Gale duality and homogeneous toric varieties

Toric varieties

Let Σ be a fan in a lattice N and let $X = X_\Sigma$ be the corresponding toric variety over algebraically closed field $\mathbb{K}$ of characteristic zero. Denote $M = \mathrm{Hom}_\mathbb{Z}(N, \mathbb{Z})$, $N_\mathbb{Q} = N \otimes_\mathbb{Z} \mathbb{Q}$, $M_\mathbb{Q} = M \otimes_\mathbb{Z} \mathbb{Q}$.

The primitive vectors of rays of Σ are called the *primitive vectors of* Σ. Recall that a cone σ in N is called *regular* if the primitive vectors of σ can be supplemented to a basis of N. A fan Σ is called *regular* if every cone $\sigma \in \Sigma$ is regular.

$$X \text{ is smooth} \iff \Sigma \text{ is regular}$$

A toric variety X is called *degenerate* if X is equivariantly isomorphic to the product of a nontrivial torus T_0 and a toric variety X_0 of smaller dimension. Note that X is homogeneous if and only if X_0 is homogeneous.

$$X \text{ is non-degenerate} \iff \text{the primitive vectors of } \Sigma \text{ span } N_\mathbb{Q}$$

Demazure roots

Denote by n_ρ the primitive vector of a ray $\rho \in \Sigma(1)$. Let $\langle \cdot, \cdot \rangle : N \times M \to \mathbb{Z}$ be the pairing of dual lattices $\langle v, u \rangle = u(v)$. For $\rho \in \Sigma(1)$ consider the set $\mathfrak{R}_\rho$ of all vectors $e \in M$ such that

1. $\langle n_\rho, e \rangle = -1$ and $\langle n_{\rho'}, e \rangle \geq 0$ for all $\rho' \in \Sigma(1) \setminus \{\rho\}$;
2. if $\sigma \in \Sigma$ and $\langle v, e \rangle = 0$ for all $v \in \sigma$, then the cone generated by σ and ρ is in Σ as well.

The elements of the set $\mathfrak{R} = \bigsqcup_{\rho \in \Sigma(1)} \mathfrak{R}_\rho$ are called the *Demazure roots* of the fan Σ.

Elements of $\mathfrak{R} \leftrightarrow \mathbb{G}_a$-actions on X normalized by the acting torus T

For $e \in \mathfrak{R}$ let H_e be the corresponding $\mathbb{G}_a$-subgroup of $\mathrm{Aut}(X)$ and R_e be the one-parameter subgroup of T corresponding to the primitive vector of the distinguished ray of e.

Strongly regular fans I

Recall the orbit-cone correspondence

$$\text{Cones } \sigma \in \Sigma \leftrightarrow T\text{-orbits } \mathcal{O}_\sigma \text{ on } X$$

and

$$\sigma_1 \subseteq \sigma_2 \iff \mathcal{O}_{\sigma_2} \subseteq \overline{\mathcal{O}_{\sigma_1}}, \ \dim \mathcal{O}_\sigma = \dim X - \dim \langle \sigma \rangle_{\mathbb{Q}}$$

Proposition 1

For every point $x \in X \setminus X^{H_e}$ the orbit $H_e \cdot x$ meets exactly two T-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ on X with $\dim \mathcal{O}_1 = \dim \mathcal{O}_2 + 1$. The intersection $\mathcal{O}_2 \cap (H_e \cdot x)$ consists of a single point, while $\mathcal{O}_1 \cap (H_e \cdot x)$ is an R_e-orbit.

A pair of T-orbits $(\mathcal{O}_1, \mathcal{O}_2)$ is said to be *H_e-connected* if $H_e \cdot x \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ for some $x \in X \setminus X^{H_e}$ (it implies that $\mathcal{O}_2 \subseteq \overline{\mathcal{O}_1}$ and $\dim \mathcal{O}_1 = \dim \mathcal{O}_2 + 1$).

