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Two models for wave turbulence and the method of quasisolutions

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A. Introduction: mathematical models for water turbulence

Below I always assume periodic boundary conditions.

The stochastic Navier–Stokes equation model for turbulence.

$$\text{(NSE)}\qquad \begin{aligned} &\partial_t u(t,x) - \nu\Delta u(t,x) + (u\cdot\nabla)u + \nabla p = \partial_t \eta^\omega(t,x),\\ &x \in \mathbb{T}^d_L = \mathbb{R}^d/L\mathbb{Z}^d,\ L\geq 1;\quad \text{div}\, u = 0,\quad 0<\nu\ll 1. \end{aligned}$$

Usually $d=3$. Here η^ω is a Wiener process in the space of functions, smooth as a function of x. I write it as Fourier series

$$\text{(*)}\qquad \eta^\omega(t,x) = L^{-d/2}\sum_{s\in L^{-1}\mathbb{Z}^d} b(s)\beta^\omega_s(t)e^{2\pi i s\cdot x}.$$

• The pre-factor $L^{-d/2}$ is convenient to write Fourier series on $\mathbb{T}^d_L = \mathbb{R}^d/L\mathbb{Z}^d$ with, possibly, large L. So in this lecture I write Fourier series for a solution $u(t,x)$ as

$$L^{-d/2}\sum_{s\in L^{-1}\mathbb{Z}^d} \hat{u}_s(t)e^{2\pi i s\cdot x}.$$

• For a function $u(x)$ on $\mathbb{T}^d_L$, I denote by $\|u(t)\|$ its normalised L_2-norm:

$$\|u(t)\|^2 = L^{-d} \int_{\mathbb{T}^d_L} |u(t,x)|^2 dx.$$

Note that $\|u\|^2$ is the double energy per volume of the flow u. For eq. (NSE) to be a model for turbulence its solutions u should satisfy

$$\|u(t,\cdot)\|^2 \sim 1, \quad \text{Re}\, u \gg 1.$$

• For 3d, due to the heuristic 0-th law of turbulence, the solutions $u(t,x)$ of (NSE) satisfy $\mathbf{E}\|u(t,\cdot)\|^2 \sim 1$ as $\nu \to 0$ – this is how it should be. Note that the 0-th law is not proved rigorously (especially for $L \gg 1$). Due to this law, for $L \sim 1$,

(*) $$Re \sim \nu^{-1}.$$

• For other equations, to achieve the normalisation $\|u(t,\cdot)\|^2 \sim 1$ we may need to put a factor ν^a in front of the random force. E.g. for 2d NSE and $L \sim 1$, $a = 1/2$.

• Due to (*), the task is to study solutions $u(t,x)$ for $\nu \ll 1$. In particular, to examine the energy spectrum $\mathbf{E}|\hat{u}_s|^2$ and recover the Kolmogorov–Obukhov law, since if this problem starts to move, then many others also will.

- For $d = 1$ (with the Burgers equation instead of the Navier-Stokes) and $L \sim 1$ the task, discussed above, is done (in some good sense).

Boritchev, Kuksin “One-dimensional turbulence and the stochastic Burgers equation”, AMS 2021.

- It is reasonable to study also the hypeviscosity models, replacing in (NSE) $-\Delta$ by $(-\Delta)^r$ with some $r \geq 5/4$:

$$\partial_t u(t,x) + \nu(-\Delta)^r u(t,x) + (u \cdot \nabla)u + \nabla p = \partial_t \eta^\omega(t,x).$$

Then the 3d equation is well posed.

Some people believe that the Euler equation makes another model for 3d water turbulence:

The Euler equation model:

(Euler)
$$\partial_t u(t,x) + (u\cdot\nabla)u + \nabla p = 0, \quad \text{div}\, u = 0, \quad x \in \mathbb{T}^3_L$$
$$u(0,x) = u_0^\omega(x) \ \text{ is a random field.}$$

I think that (Euler) MAY be a model for water turbulence for short times, or if the initial data u_0^ω is distributed as AN invariant measure for the equation.

