y \in B^{\phi}$ and $\operatorname{tr.deg}_k B^{\phi} = 2$)

• D fact. closed in $k[z,t]$ (as B^{ϕ} fact. closed in B)

• B^{ϕ} graded subring of $k[y,z,t] = \bigoplus_{n\in\mathbb{Z}} k[z,t]y^n$ and $\operatorname{tr.deg}_k B^{\phi} = 2 \Rightarrow k \subsetneq D$. So $\operatorname{tr.deg}_k D = 1$.

• **Theorem AEH3** $\Rightarrow D = k[w] = k^{[1]}$, $w \in k[z,t]$.

• $B^{\phi} = k[y,w]$. Set $L = k(y,w)$, the field of fractions of B^{ϕ} and $S = B^{\phi}$. Then $B \otimes_S L = L^{[1]}$.

• Since $L \subseteq k[y,z,t] \otimes_S L \subseteq B \otimes_S L = L^{[1]}$ and $k[y,z,t] \otimes_S L$ is a normal domain, by **Theorem AEH1** $k[y,z,t] \otimes_S L = L^{[1]}$.

• By **Theorem RS3**, $k[z,t] = k[w]^{[1]}$ (since $D = k[w]$ is alg. closed in $k[z,t]$).

Proof of Proposition IV

Case II : $B^{\phi} \not\subseteq k[y,z,t]$

- $x \in B^{\phi}$ (B^{ϕ} graded subring and fact. closed).
- ϕ induces non-trivial exp. map ϕ_1 on $\tilde{B} := B \otimes_{k[x]} k(x) = k(X)[Y,Z,T]/(X^mY - f(Z,T)) = k(x)[z,t]$. and $\tilde{B}^{\phi_1} = B^{\phi} \otimes_{k[x]} k(x)$.
- By **Theorem AEH3**, $\tilde{B}^{\phi_1} = k(x)[w_1]$, $w_1 \in k[z,t]$ (since $\operatorname{tr.deg}_{k(x)} \tilde{B}^{\phi_1} = 1$ and $\tilde{B}^{\phi_1}$ is fact. closed in $\tilde{B}$).
- By **Theorem RS1**, $k(x)[z,t] = k(x)[w_1]^{[1]}$ (since $k(x)[z,t] \otimes_{k[x,w_1]} k(x,w_1) = k(x,w_1)^{[1]}$).
- Let $w_2 \in k[x,z,t]$ s.t. $k(x)[w_1] = k(x)[w_2]$. Then $w_2 \in \tilde{B}^{\phi_1} \cap k[x,z,t] \subseteq B^{\phi}$. Moreover, if $w_2 = \sum_i h_i(z,t)x^i$, then $h_i(z,t) \in B^{\phi}$ (B^{ϕ} graded subring and fact. closed).

Proof of Proposition IV

• Set $E := B^{\phi} \cap k[z,t]$. Now $\operatorname{tr.deg}_k B^{\phi} = 2$ and $x \in B^{\phi} \Rightarrow E \subsetneq k[z,t]$ and $h_i(z,t) \in E \Rightarrow k \subsetneq E$. So $\operatorname{tr.deg}_k E = 1$ and by **Theorem AEH3**, $E = k[w]$ for some $w \in k[z,t]$. •

$E = k[w] \subseteq B^{\phi} \subseteq \tilde{B}^{\phi_1} = k(x)[w_2]$ and $k(x)[w_2] = \tilde{B}^{\phi_1} = B^{\phi} \otimes_{k[x]} k(x) \subseteq k(x)[w]$ (since $E = k[w]$).
• So $k(x)[w] = k(x)[w_2] = k(x)[w_1]$. • Since $E = k[w]$ is alg.

closed in $k[z,t]$ and $k(x)[z,t] = k(x)[w_1]^{[1]} = k(x)[w]^{[1]}$, by **Theorem RS1**, $k[z,t] = k[w]^{[1]}$.

Lemma V

The following result was proved by Makar-Limanov (2001) for ch. $k = 0$. Modifying his arguments, Gupta (2014) has proved

Lemma V : Let k be a field, $p(Z) \in k[Z]$ be such that $deg_Z p(Z) > 1$ and

$$D := \frac{k[X, Y, Z]}{(X^m Y - p(Z))} \quad \text{where } m \geqslant 2.$$

Let x, y, z denote the images of X, Y and Z in D. Then there does not exist any exponential map ϕ on D such that $y \in D^{\phi}$.

