Let us consider the case of slowly varying (adiabatic) Hamiltonian $\hat{H}(t)$. We will assume that the energy levels $E_n(t)$ are distinct and vary continuously with t , corresponding eigenvectors will be denoted by $\phi_n(t)$. We assume that these eigenvectors constitute an orthonormal system.Then it is easy to prove that in the first approximation the evolution operator $\hat{U}(t)$ transforms eigenvector into eigenvector:

$$\hat{U}(t)\phi_n(0) = e^{-i\alpha_n(t)}\phi_n(t), \frac{d\alpha_n(t)}{dt} = E_n(t)$$

To verify this fact we check that the RHS satisfies the equation of motion if we neglect $\dot{\phi}_n(t)$.

More accurate consideration

Consider a smooth family of Hamiltonians $\hat{H}(g)$ with eigenvectors $\phi_n(g)$ and eigenvalues $E_n(g)$ smoothly depending on g. The family
$\hat{H}(t) = \hat{H}(g(t))$
is an adiabatic family if $\dot{g}(t)$ can be neglected. (For example, we can fix a function $g(t)$ and consider a family of functions $g_a(t) = g(at)$ with $a \to 0$.) Then

$$\hat{U}(t)\phi_n(g(0)) = e^{-i\alpha_n(t)}\phi_n(g(t)), \frac{d\alpha_n(t)}{dt} = E_n(g(t))$$

because $\dot{\phi}_n(g(t))$ can be neglected.

$U(t)$-evolution operator of density matrix K.
$\frac{dK}{dt} = H(t)K(t) = \frac{1}{i\hbar}[\hat{H}(t), K]$
Introduce operators $\psi_{mn}(t)$ by the formula $\psi_{mn}(t)x = \langle x, \phi_n(t)\rangle \phi_m(t)$. These operators are eigenvectors of the operator $H(t)$. In adiabatic approximation $U(t)$ transforms eigenvector into eigenvector:

$$U(t)\psi_{mn}(0) = e^{-i\beta_{mn}(t)}\psi_{mn}(t), \frac{d\beta_{mn}(t)}{dt} = E_m(t) - E_n(t)$$

(this equation is true up to terms that can be neglected for slowly varying Hamiltonian). Notice that β_{mm} does not depend on t.

It follows that $K(t) = U(t)K$ where $K = \sum k_{mn}\psi_{mn}$ is given by the formula $K(t) = \sum k_{mn}(t)\psi_{mn}(t)$ where
$k_{mn}(t) = e^{-i\beta_{mn}(t)}k_{mn}$,
$\beta_{mn}(t) = \int_0^t (E_m(\tau) - E_n(\tau))d\tau$
If $\hat{H}(T) = \hat{H}(0)$ we can assume that $\phi_n(T) = \phi_n(0)$. Then we see that diagonal entries of K do not change, but non-diagonal entries are multiplied by a phase factor.

If $\hat{H}(t)$ is an unknown adiabatic deformation of Hamiltonian $\hat{H}$ the non-diagonal entries of $K(T)$ are unpredictable.
If we consider linear combination (superposition) $\alpha_0\phi_0 + \alpha_1\phi_1$ of eigenvectors of $\hat{H} = \hat{H}(0)$ then evolving it with respect to the Hamiltonian $\hat{H}$ we get phase factors : $\alpha_k(t) = e^{-iE_k t}\alpha_k$. If we evolve this linear combination with respect to unknown adiabatic deformation $\hat{H}(t)$ the absolute value $|\alpha_k(t)|$ remains constant ,but the phase factors are unpredictable.
Decoherence.

We model interaction with environment by random adiabatic perturbation of Hamiltonian $\hat{H}$. We assume that random time-dependent Hamiltonian depends on some parameters $\lambda \in \Lambda$ with some probability distribution on Λ . If we start with density matrix (with matrix entries k_{mn} in $\hat{H}$ -representation) then the density matrix $K_\lambda(T)$ is equal to $\sum C_{mn}(\lambda, T) k_{mn} \psi_{mn}$, i.e. the matrix entries acquire phase factor $C_{mn}(\lambda, T)$. Now we should take the mixture $\bar{K}(T)$ of states $K_\lambda(T)$ (this means that we should take the average of phase factors). It is obvious that non-diagonal entries of $\bar{K}(T)$ are smaller by absolute value than corresponding entries of K.

