An analogue of Steinberg theory for symmetric pairs

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Outline:

- The Robinson-Schensted corr.:

$$RS : \mathfrak{S}_n \xrightarrow{\sim} \{\text{pairs of Y. tableaux of same shape} \vdash n\} = \bigsqcup_{\lambda \vdash n} Tab(\lambda) \times Tab(\lambda)$$

- The Steinberg correspondence

$$St : W \simeq \left\{ \begin{matrix} \text{Bruhat cells} \\ \text{of } G \text{ red.} \end{matrix} \right\} \longrightarrow \text{nilpotent orbits of } Lie(G)$$

If $G = GL_n$: geom. interpretation of RS.

- Goal: analogue of St for symmetric pair (G, K)
  analogue of RS.

Multiple flag varieties

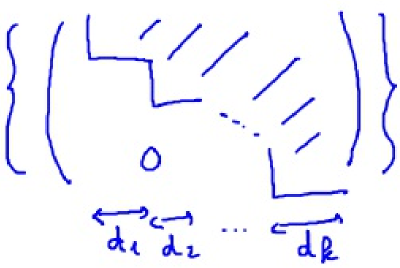
Setting: let G be connected reductive $/ \mathbb{C}$, $\mathfrak{g} = Lie(G)$.

Partial flag variety:

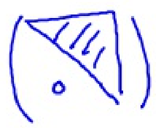
$$G/P = \{ P_1 \subset G \mid \text{conjugate to } P \} = \{ \mathfrak{p}_1 \subset \mathfrak{g} \mid \text{conjugate to } Lie(P) \}$$

P parabolic, G-homogeneous

If $G = GL_n$:

$P = \left\{ \right.$  $\left. \right\} \rightsquigarrow G/P = \{ (V_0 \subset V_1 \subset \cdots \subset V_k = \mathbb{C}^n) : \dim V_i / V_{i-1} = d_i \}$

2 special cases:

- $P = B_n^+ =$  : $G/B_n^+ = \{ (V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n) : \dim V_i = i \} = Flags(\mathbb{C}^n)$

- $P = P_{max}(r) =$  : $G/P_{max}(r) = Grass(\mathbb{C}^n, r)$

"Mirabolic" $P = P_{max}(1)$ $\quad G/P_{max}(1) = \mathbb{P}(\mathbb{C}^n)$.

- Double flag variety

$$G/P_1 \times G/P_2 \curvearrowleft G \text{ acts diagonally}$$

Finite number of orbits $\xleftrightarrow{\sim}$ $W_{P_1} \backslash W / W_{P_2}$

- Triple flag variety

$$G/P_1 \times G/P_2 \times G/P_3 \curvearrowleft G \text{ acts diagonally}$$

in general infinitely many orbits, sometimes finitely many

classified by
Magyar-Weymann-Zelevinsky (A, C)
Matsuki (B, D)

- >3 factors: always an infinite number of orbits.

Symmetric pairs

Let $\theta : G \to G$ involution

$K := G^\theta$ symmetric subgroup, (G, K) is a sym. pair

Notation: $\theta \in \mathrm{Aut}(G) \rightsquigarrow \theta \in \mathrm{Aut}(\mathfrak{g})$ involution

Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, $\mathfrak{k} = \mathfrak{g}^\theta = \mathrm{Lie}(K)$, $\mathfrak{s} = \mathfrak{g}^{-\theta}$

$$\mathfrak{g} \ni x = x^\theta + x^{-\theta}$$

Examples:
- (G, G) is a "trivial" sym. pair, $\theta = \mathrm{id}_G$.
- If $G = GL_n$: θ conjugation by $\left(\begin{array}{c|c} I_p & 0 \\ \hline 0 & -I_q \end{array}\right)$ $\quad p+q=n$

$$K = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right) \simeq GL_p \times GL_q$$

$(GL_n, GL_p \times GL_q)$ is a sym. pair of type A3.
- (GL_{2n}, Sp_{2n}) is a sym. pair of type A2.
- (SL_n, SO_n) is a sym. pair of type A1.
- (Sp_{2n}, GL_n) is a sym. pair of type C1.