Sketch of proof of Lemma V

• Consider the **proper** $\mathbb{Z}$ filtration $D_{n n\in\mathbb{Z}}$ on D, given by $D_n := D \cap \bigoplus_{i\leqslant n} C_i$, where $C_i = k[x, x^{-1}]z^i$ and $D \hookrightarrow k[x, x^{-1}, z] = \bigoplus_{i\in\mathbb{Z}} C_i$.

• This filtration on D is **admissible** with generating set $\Gamma = \{x, y, z\}$ and $E := \mathrm{gr}(D) \cong \dfrac{k[X, Y, Z]}{(X^m Y - \lambda Z^r)}$ where λZ^r is the leading term in $p(Z)$.

• Suppose $\exists$ non-trivial $\phi \in \mathrm{Exp}(D)$ such that $y \in D^{\phi}$. Then, by **Theorem DHM**, ϕ induces a non-trivial **homogeneous** exp. map $\bar{\phi}$ on E such that $k[\bar{y}] \subseteq E^{\bar{\phi}}$ (for $g \in D$, $\bar{g}$ is its image in E).

• So $\bar{\phi}$ induces a **non-trivial** exp. map on $E \otimes_{k[\bar{y}]} k(\bar{y})$. But it is **not normal** and has tr. deg. 1 over k – a contradiction!!

Lemma VI : Let k be a field and A the integral domain defined by

$$A := \frac{k[X, Y, Z, T]}{(X^m Y - F(X, Z, T))} \quad \text{where } m \geqslant 1.$$

Let x, y, z, t denote the images of X, Y, Z, T in A. Then $k[x, z, t] \subseteq \mathrm{DK}(A)$ and $\mathrm{ML}(A) \subseteq k[x]$.

Proof of Lemma VI :

Proof : Define two exp. maps ϕ_1 and ϕ_2 on A as follows :
$\phi_1(x) = x, \ \ \phi_1(z) = z, \ \ \phi_1(t) = t + x^m U,$ and

$$\phi_1(y) = \frac{F(x, z, t + x^m U)}{x^m} = y + U\alpha(x, z, t, U);$$

Proof of Lemma VI :

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$\phi_2(x) = x, \;\; \phi_2(t) = t, \;\; \phi_2(z) = z + x^m U,$ and

$$\phi_2(y) = \frac{F(x, z + x^m U, t)}{x^m} = y + U\beta(x, z, t, U).$$

• Note that, $\alpha, \beta \in k[x, z, t, U]$.
• $k[x, z]$ and $k[z, t]$ are algebraically closed subrings of A of transendence degree 2.
• So $A^{\phi_1} = k[x, z]$ and $A^{\phi_2} = k[z, t]$ and ϕ_1, ϕ_2 are non-trivial.
• $k[x, z, t] \subseteq \mathrm{DK}(A)$ and $k[x, z] \cap k[z, t] = k[x] \subseteq \mathrm{ML}(A)$.

Lemma VIIb

We shall use a lemma proved by Gupta in **[NG2]**.

Lemma VIIb : Let B be an affine domain over an infinite field k and $f \in B$ be such that $f - \lambda$ is a prime element in B for **infinitely** many $\lambda \in k$. Let ϕ be a non-trivial exponential map on B such that $f \in B^{\phi}$. Then there exist **infinitely** many $\beta \in k$ such that $f - \beta$ is prime in B and ϕ induces a non-trivial exponential map on $B/(f - \beta)$.

Proposition VII: Let

$$\mathbf{A} = \mathbf{k}[\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}]/(\mathbf{X}^{\mathbf{m}}\mathbf{Y} - \mathbf{F}(\mathbf{X}, \mathbf{Z}, \mathbf{T})), \quad \text{where} \quad m \geq 2.$$

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Moreover, $k[Z, T]/(f) = k^{[1]} \Rightarrow k[Z, T] = k[f]^{[1]}$.

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Proof: Choose ϕ an exponential map such that $A^{\phi} \not\subseteq k[x, z, t]$.