Strongly regular fans II

We say that a cone $\sigma_2 \in \Sigma$ is *connected* with its facet σ_1 by a root $e \in \mathfrak{R}$ if $e\,|_{\sigma_2} \leq 0$ and σ_1 is given by the equation $\langle \cdot, e \rangle = 0$ in σ_2.

$$\text{A pair } (\mathcal{O}_{\sigma_1}, \mathcal{O}_{\sigma_2}) \text{ is } H_e\text{-connected } \iff$$

$$\iff \sigma_2 \text{ is connected with } \sigma_1 \text{ by the root } e$$

A fan Σ is called *strongly regular* if every nonzero cone $\sigma \in \Sigma$ is connected with some of its facets by a root.

Let $S(X) \subseteq \mathrm{Aut}(X)$ be the subgroup generated by root subgroups $H_e, e \in \mathfrak{R}$. A toric variety X is said to be *S-homogeneous* if $S(X)$ acts on X transitively.

Proposition 2

A non-degenerate toric variety X_Σ is S-homogeneous if and only if Σ is strongly regular.

Denote by $G(X)$ the subgroup of $\mathrm{Aut}(X)$ generated by $S(X)$ and T.

Lemma 1

The group $G(X)$ acts on X transitively if and only if Σ is strongly regular.

WLOG we may assume that X is non-degenerate. If Σ is strongly regular, then we can send every point $x \in X$ to an orbit of higher dimension with H_e. After we reach the open orbit, we use T.
Conversely, assume that Σ is not strongly regular. Let $\sigma \in \Sigma$ be a nonzero cone which is not connected with any facet by a root. Since $H_e \cdot \mathcal{O}_\sigma \subset \overline{\mathcal{O}_\sigma}$ and $T \cdot \overline{\mathcal{O}_\sigma} = \overline{\mathcal{O}_\sigma}$, we obtain that $\overline{\mathcal{O}_\sigma}$ is $G(X)$-invariant. Lemma 1 is proved.

Proof of Proposition 2 II

It remains to show that the group $S(X)$ acts on X transitively if X is non-degenerate and Σ is strongly regular. Each ray $\rho \in \Sigma(1)$ is connected with its facet $\{0\}$ by some root e_ρ (and ρ is the distinguished ray of e_ρ). So, the intersection of the open T-orbit and an H_{e_ρ}-orbit is an R_{e_ρ}-orbit. Recall that R_{e_ρ} is the one-parameter subgroup given by the vector $n_\rho \in N$.

Since X is non-degenerate, the collection $\{n_\rho,\ \rho \in \Sigma(1)\}$ has full rank in N. It implies that there is an $S(X)$-orbit which contains the open T-orbit. Thus, this $S(X)$-orbit is T-invariant and by Lemma 1 it coincides with X. Proposition 2 is proved.

Corollary 1

Every strongly regular fan is regular.

Examples

1. If X_Σ is a non-degenerate smooth affine toric variety, then Σ consists of a regular cone σ and all its faces. Therefore, $X_\Sigma = \mathbb{A}^n$. It easy to see that $\mathbb{A}^n$ is S-homogeneous. So, a regular cone together with all of its faces is a strongly regular fan.
2. If X_Σ is a complete toric variety, then X_Σ is homogeneous if and only if X_Σ is S-homogeneous. The only complete homogeneous toric varieties are the products of projective spaces. Therefore, every complete strongly regular fan is the product of fans of projective spaces.

Admissible collections

Let P be an abelian group and let $\mathcal{A} = (a_1, \ldots, a_r)$ be a collection of elements (possibly with repetitions) of P. The collection $\mathcal{A}$ is called *admissible* if $\mathcal{A}$ generates P and for any $a_i \in \mathcal{A}$ the element a_i is contained in the semigroup generated by $\mathcal{A} \setminus \{a_i\}$. A pair $(P, \mathcal{A})$ is said to be *equivalent* to a pair $(P', \mathcal{A}')$ if there is an isomorphism of abelian groups $\gamma : P \to P'$ such that $\gamma(\mathcal{A}) = \gamma(\mathcal{A}')$ (element-wise).