Return to a general (G,K).

Definition (Nishiyama-Ochiai):
Double flag variety $X = G/_P \times K/_Q$

$P \subset G$, $Q \subset K$ parabolic.

endowed with diag. action of K.

Infinitely many orbits in general, sometimes fin. many

Examples: . $G/_{P_1} \times G/_{P_2} = X$ for $(G,K) = (G,G)$.

. $G/_{P_1} \times G/_{P_2} \times G/_{P_3} = G\times G/_{P_1 \times P_2} \times G/_{P_3}$

$= X$ for the sym. pair $(G \times G, \Delta G)$.

. In general $X = G/_P \times K/_Q \hookrightarrow G/_P \times G/_{\theta(P)} \times G/_{Q'}$ ← θ-stable s.t. $Q' \cap K = Q$

NO: fin. many K-orbits $\Longleftarrow$ fin. many G-orbits

Example: In type A 3: $X/_K$ finite whenever P is maximal or Q is mirabolic.

"Our case": type A 3 $\quad G = GL_n, \quad K = GL_p \times GL_q$

$P = P_{max}(r)$

$Q = B_K = B_p^+ \times B_q^+$

Then: $X = G/_P \times K/_Q = Grass(\mathbb{C}^n, r) \times Flags(\mathbb{C}^p) \times Flags(\mathbb{C}^q)$

We know a priori: X has a finite number of K-orbits.

Explicit parametrization:

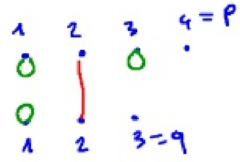
$$X/_K \xleftarrow{\sim} Grass(\mathbb{C}^n, r)/_{B_K} \xleftarrow{\sim} \Omega = \left\{ \begin{array}{l} p\updownarrow \\ q\updownarrow \end{array} \overset{r \text{ columns}}{\begin{pmatrix} \tau_1 \\ \hline \tau_2 \end{pmatrix}} \text{ of rank } r \right\} /_{\mathfrak{S}_r}$$

$$\mathbb{O}_\omega \longleftarrow B_K \cdot (\mathrm{Im}\,\omega) \longleftarrow \omega$$

"partial permutations"
0's except at most one 1 in each row/column

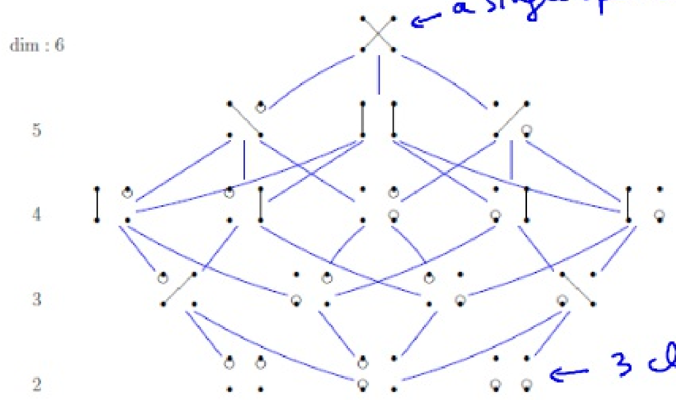
example: $\quad \omega = \begin{array}{c} 1 \\ 2 \\ 3 \\ \\ 1 \\ 2 \\ \\ \end{array} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} p = 4 = r \\ q = 3 \end{array}$

represent it by a graph:



Dimension formula, and closure relations

For $p=q=r=2$: Hasse diagram



$\leftarrow$ a single open orbit in X

$\leftarrow$ 3 closed orbits

Return to a general sym. pair (G,K)

Action of K on $X = G/P \times K/Q$

Yields an action of K on
$$T^*X = T^*G/P \times T^*K/Q = \left\{ (\underset{G/P}{p_1} \ni, \underset{\mathfrak{g}}{x} \in, \underset{K/Q}{q_1} \ni, \underset{\mathfrak{k}}{y} \in) : x \in \mathrm{nil}(p_1),\ y \in \mathrm{nil}(q_1) \right\}$$