- Consider A as a subring of $k[x, x^{-1}][z, t]$ and define a filtration $A_n = A \cap \bigoplus_{i \geq -n} k[z, t]x^i$ on A. Then $\mathrm{gr}(A) \cong k[X, Y, Z, T]/(X^m Y - f(Z, T)) = B$.

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- By **Theorem DHM**, ϕ induces $\bar{\phi}$ on B satisfying $\rho(A^{\phi}) \subseteq \mathrm{gr}(A)^{\bar{\phi}}$.

Proof of Prop. VII

• Show that $\bar{y} \in B^{\bar{\phi}}$ (for $g \in A^{\phi} \setminus k[x, z, t]$, $\bar{g} = g_{ij}(\bar{z}, \bar{t})\bar{x}^i \bar{y}^j$ s.t. $0 \leqslant i < m$ and $j > 0$; $B^{\bar{\phi}}$ is fact. closed)

• By **Proposition IV**, $\exists\ \bar{z}_1 \in k[\bar{z}, \bar{t}]$ s.t. $k[\bar{z}, \bar{t}] = k[\bar{z}_1]^{[1]}$ and $\bar{z}_1 \in B^{\bar{\phi}}$. Then $k[Z, T] = k[Z_1, T_1]$ (where Z_1= preimage of $\bar{z}_1$ in $k[Z, T]$).

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- Let $h(Z_1, T_1) = f(Z, T) = \sum_{i=1}^{n} a_i(Z_1) {T_1}^i$ and $\tilde{k}$ = alg. closure of k. Then $\bar{\phi}$ induces a **non-trivial exponential map** $\tilde{\phi}$ on $\tilde{B} := B \otimes_k \tilde{k} = \tilde{k}[X, Y, Z_1, T_1]/((X^m Y - h(Z_1, T_1)) = \tilde{k}[\bar{x}, \bar{y}, \bar{z_1}, \bar{t_1}]$, such that $\tilde{k}[\bar{y}, \bar{z_1}] \subseteq \tilde{B}^{\tilde{\phi}}$.

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- There exists **infinitely** many $\beta \in \tilde{k}$ such that $\bar{z_1} - \beta$ is prime in $\tilde{B}$. By **Lemma VIIa**, we can choose $\beta \in \tilde{k}$ such that $\tilde{\phi}$ induces a **non-trivial exponential map** on $\tilde{B}/(\bar{z_1} - \beta)$ and $a_n(\beta) \neq 0$.

Proof of Prop. VII

- Thus there exists a non-trivial exponential map on $\dfrac{\tilde{B}}{(z_1 - \beta)} = \dfrac{\tilde{k}[X, Y, T_1]}{(X^m Y - (a_0(\beta) + a_1(\beta) T_1 + \cdots + a_n(\beta) {T_1}^n)}$ with image of $\bar{y}$ in its ring of invariants.

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- We have $f(Z, T) = a_0(Z_1) + a_1(Z_1)T_1$. If $a_1 = 0$, then f is a linear polynomial in Z_1, since $k[Z_1, T_1]/(f) = k[Z, T]/(f) = k^{[1]}$. So f is a **variable** in $k[Z, T]$.
- If $a_1 \neq 0$, a_0 and a_1 are coprime in $k[Z_1]$. So $A/xA \cong k[Z, \frac{1}{a_1(Z_1)}]^{[1]}$. But $(A/xA)^* = k^*$, since f is a line. So $a_1(Z_1) \in k^*$. Hence f is a **variable**, being monic in T_1.

Non-triviality of $x^m y = F(x, z, t)$ for $m > 1$

Theorem B : Let k be any field of characteristic $p > 0$ and $f(Z, T) \in k[Z, T]$ be such that

$$k[Z, T]/(f) = k^{[1]} \text{ but } k[Z, T] \neq k[f]^{[1]}.$$

Let

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Proof : Follows from **Proposition VII** as $\mathrm{DK}(A) \neq A$.

A is stably polynomial

With the usual notations, if $f(Z, T)$ is a line in $k[Z, T]$, then **by Lemma II**, A is an $\mathbb{A}^2$-fibration over $k[x]$. It follows from a result of Asanuma (1987) [Proposition 2.5][Asa87] that $A^{[l]} = k[x]^{[l+2]}$ for some $l \geqslant 0$.