Eq. (Euler) is a 2-homogeneous Hamiltonian PDE, and (NSE) is a damped/driven 2-homogeneous Hamiltonian PDE. Small solutions of (Euler) may be scaled to solutions of size 1. Accordingly, there are no quasi-linear regimes in (Euler) and no interesting quasi-linear regimes in (NSE). So there is no easier "quasi–linear" problem, related with water turbulence, and modelled by (Euler) or by (NSE).

In order to find turbulence-like problems which allow quasi-linear regimes, let us replace (Euler) with some quasi-linear Hamiltonian PDE, and (NSE) – with its damped-driven version. We should look at solutions $u(t,x)$ with $\|u(t)\| \sim 1$, and examine for them analogies of the bullets • above.

Let us take for this pair of equations the NLS and the damped-driven NLS.

B. NLS equation (as a quasi-linear Hamiltonian PDE):

$$\partial_t u(t,x) + i\Delta u(t,x) - i\lambda(|u|^2 - 2\|u\|^2)u = 0,$$

(NLS) $$x \in \mathbb{T}^d_L = \mathbb{R}^d/L\mathbb{Z}^d, \quad d \geq 2, \qquad \|u\|^2 = L^{-d}\int_{\mathbb{T}^d_L} |u(x)|^2 dx,$$

$$u(0,x) = u_0^\omega(x) - \text{a random field.}$$

As before, I write $u(t,x) = L^{-d/2}\sum_{s \in L^{-1}\mathbb{Z}^d} \hat{u}_s(t) e^{2\pi i s\cdot x}$.

Wave turbulence (WT), starting from 1960's, studies solutions of (NLS) and of similar quasi-linear Hamiltonian PDEs under the WT limit:

(WT Limit) $L \to \infty$ while λ is properly scaled in terms of L.

In particular, WT studies the *energy spectrum of a solution* u, which is the mapping $s \mapsto \mathbf{E}|\hat{u}_s(t)|^2$. This study is closely related with the *heart conduct in crystals*, and in fact originated in a paper by Rudolf Pierls (1929) on that topic. So progress in the study of the WT limit should imply progress in the rigorous study of the heat conduct, which is essentially an open problem.

There are thousands of physical publications on WT, but just a few mathematical works. First of them, dealing with a modification of eq. (NLS), is due to Spohn-Lukkarinen (2015).

In the last 4-5 years a set of rigorous mathematical works on the WT Limit for (NLS) has been made by the group of 5 people J.Shatah, P.Germain, T.Buckmaster, Y.Deng, Z.Hani; I will refer to it as *the Courant group*. In total in various combinations of the authors, they have written at least 6 linked works. I talk about a result, stated in one of the last preprints of the group, written by Deng-Hani.

SETTING OF THE RESULT. Let τ be the time t, properly scaled with L. The task is to study the energy spectrum $\mathbf{E}|\hat{u}_s(\tau)|^2$, properly scaled with L to be of order one. Denote the scale spectrum $y_s(\tau)$. Note that $y_s(\tau)$ is defined for $s \in L^{-1}\mathbb{Z}^d.$ So when $L \to \infty$, $y_s(\tau)$ becomes defined for $s \in \mathbb{R}^d$.

Consider the standard cubic WKE on $\mathbb{R}^d$:

(WKE) $$\partial_\tau w(\tau, s) = K_s\big(w(\tau, \cdot)\big), \quad \tau \geq 0, \; s \in \mathbb{R}^d; \qquad w(0, s) = y_{0s},$$

where the initial data y_{0s} is the energy spectrum $\mathbf{E}|\hat{u}_{0s}|^2$, properly scaled with L.