Moment map: $\mu_X : T^*X \longrightarrow \mathfrak{k}^* \simeq \mathfrak{k}$
$$(p_1, x, q_1, y) \longmapsto x^\theta + y$$

Conormal variety $\mathcal{Y} = \mu_X^{-1}(0) = \left\{ (\underset{G/P}{p_1}, \underset{K/Q}{q_1}, \underset{\mathfrak{g}}{x}) : x \in \mathrm{nil}(p_1),\ x^\theta \in \mathrm{nil}(q_1) \right\}$

If X/K finite:

$$\mathcal{Y} = \bigcup_{\mathbb{O} \in X/K} \overline{T^*_{\mathbb{O}} X}$$

$\leftarrow$ irreducible of same dim. as X

$\uparrow$ the irred. components of $\mathcal{Y}$

$\mathrm{Irr}(\mathcal{Y}) \simeq X/K$.

The <u>Steinberg correspondence</u> for the trivial sym. pair $(G,K)=(G,G)$

$$X = G/B \times G/B \quad \text{where } B \subset G \text{ Borel}$$

Then: $\mathcal{Y} = \{ (\underset{G/B}{b_1}, \underset{G/B}{b_2}, \underset{\mathfrak{g}}{x}) : x \in \mathrm{nil}(b_1) \cap \mathrm{nil}(b_2) \} \supset \overline{T^*_{O_w} X}$, $w \in W \simeq X/G$

$$\mathcal{Y} \xrightarrow{pr_3} \mathcal{N} \text{ nilp. cone of } \mathfrak{g}, \text{ finitely many } G\text{-orbits}$$

$$\overline{T^*_{O_w} X} \xrightarrow{pr_3} \text{closed irreducible } G\text{-stable } \overline{\mathcal{O}_w} \text{ nilp. orbit}$$

the Steinberg map:

$$St: \quad W \simeq X/G \longrightarrow \mathcal{N}/G$$
$$w \longmapsto \mathcal{O}_w$$

More precisely:

$$\underbrace{W \simeq \mathrm{Irr}(\mathcal{Y}) = \bigsqcup_{x \in \hat{\mathcal{N}}} \mathrm{Irr}\left(pr_3^{-1}(G \cdot \overset{\mathcal{N}}{\underset{\psi}{x}}) \right) \simeq \bigsqcup_{x \in \hat{\mathcal{N}}} \mathrm{Irr}(\mathcal{B}_x) \times \mathrm{Irr}(\mathcal{B}_x) / Z_x}_{\text{Steinberg correspondence}}$$

$x \in \hat{\mathcal{N}}$: representatives of nilp. orbits

$\mathcal{B}_x = \{ \mathfrak{b} \subset \mathfrak{g} : x \in \mathfrak{b} \}$ Springer fiber

If $G = GL_n$: $\quad \mathfrak{S}_n \xrightarrow{\sim} \bigsqcup_{\lambda \vdash n} \mathrm{Tab}(\lambda) \times \mathrm{Tab}(\lambda)$

coincides with RS.

An analogue for symmetric pair:

let (G,K) be general sym. pair

$$X = G/_P \times K/_Q$$

Assume: $X/_K$ finite

The conormal variety: $\mathcal{Y} = \{ (\underset{G/P}{p_1} \in, \underset{K/Q}{q_1} \in, \underset{\mathfrak{g}}{x} \in) : x \in \mathrm{nil}(p_1),\ x^\theta \in \mathrm{nil}(q_1) \} \curvearrowleft K$

$$\begin{tikzcd}
 & (p_1, q_1, x) \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\
x^\theta \in N(\mathfrak{k}) & & x^{-\theta} \in N(\mathfrak{s})
\end{tikzcd}$$

fin. many K-orbits

assume this

We get 2 maps

$$\Phi_1 : X/_K \simeq \mathrm{Irr}(\mathcal{Y}) \longrightarrow N(\mathfrak{k})/_K$$

$$\Phi_2 : X/_K \simeq \mathrm{Irr}(\mathcal{Y}) \longrightarrow N(\mathfrak{s})/_K$$