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Gupta has improved Asanuma's results by showing [Theorem 4.2][NG]
Theorem C : Let k be any field and

$$A = \frac{k[X,Y,Z,T]}{(X^mY - F(X,Z,T))} \quad \text{where} \quad m \geqslant 1.$$

Let $f(Z,T) = F(0,Z,T)$ be such that $k[Z,T]/(f) = k^{[1]}$.
Then

$$A^{[1]} \cong_{k[x]} k[x]^{[3]} \cong_k k^{[4]}.$$

ZCP for $n = 3$ and ch. $k > 0$

As a consequence of **Theorems B** and **C**, we have

Corollary : Zariski's cancellation conjecture does not hold for any threefold A defined by $A = \frac{k[X, Y, Z, T]}{(X^m Y - F(X, Z, T))}$, where $m > 1$ and $f(Z, T) = F(0, Z, T)$ is a non-trivial line in $k[Z, T]$.

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Remark : The *hypotheses* of the above Corollary are fulfilled only when ch. $k > 0$. By the famous result of Abhyankar-Moh, Suzuki (1975), there are **no non-trivial** lines when ch. $k = 0$. As mentioned earlier, when ch. $k = p > 0$, we do have non-trivial lines (e.g. the **Segre-Nagata** lines $f(Z, T) = Z^{p^e} + T + T^{sp}$ where $p^e \nmid sp$ and $sp \nmid p^e$.

A recapitulation of the results

Theorem RS2 : Let $R \subset D$ be domains such that D is a f.g. R-algebra. Suppose there exists a prime element $\pi \in R$ s.t. π remains prime in D, $D[\frac{1}{\pi}] = R[\frac{1}{\pi}]^{[1]}$, $\pi D \cap R = \pi R$ and $R/\pi R$ is alg. clsd. in $D/\pi D$. Then $D = R^{[1]}$.

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- $(m > 1)$ $k[x,z,t] \subseteq \mathrm{DK}(A)$ **(Lemma VI)**.

Some results from K-theory

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Theorem KT1 : Let R be a regular ring and U an indeterminate. Then
(i) The inclusion map $R \hookrightarrow R[U]$ induces an isomorphism from $K_i(R)$ to $K_i(R[U])$ for each $i \geqslant 0$.
(ii) For each $i \geqslant 1$, the sequence

$$0 \longrightarrow K_i(R[U]) \longrightarrow K_i(R[U, U^{-1}]) \longrightarrow K_{i-1}(R) \longrightarrow 0$$

is a split short exact sequence, where the map $K_i(R[U]) \longrightarrow K_i(R[U, U^{-1}])$ is induced by the inclusion $R[U] \hookrightarrow R[U, U^{-1}]$.

Some results from K-theory

We shall also need the following long exact sequence [Sri08, Proposition 5.15, 5.6 (pg 52), 5.16 (pg 61)].

Theorem KT2 : Let R be a regular ring and $x \in R$ be a non-zero-divisor such that $R/(x)$ is a regular ring. Let $j : R \to R[x^{-1}]$ is the inclusion map. Then we have the following long exact sequence of K-groups

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Moreover, if $\phi : R \to S$ is a flat ring homomorpshism such that S and $S/(u)$ are regular ($\phi(x) = u$), then we have the following natural commutative diagram.

$$\begin{array}{ccccccc} \to K_i(R/(x)) & \to & K_i(R) & \to & K_i(R[x^{-1}]) & \xrightarrow{\delta} & K_{i-1}(R/(x)) \to \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \to K_i(S/(u)) & \to & K_i(S) & \to & K_i(S[u^{-1}]) & \xrightarrow{\delta} & K_{i-1}(S/(u)) \to \end{array}$$

Some results from K-theory

We shall also need an elementary result.

Lemma KT3 : Let $\phi : R \to B$ be an injective ring homomorphism. Then the map $\phi_* : K_1(R) \to K_1(B)$, induced by ϕ, maps the subgroup R^* of $K_1(R)$ injectively into the subgroup B^* of $K_1(B)$.

Main Theorem : Theorem A

Let k be a field and $A = \dfrac{k[X, Y, Z, T]}{(X^m Y - F(X, Z, T))}$ where $m > 1$.
Let x, y, z, t denote images of X, Y, Z, T in A. Set $G := X^m Y - F(X, Z, T)$ and $f(Z, T) = F(0, Z, T)$. Then TFAE :

(i) $k[X, Y, Z, T] = k[X, G]^{[2]}$
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(ix) $k[Z, T] = k[f]^{[1]}$, i.e., f is a **variable**

(x) $k[Z, T]/(f) = k^{[1]}$ and $\mathrm{DK}(A) = A$.