Imposing some mild conditions on the probability distribution on Λ one can prove that the non-diagonal entries of $\bar{K}(T)$ tend to zero when the adiabatic parameter α tends to zero. In other words the matrix $\bar{K}(T)$ tends to a diagonal matrix $\bar{K}$ having the same diagonal entries as K. The matrix $\bar{K}$ can be considered as a mixture of pure states, corresponding to the vectors ϕ_n with probabilities k_{nn}.

Let us include the Hamiltonian $\hat{H}$ in a family of Hamiltonians $\hat{H}(g)$, where $g \in \Lambda$. We assume that $\hat{H}(0) = \hat{H}$. Let us consider a time dependent Hamiltonian $\hat{H}(g(t))$ where $g(0) = 0, g(1) = 0$. We can construct an adiabatic Hamiltonian by the formula
$\hat{H}_\alpha(t) = \hat{H}(g(\alpha t))$ where $\alpha \to 0$.
It is clear that $\hat{H}_\alpha(0) = \hat{H}_\alpha(T) = \hat{H}$ where $T = \alpha^{-1}$.

Denote by $E_n(g)$ the eigenvalues of $\hat{H}(g)$. Then the eigenvalues of $\hat{H}_\alpha(t)$ are equal to $E_n(g(\alpha t))$ The evolution of matrix entries of the density matrix with respect to the Hamiltonian $\hat{H}_\alpha(t)$ is governed by phase factors $e^{-i\beta_{mn}(t)}$ where for $t = T$

$\beta_{mn} = \int_0^T d\tau (E_m(g(\alpha\tau)) - E_n(g(\alpha\tau))) =$
$\frac{1}{\alpha} \int_0^1 d\tau (E_m(g(\tau)) - E_m(g(\tau)))$

Introducing probability distribution on the set Λ we obtain random adiabatic Hamiltonian.

The average phase factor vanishes for $m \neq n$.

Geometric approach to quantum theory

We start with a bounded closed convex set $\mathcal{C}_0 \subset \mathcal{L}$ (set of states) and a subgroup $\mathcal{V}$ of the automorphism group of $\mathcal{C}_0$

Here $\mathcal{L}$ is a Banach space and automorphisms of $\mathcal{C}_0$ are invertible linear operators in $\mathcal{L}$ mapping $\mathcal{C}_0$ onto itself.

The evolution operator $\sigma(t) \in \mathcal{V}$ transforms the state in the moment 0 into the state in the moment t

Equation of motion

$$\frac{d\sigma}{dt} = H(t)\sigma(t)$$

This formula can be considered as a definition of $H(t)$ ("Hamiltonian"), but usually we want to find the evolution operator knowing the "Hamiltonian". If H does not depend on time then $\sigma(t) = \exp(Ht)$.
Observable - a pair (A, a) where $A \in Lie(\mathcal{V})$ and a is an A-invariant linear functional on $\mathcal{L}$
$A \in Lie(\mathcal{V})$ if A generates a one-parameter subgroup of $\mathcal{V}$ denoted by $\sigma_A(t) = \exp(At)$

In textbook QM $\mathcal{C}_0$ consists of density matrices, $\mathcal{V}$ is unitary group.
Observables are pairs (A, a) where A is self-adjoint operator, $a(K) = tr(AK)$.
In algebraic approach $\mathcal{C}_0$ consists of positive linear functionals ω on unital associative algebra with involution denoted by $\mathcal{A}$ (positive means that $\omega(A^*A) \geq 0$).
Observable is a pair (A, a) where A is a self-adjoint elements of $\mathcal{A}$ and $a(\omega) = \omega(A)$.

Decoherence in geometric approach

Let us fix a time independent "Hamiltonian" H.