K in (WKE) is the *kinetic integral of 4 waves interaction.* For a function y_s on $\mathbb{R}^d$ its image under the operator K is a function $K_s(y.)$, $s \in \mathbb{R}^d$, defined by the following integral :

$$K_s(y.) =$$

$$\text{Const} \int_{\Sigma^s \subset \mathbb{R}^{2d}} \frac{\left(y_{s_1} y_{s_2} y_{s_3} + y_{s_1} y_{s_2} y_s - y_{s_2} y_{s_3} y_s - y_{s_1} y_{s_3} y_s\right) \mu^s(ds_1 ds_2)}{\sqrt{|s_1 - s|^2 + |s_2 - s|^2}}.$$

Here $s_3 = s_1 + s_2 - s$, Σ^s is the quadric

$$\Sigma^s = \{(s_1, s_2) : (s_1 - s) \cdot (s_2 - s) = 0\},$$

and μ^s is the volume element on Σ^s, corresponding to the usual euclidean structure on $\mathbb{R}^{2d}$.

Let $u(t,x) = L^{-d/2}\sum_{s\in L^{-1}\mathbb{Z}^d} \hat{u}_s(t) e^{2\pi i s\cdot x}$ be a solution of

(NLS) $$\partial_t u(t,x) + i\Delta u(t,x) - i\lambda(|u|^2 - 2\|u\|^2)u = 0, \quad u(0,x) = u_0(x).$$

THEOREM (the Courant group). *Let the initial data u_0^ω be a Gaussian random field. Then there exist a small $\varepsilon_* > 0$ and:*

1) a "good" event $\Omega_L \subset \Omega$ such that $\mathbf{P}(\Omega_L) \to 1$ as $L \to \infty$,

2) proper scalings with L of t (scaled t is denoted τ) and of λ,

3) proper scaling $y_s(\tau)$ of the "modified energy spectrum", defined as $\mathbf{E}\big(\chi_{\Omega_L}|\hat{u}_s(\tau)|^2\big)$,

such that we have

$$\big|y_s(\tau) - w(\tau,s)\big| \to 0 \quad \textit{as } L \to \infty, \quad \textit{for } 0 \le \tau \le \varepsilon_* \textit{ and all } s,$$

where $w(\tau,s)$ solves (WKE).

How the authors prove their result. I recall the (NLS) equation:

$$\text{(NLS)} \qquad \partial_t u(t,x) + i\Delta u(t,x) - i\lambda(|u|^2 - 2\|u\|^2)u = 0, \quad u(0,x) = u_0(x).$$

Let us write its solution u as a series in λ, $u(\tau,x) = \sum_{k=0}^{\infty} \lambda^k u^{(k)}(\tau,x)$. Then

$$u^{(0)}(\tau,x) = e^{-i\tau\Delta} u_0^{\omega}(x),$$

and we recursively write the components $u^{(r)}$ with $r \geq 1$ as Duhamel integrals:

$$u^{(r)}(\tau,x) = i\int_0^{\tau} e^{-it(\tau-l)\Delta} P_r^3\big(u^{(k)}(l,x), \bar{u}^{(k)}(l,x); k < r\big) dl, \quad r \geq 0,$$

where P_r^3 is a cubic polynomial. As Spohn-Lukkarinen (2015), the authors examine the terms $u^{(r)}$ in Fourier presentation and recursively in r write their Fourier coefficients $\hat{u}_s^{(r)}(\tau)$ as sums, parametrised by Feynman diagrams. Complicated combinatorics is developed to control the sums and cancellations in them, using the Gaussian techniques and technique of oscillating integrals. This allows to show that the sum for $\hat{u}_s^{(r)}(\tau)$ converges for small τ and "good" $\omega \in \Omega_L$, and to prove the theorem.

Certainly this is a good result. Its shortcomings are that the paper with a proof is 136 pages long and refers to facts from the previous publications of the group.
The restriction that time is small cannot be removed, and the proof significantly uses that the nonlinearity in the equation is exactly cubic.

C. Damped-driven NLS equation. A damping and forcing often are added to Hamiltonian systems in physical works on WT, implicitly or explicitly. Implicitly this is done when an author talks about "adding energy to low modes and extracting it from high modes". Explicitly the damping and forcing were added to equations e.g. in some works of V. Zakharov and his students (Falkovich and others).