Proof of Theorem A

Easy implications : $(i) \Rightarrow (ii) \Rightarrow (iv) \Rightarrow (v)$ and $(i) \Rightarrow (iii) \Rightarrow (vi)$.

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$(v) \Rightarrow (vii)$: By **Theorem KT1** and since $K_1(k) = k^*$ for any field k.

$(vii) \Rightarrow (viii)$: Since $\mathrm{DK}(A) = A$, by **Prop. VII**, we can assume $f(Z, T) = a_0(Z) + a_1(Z)T$. Let us consider two cases

Case : $a_1(Z) = 0$

- If $\bar{k}$ is an alg. closure of k, then $A \otimes_k \bar{k}$ is a UFD.
- By **Lemma I**, $a_0(Z)$ is either irreducible in $\bar{k}[Z, T]$ or $a_0(Z) \in \bar{k}^*$.
- If $a_0(Z) \in \bar{k}^*$, then $A = k[x, x^{-1}, z, t]$ and hence $K_1(A) \neq k^*$ (a contradiction!). So $a_0(Z)$ is irreducible and hence linear in Z. Thus f is a **variable** in $\bar{k}[Z, T]$. Now $A/(x) = (k[Z, T]/(f))^{[1]} = k^{[2]}$. So $(A/(x))^* = k^*$.

Proof of Theorem A

Case : $a_1(Z) \neq 0$

• By **Lemma I**, f is irreducible in $k[Z, T]$ (being linear in T). So $(a_0(Z), a_1(Z))_{k[Z]} = 1$. Hence $A/(x) = k[Z, T, Y]/(a_0(Z) + a_1(Z)T) = k[Z, \frac{1}{a_1(Z)}][Y]$. Also $A[x^{-1}] = k[x, x^{-1}]^{[2]}$. Since both $A/(x)$ and $A[x^{-1}]$ are regular, so is A.

• By **Theorem KT2**, we have an exact sequence :

$$\to K_2(A[x^{-1}] \xrightarrow{\delta} K_1(A/(x)) \to K_1(A) \xrightarrow{j_*} K_1(A[x^{-1}]) \to,$$

where

j_* is induced by $j : A \hookrightarrow A[x^{-1}]$ and δ is the connecting homomorphism.

• If $\eta : k \hookrightarrow A$, then, by **Lemma KT3**, $j_* \circ \eta_*$ maps k^* injectively into $(A[x^{-1}])^*$. Since $K_1(A) = \eta_*(K_1(k)) \cong k^*$, j_* maps $K_1(A)$ injectively into $K_1(A[x^{-1}])$.

Proof of Theorem A

So we have the following exact sequence :

$\to K_2(A[x^{-1}] \xrightarrow{\delta} K_1(A/(x)) \to 0$

• Since $A[x^{-1}] = k[x, x^{-1}]^{[2]}$, by **Theorem KT1**, $K_2(k[x, x^{-1}]) \cong K_2(A[x^{-1}])$.

• Again, by **Theorem KT1**, we have the following split short exact sequence : $0 \to K_2(k[x]) \to K_2(k[x, x^{-1}]) \to K_1(k) \to 0$

• Since A is a torsion-free module over $k[x]$ and hence free, by **Theorem KT2**, we have the following commutative diagram

$$
\begin{array}{ccc}
K_2(k[x, x^{-1}] & \xrightarrow{\delta} & K_1(k) \to 0 \\
\cong \downarrow & & \phi_* \downarrow \\
K_2(A[x^{-1}] & \xrightarrow{\delta} & K_1(A/(x)) \to 0
\end{array}
$$

ϕ_* is induced by the inclusion $\phi : k \hookrightarrow A/(x)$.

Proof of Theorem A

• From prev. diag., ϕ_* is surjective. Again, by **Lemma KT3**, ϕ_* maps k^* injectively into $(A/(x))^* \leqslant K_1(A/(x))$. Hence $(A/(x))^* = k^* = K_1(A/(x))$.