Evolution operator $\sigma(t) = e^{tH}$

We assume that H is diagonalizable (there exists a basis of $\mathcal{L}$ consisting of eigenvectors of H). Let us denote by (ψ_j) such a basis:

$$H\psi_j = \epsilon_j \psi_j$$

The eigenvalues ϵ_j are purely imaginary.

Let us assume that $H(g)$ is a family of "Hamiltonians" such that $H(0) = H$ and there exists a basis $(\psi_j(g))$ depending smoothly on g in such a way that $\psi_j(0) = \psi_j$ and

$$H\psi_j(g) = \epsilon_j(g)\psi_j(g)$$

We say ψ_j *is a robust zero mode of* H *if* $\epsilon_j(g) \equiv 0$.

Let us model the interaction with environment by random "Hamiltonian" $H(g(t))$. Then neglecting $\dot{g}(t)$ (in the adiabatic approximation) we obtain

$$\sigma(t)\psi_j = e^{\rho_j(t)}\psi_j(g(t)),$$

where $\frac{d\rho_j}{dt} = \epsilon_j(g(t))$.
If ψ_j is a robust zero mode of H then $\sigma(t)\psi_j = \psi_j(g(t))$. If $g(T) = g(0)$ we have $\sigma(T)\psi_j = \psi_j$.
All other modes acquire phase factors. If we have an unknown adiabatic perturbation it is impossible to predict these phase factors.

Imposing some conditions on the random Hamiltonian $H(t)$ we can prove that in average the random phase factors $e^{\rho_j(t)}$ vanish unless ϕ_j is a robust zero mode of H.
In textbook quantum mechanics robust zero modes of H are diagonal entries of density matrix in $\hat{H}$-representation.
Decoherence

Let us denote by P' a linear operator leaving intact robust zero modes of H and sending to zero all other eigenvectors of H. If $x \in \mathcal{C}_0$ one can prove that $P'x \in \mathcal{C}_0$. One can represent $P'x$ as a mixture of extreme points u_i of $P'(\mathcal{C}_0)$ (of pure robust zero modes): $P'x = \sum p_i u_i$. The coefficients p_i should be interpreted as probabilities.
If all zero modes of H are robust $P' = P$ where

$$P = \lim_{T \to \infty} \frac{1}{T} \int_0^T \sigma(t) dt$$

Take observable (A, a) where $A \in \mathcal{V}$, a is a functional obeying $a(Ax) = 0$.
x is a robust zero mode of A if $Ax = 0$ and for every A' that is close to A there exists x' that is close to x and obeys $A'x' = 0$.
Assume that all zero modes of A are robust.
Consider the set $\mathcal{C}_A$ of all states that are zero modes of A:
$\mathcal{C}_A = (KerA) \cap \mathcal{C} = Im(P_A)$ where
$P_A = \lim_{T\to\infty} \frac{1}{T} \int_0^T dte^{At}$.
Represent $P_A(x)$ as a mixture of pure zero modes of A (of extreme points of $\mathcal{C}_A$):
$P_A(x) = \sum p_i u_i$.
In non-degenerate case p_i is the probability of $a(u_i)$ in the state x.

L-functionals

Weyl algebra with generators $\hat{u}^i$ obeying

$\hat{u}^k \hat{u}^l - \hat{u}^l \hat{u}^k = i\hbar\sigma^{k,l}$

K-density matrix in representation of Weyl algebra

$L_K(\alpha) = tre^{i\alpha_k \hat{u}^k} K = trV_\alpha K$

$V_\alpha = e^{i\alpha_k \hat{u}^k} = e^{i\alpha u}$ where α_k are real is a unitary operator

$V_\alpha V_\beta = e^{-i\frac{\hbar}{2}\alpha\sigma\beta} V_{\alpha+\beta}$ where $\alpha\sigma\beta = \alpha_k \sigma^{k,l} \beta_k$.

$\mathcal{L}$ -space of all linear functionals on Weyl algebra

$L_K \in \mathcal{L}$ specifies a positive functional (state). It is normalized: $L_K(0) = 1$.