An S-homogeneous toric variety X_Σ is said to be *maximal* if it does not admit a proper open toric embedding $X_\Sigma \hookrightarrow X_{\Sigma'}$ into an S-homogeneous toric variety $X_{\Sigma'}$ with $\mathrm{codim}_{X_{\Sigma'}}(X_{\Sigma'} \setminus X_\Sigma) \geq 2$. This corresponds to a *maximal* strongly regular fan Σ i.e., Σ cannot be realized as a proper subfan of a strongly regular fan Σ' with $\Sigma'(1) = \Sigma(1)$.

Theorem 1

There is a one-to-one correspondence between maximal S-homogeneous toric varieties and equivalence classes of pairs $(P, \mathcal{A})$, where P is an abelian group and $\mathcal{A}$ is an admissible collection of elements of P.

Linear Gale duality I

By a *vector configuration* in a vector space V we mean a finite collection $v_1, \ldots, v_r \in V$ (possibly with repetitions) that spans V. A vector configuration $\mathcal{V} = (v_1, \ldots, v_r)$ in a rational vector space V and a vector configuration $\mathcal{W} = (w_1, \ldots, w_r)$ in a rational vector space W are *Gale dual* to each other (or $\mathcal{W}$ is the *Gale transform* of $\mathcal{V}$) if for any tuple $(a_1, \ldots, a_r) \in \mathbb{Q}^r$ one has

$$a_1 w_1 + \cdots + a_r w_r = 0 \iff l(v_i) = a_i \text{ for } i = 1, \ldots, r \text{ with some } l \in V^*.$$

Linear Gale duality II

Given a vector configuration $\mathcal{V} = (v_1, \ldots, v_r)$ in a space V one can produce its Gale dual as follows. Consider a surjective linear map $\alpha : \mathbb{Q}^r \to V$ given on the standard basis $e_1, \ldots, e_r$ of $\mathbb{Q}^r$ by $\alpha(e_i) = v_i, \; i = 1, \ldots, r$. Consider two dual short exact sequences

$$0 \longrightarrow \mathrm{Ker}(\alpha) \longrightarrow \mathbb{Q}^r \xrightarrow{\alpha} V \longrightarrow 0$$

$$0 \longleftarrow (\mathrm{Ker}(\alpha))^* \xleftarrow{\beta} (\mathbb{Q}^r)^* \longleftarrow V^* \longleftarrow 0$$

Let $e_1^*, \ldots, e_n^*$ be the dual basis in $(\mathbb{Q}^r)^*$. Setting $W = (\mathrm{Ker}(\alpha))^*$ and $w_i = \beta(e_i^*)$ for $i = 1, \ldots, r$ we obtain the Gale dual configuration $\mathcal{W} = (w_1, \ldots, w_r)$.

Lattice Gale transform

A *vector configuration* $\mathcal{N}$ in a lattice N is a finite collection of vectors $n_1, \ldots, n_r \in N$ that spans the vector space $N_{\mathbb{Q}}$. Consider the lattice $\mathbb{Z}^r$ with the standard basis $e_1, \ldots, e_r$ and the exact sequence

$$0 \longrightarrow L \longrightarrow \mathbb{Z}^r \xrightarrow{\alpha} N$$

defined by $\alpha(e_i) = n_i,\ i = 1, \ldots, r$. We identify the dual lattice $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^r, \mathbb{Z})$ with $\mathbb{Z}^r$ using the dual basis $e_1^*, \ldots, e_r^*$. Let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. The homomorphism $M \to \mathbb{Z}^r$ dual to α gives rise to the short exact sequence

$$0 \longleftarrow P \xleftarrow{\beta} \mathbb{Z}^r \longleftarrow M \longleftarrow 0$$

Let $a_i = \beta(e_i^*)$ for $i = 1, \ldots, r$. We call the collection $\mathcal{A} = (a_1, \ldots, a_r)$ the *lattice Gale transform* of the configuration $\mathcal{N}$. Conversely, given elements $a_1, \ldots, a_r$ that generate a group P, we can reconstruct lattices M and $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, the dual homomorphism $\mathbb{Z}^r \to N$, and the vectors $n_1, \ldots, n_r$.