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(viii) $\Rightarrow$ *(ix)* : As before, we may assume $f(Z,T) = a_0(Z) + a_1(Z)T$.

• If $a_1(Z) = 0$, then $A/(x) = k[Y,Z,T]/(a_0(Z))$. As $(A/(x))^* = k^*$, we have $a_0(Z) \notin k$. Then, as before, $a_0(Z)$ is irreducible in $\bar{k}[Z,T]$ and hence linear in Z. So $f = a_0(Z)$ is a **variable** in $k[Z,T]$.

• If $a_1(Z) \neq 0$, then, as before, $A/(x) = k[Z, \frac{1}{a_1(Z)}][Y]$. Since $(A/(x))^*$, we have $a_1(Z) \in k^*$. Thus f is a **variable** in $k[Z,T]$.

Proof of Theorem A

$(ix) \Rightarrow (i)$: WLOG, assume $f(Z,T) = Z$. Set $D := k[X,Y,Z,T]$ and $R = k[X,G,T]$. Then $D[X^{-1}] = R[X^{-1}][Z]$ and $D/XD = (R/XR)^{[1]}$. By **Theorem RS2**, $D = R^{[1]}$, i.e., $k[X,Y,Z,T] = k[X,G]^{[2]}$.

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$(x) \Rightarrow (ix)$: By **Proposition VII**.

Several implications of Theorem A go through when $m = 1$.
However, when $m = 1$,
"A geo. fact. over k, $\mathrm{DK}(A) = A$ and $K_1(A) = k^*$" (vii)
$\not\Rightarrow$ $k[Z, T] = k[f]^{[1]}$ (ix); and
"A geo. fact. over k, $\mathrm{DK}(A) = A$ and $(A/(x))^* = k^*$" (vii)
$\not\Rightarrow$ $k[Z, T] = k[f]^{[1]}$ (ix).

Remark (NG) : Let $A = k[X, Y, Z, T]/(XY - F(X, Z, T))$.
Then $\mathrm{DK}(A) = A$ and $\mathrm{ML}(A) = k$.

A modified version of Theorem A

Theorem A2 : Let k be a field and

$$A = \frac{k[X, Y, Z, T]}{(X^m Y - F(X, Z, T))} \quad \text{where} \quad m > 1.$$

Let x, y, z, t denote images of X, Y, Z, T in A. Set $G := X^m Y - F(X, Z, T)$ and $f(Z, T) = F(0, Z, T)$. Further assume that f is a line in $k[Z, T]$, i.e., $k[Z, T]/(f) = k^{[1]}$. Then TFAE

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Proof of Theorem A2

The following implications are obvious :

$$(i) \Rightarrow (ii) \Rightarrow (iv) \quad \text{and} \quad (i) \Rightarrow (iii) \Rightarrow (v).$$

$(iv) \Rightarrow (vi)$: As $A = k^{[3]}$, $\mathrm{DK}(A) = A$. Now apply **Proposition VII**.

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Theorem C

Theorem C : Let k be any field and

$$A = \frac{k[X, Y, Z, T]}{(X^m Y - F(X, Z, T))} \quad \text{where} \quad m \geqslant 1.$$

Let $f(Z, T) = F(0, Z, T)$ be such that $k[Z, T]/(f) = k^{[1]}$.
Then

$$A^{[1]} \cong_{k[x]} k[x]^{[3]} \cong_k k^{[4]}.$$

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A lemma we shall need :

Lemma C2 : Let k be a field and D an affine k-domain. Let $F(X) \in D[X]$ and $f := F(0)$. Suppose $D/(f) = k^{[1]}$. Then $D[X]/(X^m, F) = (k[X]/(X^m))^{[1]}$ for every $m \geqslant 1$.

Theorem C

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We shall apply **Lemma C2** with $D = k[Z, T]$ and $F = F(X, Z, T)$.

Proof of Theorem C

$y :=$ image of Y, $A = k[X, Z, T, y]$ and
$U :=$ indeterminate over $k[X]$. We have the following diagram
:

$$\begin{array}{c} K[X,U] \\ \psi \big\downarrow \\ \end{array}$$

$$\Phi : k[X, Z, T] \longrightarrow k[X, U]/(X^m), \text{ where}$$

Ψ is the **canonical surjection**, Φ is the surjection obtained by **Lemma C2** with $Ker\ \Phi = (X^m, F)$ and $\Phi(X) = \Psi(X)$.