The rigorous results below are obtained by myself jointly with A.Dymov in a number of papers. In next few slide I follow our work
A.Dymov, SK, *Comm. Math. Physics* **382**, 951-1014 (2021). It deals with the equation

(D/D NLS)
$$\partial_t u(t,x) + i\Delta u(t,x) - i\lambda(|u|^2 - 2\|u\|^2)u = \\ -\nu(-\Delta+1)^r u + \sqrt{\nu}\,\partial_t \eta^\omega(t,x), \quad u(0,x)=0 \text{ for simplicity.}$$

Here $x \in \mathbb{T}^d_L$, $0 < \nu < 1$, $d \geq 2$, $\eta^\omega(t,x)$ is a Wiener process in a functions space, like the random force for NSE, and $\lambda > 0$ should be properly scaled with ν and L. Again the goal is to study the energy spectra of solutions under the WT limit, which now reads

(WT Limit) $\nu \to 0, \quad L \to \infty$ while λ is properly scaled with ν and L.

We will see later that in order a limit for a solution u to exist, a relation between ν and L has to be imposed. Below, except the very end of my talk, I assume that

(*) $$L \gg \nu^{-2}.$$

What I will say remains true for the popular in physical works extreme case when

first $L \to \infty$, then $\nu \to 0$.

Analysis shows that a non-trivial WT limit for the energy spectrum $\mathbf{E}|\hat{u}_s(t)|^2$ may exist only if $\lambda \sim \sqrt{\nu}$. Accordingly I choose $\lambda = \varepsilon\sqrt{\nu}$, where $\varepsilon > 0$ is a FIXED constant < 1. This ε is a properly scaled amplitude of solutions. As a good example one may take

$$\varepsilon \sim 1/20, \quad \nu \sim 10^{-10}.$$

Passing to the slow time

$$\tau = \nu t,$$

I write the damped/driven equation above with $\lambda = \varepsilon\sqrt{\nu}$ as

$$\begin{aligned} \partial_\tau u(\tau,x) + i\nu^{-1}\Delta u(\tau,x) - i\varepsilon\nu^{-1/2}(|u|^2 - 2\|u\|^2)u \\ = -(-\Delta+1)^r u + \partial_\tau \eta^\omega(\tau,x), \quad u(0,x)=0; \quad x \in \mathbb{T}^d_L. \end{aligned} \tag{1}$$

Here

$$\eta(\tau,x) = L^{-d/2}\sum_{s\in L^{-1}\mathbb{Z}^d} b(s)\beta^\omega_s(\tau)e^{2\pi i s\cdot x},$$

where $b(s)$ is a Schwartz function on $\mathbb{R}^d$ and $\{\beta^\omega_s(\tau), s \in L^{-1}\mathbb{Z}^d\}$ are standard independent Wiener processes.

In Fourier equation (1) reeds:

$$\begin{aligned} \partial_\tau \hat{u}_s(\tau) - i\nu^{-1}|s|^2|\hat{u}_s|^2 - i\varepsilon\nu^{-1/2}L^{-d} \sum_{L^{-1}\mathbb{Z}^d \ni s_1,s_2 \neq s} \hat{u}_{s_1}\hat{u}_{s_2}\bar{\hat{u}}_{s_1+s_2-s} \\ = -(|s|^2+1)^r\hat{u}_s + b(s)\partial_\tau\beta^\omega_s(\tau), \qquad s \in L^{-1}\mathbb{Z}^d. \end{aligned}$$

Why this equation is difficult? Below we will see that under the imposed scaling, $\hat{u}_s(\tau) \sim 1$ for $|s| \leq 1$, uniformly in ν, L. If I naively estimate each $\hat{u}_s(\tau)$ by 1, then I get the following bound for the part of the nonlinearity, corresponding to $|s_j| \leq 1$:

$$\nu^{-1/2} L^{-d} \left| \sum_{|s_1|,|s_2| \leq 1} \hat{u}_{s_1} \hat{u}_{s_2} \bar{\hat{u}}_{s_1+s_2-s} \right| \lesssim \nu^{-1/2} L^d,$$

which is absolutely useless. So we must carefully take into account various cancellations. They are numerous and different in nature (certainly the same difficulty arises when working with the WT limit for (NLS)).