Example

Let $\mathcal{N} = (n_1, n_2)$ in $N = \mathbb{Z}^2$ with $n_1 = (1,0),\ n_2 = (1,2)$. Then $\alpha : \mathbb{Z}^2 \to N$ is given by matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

and its dual $M \to \mathbb{Z}^2$ is given by matrix

$$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

Therefore, $P = \mathbb{Z}/2\mathbb{Z}$ and $\mathcal{A} = (a_1, a_2)$, where $a_1 = a_2 = \bar{1}$. At the same time, the linear Gale transform would be $(0,0)$ in the space $\{0\}$.

Proof of Theorem 1 I

A vector configuration $\mathcal{N} = (n_1, \dots, n_r)$ in a lattice N is called *suitable* if for any $i = 1, \dots, r$ there is a vector $e_i \in M$ such that $\langle n_i, e_i \rangle = -1$ and $\langle n_j, e_i \rangle \geq 0$ for all $j \neq i$.

Lemma 2

A vector configuration $\mathcal{N} = (n_1, \dots, n_r)$ in a lattice N is suitable if and only if its lattice Gale transform $\mathcal{A}$ in P is an admissible collection.

An element $a_i \in \mathcal{A}$ is contained in the semigroup generated by $\mathcal{A} \setminus \{a_i\}$ if and only if $a_i = \sum_{j \neq i} \alpha_j a_j$ for some non-negative integers α_j. The latter condition is equivalent to existence of an element $e_i \in M$ such that

$$\langle n_i, e_i \rangle = -1, \ \langle n_j, e_i \rangle = \alpha_j.$$

Proof of Theorem 1 II

A collection of rays $\rho_1, \ldots, \rho_r$ in the space $N_{\mathbb{Q}}$ is called *suitable* if the set of primitive vectors of these rays is a suitable vector configuration in N.

Lemma 3

The collection of rays $\Sigma(1)$ of a strongly regular non-degenerate fan Σ is suitable.

By definition, every ray ρ_i is connected with its facet $\{0\}$ by a root e_i.

Proof of Theorem 1 III

Proposition 3

For every suitable collection of rays $\rho_1, \dots, \rho_r$ in $N_\mathbb{Q}$ there is a unique maximal strongly regular fan Σ with $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$. Moreover, every strongly regular fan $\hat{\Sigma}$ with $\hat{\Sigma}(1) = \Sigma(1)$ is a subfan of Σ.

Let Ω be the set of strictly convex polyhedral cones σ in $N_\mathbb{Q}$ with $\sigma(1) \subseteq \{\rho_1, \dots, \rho_r\}$. With every $\sigma \in \Omega$ we associate a subset $I \subseteq \{1, \dots, r\}$ such that $\sigma(1) = \{\rho_i, i \in I\}$.
Let $\mathcal{A} = (a_1, \dots, a_r)$ be the lattice Gale transform of the vector configuration $\mathcal{N} = \{n_1, \dots, n_r\}$. Denote by $\Gamma(\sigma)$ the semigroup in P generated by $a_j, j \notin I$. In particular, $\Gamma(\{0\}) = A$, where A is the semigroup generated by $\mathcal{A}$.
Let

$$\Sigma = \Sigma(P, \mathcal{A}) = \{\sigma \in \Omega \mid \Gamma(\sigma) = A\}.$$

Note that the group P can be interpreted as the divisor class group $\mathrm{Cl}(X)$. Indeed, for a toric variety we have the exact sequence

$$0 \longleftarrow \mathrm{Cl}(X) \longleftarrow \mathbb{Z}^r \longleftarrow M \longleftarrow 0$$

and the inclusion $M \hookrightarrow \mathbb{Z}^r$ is also dual to the map α. Moreover, the admissible collection $\mathcal{A}$ is the set of classes of T-invariant prime divisors $[D_1], \ldots, [D_r]$ on X, corresponding to rays $\rho_1, \ldots, \rho_r$ of Σ.

Recall the example with $n_1 = (1, 0)$, $n_2 = (1, 2)$ and $P = \mathbb{Z}/2\mathbb{Z}$, $a_1 = a_2 = \bar{1}$. Then

$$\Sigma(P, \mathcal{A}) = \{\mathrm{Cone}((1, 0)), \mathrm{Cone}((1, 2)), \{0\}\},$$

so $X = Y_\sigma^{\mathrm{reg}}$, where $\sigma = \mathrm{Cone}((1, 0), (1, 2))$ (Y_σ is the surface $z^2 = xy$).