In our case: $(G,K) = (GL_n, GL_p \times GL_q)$

$$X = \mathrm{Grass}(\mathbb{C}^n, r) \times \mathrm{Flags}(\mathbb{C}^p) \times \mathrm{Flags}(\mathbb{C}^q)$$

$$X/_K \simeq \Omega$$

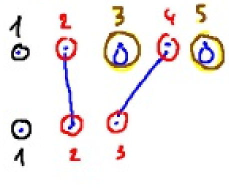
we have computed:

$$\Phi_1 : \Omega \longrightarrow N(\mathfrak{k})/_K = N(\mathfrak{gl}_p)/_{GL_p} \times N(\mathfrak{gl}_q)/_{GL_q}$$

$$= \{ (\lambda, \mu) : \lambda \vdash p,\ \mu \vdash q \}$$

$$\Phi_2 : \Omega \longrightarrow N(\mathfrak{s})/_K = \{\text{signed Young diagrams of signature } (p,q)\}$$

Example: for Φ_1:

$w =$ 

$p = 5$

$q = 3$

$r = 4$

$$\begin{array}{|c|}\hline 3 \\ \hline 5 \\ \hline \end{array} * RS_1 \begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix} * \boxed{1}$$

$$= \begin{matrix} 2 & 4 \\ 3 & \\ 5 & \end{matrix} = \begin{matrix} 1 & 4 \\ 2 & \\ 3 & \\ 5 & \end{matrix} = T_1$$

$$\Phi_1(w) = (\lambda, \mu) \quad \begin{matrix} \lambda \text{ shape of } T_1 \\ \mu \text{ shape of } T_2 \end{matrix}$$

$$\emptyset * RS_1 \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix} * \boxed{1} \qquad \begin{matrix} 1 & 3 \\ 2 & \end{matrix} = T_2$$

Fibers for Φ_1:

$$\Omega = \coprod_{\substack{\lambda \vdash p \\ \mu \vdash q}} \Phi_1^{-1}(\lambda, \mu) \xrightarrow{\sim} \coprod_{\substack{\lambda \vdash p \\ \mu \vdash q}} \left\{ (T_1, T_2, \lambda', \mu', \nu) \;\; \begin{matrix} T_1 \in Tab(\lambda) \\ T_2 \in Tab(\mu) \\ \nu \subset \lambda' \subset \lambda \\ \nu \subset \mu' \subset \mu \end{matrix} \right\}$$

means that

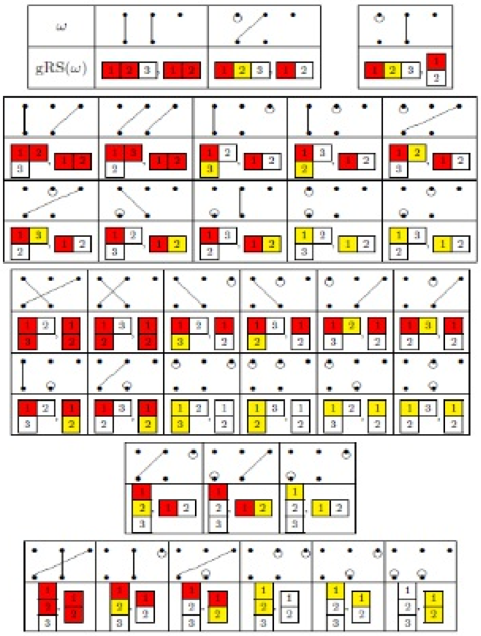
$\mu' \setminus \nu$ is column strip

This correspondence

for $p=3,\ q=r=2$

ν: boxes in red

λ', μ': in yellow



other RS correspondence in geometry:
- . Travkin
- . Henderson-Trapa (A2)
- . Singh