I recall the equation I talk about

$$
\begin{aligned}
&\partial_\tau u(\tau,x) + i\nu^{-1}\Delta u(\tau,x) - i\varepsilon\nu^{-1/2}(|u|^2 - 2\|u\|^2)u \\
&\qquad = -(-\Delta+1)^r u + \partial_\tau \eta^\omega(\tau,x), \quad u(0,x)=0;\ x\in\mathbb{T}^d_L.
\end{aligned} \tag{1}
$$

Quasisolutions. Let us decompose a solution for eq. (1) in series in $\varepsilon\nu^{-1/2}$ (which is the parameter in front of the nonlinearity):

$$
u(\tau,x) = \sum\nolimits_{k=0}^{\infty} (\varepsilon\nu^{-1/2})^k u^{(k)}(\tau,x). \tag{*}
$$

Plugging (*) to the equation we get that $u^{(0)}$ is the Gaussian process

$$
u^{(0)}(\tau,x) = \int_0^\tau e^{-(\tau-l)(i\nu^{-1}\Delta + (-\Delta+1)^r)} d_l\eta(l,x)
$$

this is a solution of the linear equation with $\varepsilon=0$). We also get recursively that for $r\geq 1$,

$$
u^{(r)} = \big(\text{Duhamel integral of a cubic polynomial of } u^{(k)}, \bar u^{(k)} \text{ with } k<r\big).
$$

Following physical works on WT we consider the quadratic part of the series (*):

$$U(\tau,x)=u^{(0)}(\tau,x)+(\varepsilon\nu^{-1/2})u^{(1)}(\tau,x)+(\varepsilon\nu^{-1/2})^2u^{(2)}(\tau,x).$$

The Fourier coefficients of U will be denoted $\hat{U}_s(\tau),\ s\in L^{-1}\mathbb{Z}^d$.

THEOREM 1. *The* $U(\tau,x)$ *satisfies eq. (1) with a disparity* $O(\varepsilon^3)$, *uniformly in* $0<\nu\le 1$, *in* $L\gg\nu^{-2}$ *and in* $\tau\ge 0$.

Due to this result and by analogy with the quasimodes from the quasiclassical approximation in quantum mechanics we call U a *quasisolution*. Consider its energy spectrum $n_s(\tau)=\mathbf{E}|\hat{U}_s(\tau)|^2$. Now consider the damped/driven WKE:

$$\partial_\tau m(\tau,s)=-2(1+|s|^2)^r m+\varepsilon^2 K_s\big(m(\tau,\cdot)\big)+2b(s)^2,\qquad m(0,s)=0.$$

Globally, this is a complicated nonlinear equation. But perturbatively it is fine: for $\varepsilon=0$ it has a unique solution $m_s^0(\tau)$, and for small ε the equation has a unique solution m^ε, which is close to $m_s^0(\tau)$:

$$|m^\varepsilon(\tau,s)-m_s^0(\tau)|=O(\varepsilon^2)\quad\forall\,\tau\ge 0,\ \ \forall\, s.$$

I recall that $U(\tau, x)$ is the quasisolution, $n_s(\tau)$ is its energy spectrum, $m^\varepsilon(\tau, s)$ is a solution of the damped-driven WKE, which is ε^2-close to the solution $m_s^0(\tau)$ for the linear equation.