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Let $h \in k[X, Z, T]$ and $P, Q \in k[X, U]$ such that
$\Phi(h) = \Psi(U)$, $\Phi(Z) = \Psi(P(X, U))$ and $\Phi(T) = \Psi(Q(X, U))$.
Let W be an indeterminate over A.

Proof of Theorem C

Set $W_1 := X^m W + h(X, Z, T)$,

$$Z_1 = \frac{Z - P(X, W_1)}{X^m} \quad \text{and} \quad T_1 := \frac{T - Q(X, W_1)}{X^m}.$$

It will be shown that $A[W] = k[X, Z_1, T_1, W_1](:= B)$.

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- $y = \frac{F(X,Z,T)}{X^m} = \frac{F(X,P(X,W_1)+X^m Z_1,Q(X,W_1)+X^m T_1)}{X^m}$
  $= \frac{F(X,P(X,W_1),Q(X,W_1))}{X^m} + \alpha(X, Z_1, T_1, W_1)$.

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- $y = \frac{F(X,Z,T)}{X^m} = \frac{F(X, P(X,W_1)+X^m Z_1, Q(X,W_1)+X^m T_1)}{X^m}$
  $= \frac{F(X,P(X,W_1),Q(X,W_1))}{X^m} + \alpha(X, Z_1, T_1, W_1)$.
- $W = \frac{W_1 - h(X,Z,T)}{X^m} = \frac{W_1 - h(X,P(X,W_1)+X^m Z_1, Q(X,W_1)+X^m T_1)}{X^m}$
  $= \frac{W_1 - h(X,P(X,W_1),Q(X,W_1))}{X^m} + \beta(X, Z_1, T_1, W_1)$.

Here $\alpha, \beta \in B$.

Proof of Theorem C

- Since $\Psi(F(X, P(X, U), Q(X, U))) = \Phi(F(X, Z, T)) = 0$, we have $F(X, P(X, W_1), Q(X, W_1)) \in X^m k[X, W_1] \subseteq X^m B$. Thus $y \in B$.

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- Since $\Psi(h(X,P(X,U),Q(X,U))) = \Phi(h(X,Z,T)) = \Psi(U)$, $h(X,P(X,W_1),Q(X,W_1)) - W_1 \in X^m k[X,W_1] \subseteq X^m B$. Thus $W \in B$.

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- Since $\Psi(F(X,P(X,U),Q(X,U))) = \Phi(F(X,Z,T)) = 0$, we have $F(X,P(X,W_1),Q(X,W_1)) \in X^m k[X,W_1] \subseteq X^m B$. Thus $y \in B$.

- Since $\Psi(h(X,P(X,U),Q(X,U))) = \Phi(h(X,Z,T)) = \Psi(U)$, $h(X,P(X,W_1),Q(X,W_1)) - W_1 \in X^m k[X,W_1] \subseteq X^m B$. Thus $W \in B$.

- $Z = X^m Z_1 + P \in B$ and $T = X^m T_1 + Q \in B$. Hence $A[W] \subseteq B$.

Proof of Theorem C

ETS : $B \subseteq A[W]$.

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- $Z_1 = \frac{Z - P(X, W_1)}{X^m} = \frac{Z - P(X, X^m W + h(X,Z,T))}{X^m}$
  $= \frac{Z - P(X, h(X,Z,T))}{X^m} + \gamma(X, Z, T, W)$, where $\gamma \in A[W]$.

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  $= \frac{Z-P(X,h(X,Z,T))}{X^m} + \gamma(X, Z, T, W)$, where $\gamma \in A[W]$.
- $T_1 = \frac{T-Q(X,W_1)}{X^m} = \frac{T-Q(X,X^m W+h(X,Z,T))}{X^m}$
  $= \frac{T-Q(X,h(X,Z,T))}{X^m} + \delta(X, Z, T, W)$, where $\delta \in A[W]$.

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- Since $\Phi(Z-P(X,h)) = 0 = \Phi(T-Q(X,h))$, we have

$Z - P(X,h) = aX^m + bF$ and $T - Q(X,h) = cX^m + dF$, where $a,b,c,d \in k[X,Z,T]$.