Homogeneous toric varieties I

Theorem 2

Let X be a non-degenerate homogeneous toric variety. Then there exists an open toric embedding $X \hookrightarrow X'$ into a maximal S-homogeneous toric variety X' with $\operatorname{codim}_{X'}(X' \setminus X) \geq 2$.

The variety X', of course, is the toric variety corresponding to the fan $\Sigma(P, \mathcal{A})$, where the admissible collection $\mathcal{A}$ is obtained from the set of rays $\Sigma(1)$ of the fan Σ, corresponding to the variety X.

Homogeneous toric varieties II

From the explicit description of maximal strongly regular fans it can be shown that every maximal non-degenerate S-homogeneous toric variety is quasiprojective.

Corollary 2

Every homogeneous toric variety is quasiprojective.

Conjecture 1

Every non-degenerate homogeneous toric variety is S-homogeneous.

Non-maximal S-homogeneous toric varieties I

Let P be an abelian group, $\mathcal{A} = (a_1, \dots, a_r)$ an admissible collection of elements of P, and A the semigroup in P generated by $\mathcal{A}$.
A *link* is a pair $(a, \mathcal{A}')$, where $\mathcal{A}'$ is a subcollection of $\mathcal{A}$, $a \in \mathcal{A} \setminus \mathcal{A}'$, and there exists an expression $a = \sum_j \alpha_j a_j$, where a_j runs through $\mathcal{A}'$ and $\alpha_j \in \mathbb{Z}_{>0}$.
We say that a subcollection $\mathcal{B} \subseteq \mathcal{A}$ is *generating*, if the elements of $\mathcal{B}$ generate the semigroup A. Let $\mathbb{G}$ be a set of generating collections in $\mathcal{A}$.
A link $(a, \mathcal{A}')$ is called a $\mathbb{G}$-*link* if for any $\mathcal{B} \in \mathbb{G}$ the condition $\mathcal{A}' \cup \{a\} \subseteq \mathcal{B}$ implies $\mathcal{B} \setminus \{a\} \in \mathbb{G}$.
A set $\mathbb{G}$ of generating collections in $\mathcal{A}$ is called *connected* if the following conditions hold

1. $\mathcal{A} \setminus \{a_i\} \in \mathbb{G}$ for all $i = 1, \dots, r$;
2. $\mathcal{B} \in \mathbb{G}$ and $\mathcal{B} \subseteq \mathcal{B}' \subseteq \mathcal{A}$ implies $\mathcal{B}' \in \mathbb{G}$;
3. if $\mathcal{B} \in \mathbb{G}$ and $\mathcal{B} \neq \mathcal{A}$ then there is a $\mathbb{G}$-link $(a, \mathcal{A}')$ with $\mathcal{A}' \subseteq \mathcal{B}$ and $a \notin \mathcal{B}$.

Non-maximal S-homogeneous toric varieties II

Let $\{\rho_1, \dots, \rho_r\}$ be a suitable collection of rays in $N_{\mathbb{Q}}$ and $\mathcal{N} = \{n_1, \dots, n_r\}$ the corresponding suitable vector configuration in N. Consider the lattice Gale transform $(P, \mathcal{A})$ of $(N, \mathcal{N})$.

Proposition 4

Strongly regular fans Σ with $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ are in bijection with connected sets $\mathbb{G}$ of generating collections in $\mathcal{A}$.

We can associate a cone $\sigma(\mathcal{B}) = \mathrm{Cone}(\rho_j \mid a_j \notin \mathcal{B})$ with any subcollection $\mathcal{B} \subseteq \mathcal{A}$. As we know, the maximal strongly regular fan $\Sigma(P, \mathcal{A})$ is the set of cones associated with every generating collection in $\mathcal{A}$.
Let $\Sigma^{\mathbb{G}}$ be the set of cones $\sigma(\mathcal{B}), \mathcal{B} \in \mathbb{G}$. The conditions from the definition of a connected set of generating collections are equivalent to the fact that $\Sigma^{\mathbb{G}}$ is a strongly regular fan.