THEOREM 2. *If ε is small, then uniformly in $\nu < 1$, $L \gg \nu^{-2}$ and in $\tau \geq 0$ the energy spectrum $n_s(\tau)$ of the quasisolution satisfies:*

1) $|n_s(\tau) - m^\varepsilon(\tau, s)| = O(\varepsilon^4) \qquad \forall s \in L^{-1}\mathbb{Z}^d;$

2) damped-driven WKE has a unique steady state m_s^ε, which is ε^2-close to the steady state m_s^0 for the linear equation (with $\varepsilon = 0$), and

$$|n_s(\tau) - m_s^\varepsilon| = O(\varepsilon^4 + e^{-\tau}) \qquad \forall s \in L^{-1}\mathbb{Z}^d, \ \ \forall \tau \geq 0.$$

To prove Theorems 1 and 2 we developed a tool-box of new analytical techniques. For example, in two earlier papers we calculated asymptotics for a new class of singular integrals with small parameter, relevant for the problem we discuss, and in

A. Dymov, SK “Formal expansions in stochastic model for wave turbulence 2: method of diagram decomposition”, 60 p., arXiv (2019)

we suggested a method to estimate a class of sums, parametrised by Feynman diagrams.

D. What next? I recall the damped/driven NLS equation which we discuss:

$$\begin{aligned}\partial_\tau u(\tau,x) + i\nu^{-1}\Delta u(\tau,x) - i\varepsilon\nu^{-1/2}(|u|^2 - 2\|u\|^2)u \\ = -(-\Delta+1)^r u + \partial_\tau\eta^\omega(\tau,x), \quad u(0,x)=0; \quad x\in\mathbb{T}^d_L.\end{aligned} \tag{1}$$

$U(\tau,x)$ is its quasisolution (quadratic in $\varepsilon\nu^{-1/2}$). It satisfied eq. (1) with a disparity $O(\varepsilon^3)$.

CONJECTURE 1. *Uniformly in $\nu<1$, $L\gg\nu^{-2}$ and $\tau\ge 0$, $U(\tau,x)$ is ε^3-close to an exact solution $u(\tau,x)$ of eq. (1).*

Then for all $\nu<1$, $L\gg\nu^{-2}$ and $\tau\ge 0$ the energy spectrum $\mathbf{E}|\hat u_s(\tau)|^2$ of the exact solution u is ε^3-close to the energy spectrum $n_s(\tau)$ of the quasisolution U. So by Theorem 2 it also is ε^3-close to the solution $m^\varepsilon(\tau,s)$ of the damped-driven WKE. – This will be exactly the WK approximation for the energy spectra of solutions.

How to prove the conjecture? Eq. (1) is a nonlinear equation on $u(\tau,x)$ with random r.h.s. By Theorem 1 $U(\tau,x)$ is its approximate solution with a disparity $\sim\varepsilon^3$. If the Implicit Function Theorem (IFTh) applies to the equation, then the result would follow.

If the force $\partial_\tau \eta^\omega(\tau, x)$ was a process with regular trajectories, then for L and ν fixed the IFTh would apply – this comes from PDEs. In our case, when the force is white, $\nu \ll 1$ and $L \gg 1$ there is no chance that IFTh may be used. But we believe that a KAM-version of the IFTh applies to the equation. If so, then it implies the existence of an exact solution, ε^3-close to the quasisolution U. By the uniqueness this exact solution must be u, and the required result follows. Our optimist is boosted by a resent success in exploiting KAM to resolve some problems from nonlinear stochastic PDEs (in my work with Shirikyan and Nersesyan on "Mixing and KAM".)

An analysis of the KAM-construction which produces the exact solution u from the approximate quasisolution U should show that, in fact, the limiting energy spectrum of u **coincides** with the solution $m^\varepsilon(\tau, s)$ of the damped-driven WKE. This is our

CONJECTURE 2. $\left|\mathbf{E}|\hat{u}_s(\tau)|^2 - m^\varepsilon(\tau, s)\right| \to 0$ as $\nu \to 0$ and $L \gg \nu^{-2}$, for all $\tau \geq 0$ and all s.