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- Since $y = \dfrac{F(X,Z,T)}{X^m}$, we have $W_1, Z_1, T_1 \in A[W]$.

Thus $B \subseteq A[W]$. Since $B = k[X]^{[3]}$, **we are done!**.

Lemma VIII : DK(A) and $\mathrm{ML}(A)$ revisited

Recall that in **Lemma VI** it was shown that when $m \geqslant 1$, $\mathrm{ML(A)} \subseteq k[x]$ and $k[x, z, t] \subseteq \mathrm{DK}(A)$. In fact, we have

Lemma VIII : Let f be a **non-trivial** line and $m \geqslant 2$. Then

$$\mathrm{DK}(A) = k[x, z, t] \text{ and } \mathrm{ML}(A) = k[x].$$

Lemma VIII : DK(A) and $\mathrm{ML}(A)$ revisited

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Proof : By **Proposition VII**, $\mathrm{DK}(A) \neq A$. Since $k[x, z, t] \subseteq \mathrm{DK}(A)$, we must have equality.

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• Let $\phi \in \mathrm{Exp}(A)$ be non-trivial. Since $\mathrm{tr.deg}_k A^\phi = 2$, there exist two alg. ind. elts $\alpha, \beta \in \mathrm{DK}(A) = k[x,z,t]$. Let

$$\alpha = x\alpha_1(x,z,t) + \alpha_0(z,t) \text{ and } \beta = x\beta_1(x,z,t) + \beta_0(z,t)$$

for suitable $\alpha_0, \alpha_1, \beta_0, \beta_1 \in k[x,z,t]$.

Proof of Lemma VIII

- Suppose, if possible, $\alpha_0(z,t)$ and $\beta_0(z,t)$ are alg. ind. over k. Consider the proper admissible $\mathbb{Z}$-filtration on A, as in **Lemma III** and the induced graded ring $B = \text{gr}(A)$.
- By **Theorem DHM**, ϕ induces a **non-trivial homogeneous** exp. map $\bar{\phi}$ on B such that $k[\bar{\alpha_0}, \bar{\beta_0}] \subseteq B^{\bar{\phi}}$.
- Since $\bar{z}$ and $\bar{t}$ are alg. ind. over k in B, $\bar{\alpha_0}$ and $\bar{\beta_0}$ are also alg. ind. over k in B.
- As $B^{\bar{\phi}}$ is alg. closed in B, $k[\bar{z}, \bar{t}] \subseteq B^{\bar{\phi}}$. Since $\bar{x}^m\bar{y} = f(\bar{z}, \bar{t}) \in B^{\bar{\phi}}$, we have $\bar{x}, \bar{y} \in B^{\bar{\phi}}$ as it is fact. closed. But $\bar{\phi}$ is non-trivial—a contradiction!

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•. So α_0 and β_0 are alg. dependent. So there exists $H \in k^{[2]}$ such that $H(\alpha_0, \beta_0) = 0$. Then $H(\alpha, \beta) \in xA$ and hence, by factorial closedness of A^{ϕ}, $x \in A^{\phi}$. Hence $\mathrm{ML}(A) = k[x]$.

Few Remarks

• If $A = \dfrac{k[X, Y, Z, T]}{(X^m Y - F(X, Z, T))}$, with $m > 1$ and $F(0, Z, T)$ a **line** in $k[Z, T]$, then it follows from **Theorem A2** and **Lemma VIII** that either $\mathrm{DK}(A) = A$ (resp. $\mathrm{ML}(A) = k$) or $\mathrm{DK}(A) = k[x, z, t]$ (resp. $\mathrm{ML}(A) = k[x]$), according as $A = k[x]^{[2]}$ or $A \neq k[x]^{[2]}$.

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- However, if $m = 1$, the DK and ML invariants of $A = k[X, Y, Z, T]/(XY - F(X, Z, T))$ are always trivial. To observe this, one should apply **Lemma VI** and interchange the roles of x and y.

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- However, if $m = 1$, the DK and ML invariants of $A = k[X,Y,Z,T]/(XY - F(X,Z,T))$ are always trivial. To observe this, one should apply **Lemma VI** and interchange the roles of x and y.

- So the question whether $A \cong k^{[3]}$ remains **OPEN** for $m = 1$, when $F(0,Z,T)$ is a line in $k[Z,T]$.

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