E. Another form of the WT limit. Consider again the damped/driven NLS equation:

$$
\begin{aligned}
&\partial_\tau u(\tau,x) + i\nu^{-1}\Delta u(\tau,x) - i\rho(|u|^2 - 2\|u\|^2)u \\
&\qquad = -(-\Delta+1)^r u + \partial_\tau \eta^\omega(\tau,x), \quad u(0,x)=0; \ x\in\mathbb{T}^d_L.
\end{aligned} \tag{1}
$$

Here I denoted by ρ the coefficient in front of the nonlinearity. I have just explained that if we choose $\rho=\varepsilon\nu^{-1/2}$, then (in particular) under the limit

$$
\text{first} \quad L\to\infty, \quad \text{then} \ \ \nu\to 0
$$

the energy spectrum of the quasisolution U of eq. (1) is ε^4-close to the solution of the damped-driven standard WKE.

From other hand, it was proved in 2015 by A.Maiocchi and myself that for any ρ and L fixed, when $\nu\to 0$ the energy spectrum of a solution for eq. (1) converges to the energy spectrum of a solution for the *averaged equation*

$$
\partial_\tau u(\tau,x) - \rho\Phi^{res}(u) = -(-\Delta+1)^r u + \partial_\tau\eta^\omega(\tau,x), \quad x\in\mathbb{T}^d_L, \quad u(0)=0,
$$

where $\Phi(u)$ is the resonant part of the nonlinearity $i(|u|^2-2\|u\|^2)u$.

Last year, in the work

Dymov, Kuksin, Maiocchi, Vladuts "The large-period limit for equations of discrete turbulence", arXiv (2021),

we examined a quasi-solution $\mathcal{A}(t,x)$ for the averaged equation above and proved

THEOREM 3. *If $\rho = \varepsilon L$, then for small ε, uniformly in $L \geq 1$, for all $\tau \geq 0$ and all $s \in L^{-1}\mathbb{Z}^d$, the energy-spectrum of the quasi-solution $\mathcal{A}(t,x)$ is ε^3-close to a solution of the damped/driven non-autonomous WKE*

$$\partial_\tau m(\tau,s) = -2(1+|s|^2)^r m + \varepsilon^2 \mathcal{K}_s\big(\tau, m(\tau,\cdot)\big) + 2b(s)^2, \qquad m(0,s) = 0.$$

Here $\mathcal{K}$ is a non-autonomous kinetic operator (so NOT the usual kinetic integral K). Our proof of this fact uses the tool-box, earlier developed by us for the "traditional" WT limit $\lim_{\nu\to 0}\lim_{L\to\infty}$, and two deep results from the number theory. One of them was amended for purposes of the project in our work

Dymov, Kuksin, Maiocchi, Vladuts "A refinement of the Heath-Brown theorem on quadratic forms", 55 p., arXiv (2021).

So the “new” WK limit first $\nu \to 0$ then $L \to \infty$ leads to ANOTHER kinetic equation, compare to the more common WK limit first $L \to \infty$ then $\nu \to 0$. Moreover, for the new limit to exist, ANOTHER scaling of the nonlinearity is required.

Concerning the quasisolution $\mathcal{A}$ of the averaged equations, we again **conjecture** that it is close to the exact solution, so the energy spectra of solutions also are ε^3-close to solutions of the damped/driven non-autonomous WKE above.

F. CONCLUSIONS. 1) The quasisolutions seem to be important by themselves since they are "almost as good" as real solutions. In particular, they suggest right scalings and right forms of kinetic limits.

2) The method of quasisolutions allows to split the problem of justifying the kinetic limit for the energy spectrum (and another problems from WT) to two: a *rigid* part which uses explicit calculation to construct quasisolutions, to calculates right scalings, right limiting equations etc., and a *soft* part which provides more estimates.

3) The method requires dramatically less calculation than the direct approach by decomposing solutions to converging series.

4) The method and its tool-box as well applies to the deterministic NLS equations (cubic and of higher degree) with random initial data.