Every element A of an algebra with involution specifies two operators on linear functionals: one (denoted by the same symbol) transforms the functional $\omega(x)$ into the functional
$(A\omega)(x) = \omega(Ax)$, another (denoted by $\tilde{A}$) transforms it into the functional
$(\tilde{A}\omega)(x) = \omega(xA^*)$.
Denote by $\mathcal{C}$ the cone of positive (not necessary normalized) linear functionals.
The operator $\tilde{A}A$ transforms $\mathcal{C}$ into $\mathcal{C}$.
If $A = e^{tH}$ then $\tilde{A} = e^{t\tilde{H}}$. Hence the evolution operator defined as solution of equation
$d\sigma/dt = (H + \tilde{H})\sigma$
also acts in $\mathcal{C}$.

It is easy to calculate that
$(V_\beta L)(\alpha) = e^{-i\frac{\hbar}{2}\alpha\sigma\beta} L(\alpha + \beta)$,
$(\tilde{V}_\beta L)(\alpha) = e^{i-\frac{\hbar}{2}\alpha\sigma\beta} L(\alpha - \beta)$.
An element of Weyl algebra $\hat{H} = \int d\beta h(\beta) V_\beta$ is self-adjoint if $h(-\beta) = h(\beta)^*$. Taking $H = -\frac{i}{\hbar}\hat{H}$ we obtain the equation of motion
$i\hbar\frac{d\sigma}{dt} = (\hat{H} - \tilde{\hat{H}})\sigma$, hence

$$i\hbar\frac{dL}{dt} =$$

$$\int d\beta h(\beta) e^{-i\frac{\hbar}{2}\alpha\sigma\beta} L(\alpha+\beta) - \int d\beta h(-\beta) e^{-i\frac{\hbar}{2}\alpha\sigma\beta} L(\alpha-\beta)$$

$$= \int d\beta h(\beta)(e^{-i\frac{\hbar}{2}\alpha\sigma\beta} - e^{i\frac{\hbar}{2}\alpha\sigma\beta}) L(\alpha+\beta)$$

Finally,

$$\frac{dL}{dt} = \int d\beta h(\beta) \frac{2\sin(\frac{\hbar}{2}\alpha\sigma\beta)}{\hbar} L(\alpha+\beta)$$

L-functionals. Another definition.

Take representation of Weyl algebra (of CCR) in Hilbert space $\mathcal{H}$. (We understand CCR as relations $[a_k, a_l^+] = \delta_{kl}, [a_k, a_l] = [a_k^+, a_l^+] = 0$, where k, l run over a discrete set M.)

To a density matrix K (or more generally to any trace class operator in $\mathcal{H}$) we can assign a functional $L_K(\alpha^*, \alpha)$ defined by the formula

$$L_K(\alpha^*, \alpha) = tre^{-\alpha a^+} e^{\alpha^* a} K = e^{\frac{1}{2}\alpha^* \alpha} tre^{-\alpha a^+ + \alpha^* a} K$$

Here αa^+ stands for $\sum \alpha_k a_k^+$ and $\alpha^* a$ for $\sum \alpha_k^* a_k$, where k

One can say that L_K is a generating functional of correlation functions.

One can consider also a more general case when CCR are written in the form

$$[a(k), a^+(k')] = \hbar\delta(k, k'), [a(k), a(k')] = [a^+(k), a^+(k')] =$$

k, k' run over a measure space M. We are using the exponential form of CCR; in this form a representation of CCR is specified as a collection of unitary operators $e^{-\alpha a^+ + \alpha^* a}$ obeying appropriate commutation relations. Here $\alpha(k)$ is a complex function on the measure space M , the expressions of the form $\alpha^* a, \alpha a^+$ can be written as integrals $\int \alpha^*(k)a(k)dk, \int \alpha(k)a^+(k)dk$ over M. We assume that α is square-integrable, then the expression for L_K is well defined.

An action of Weyl algebra $\mathcal{A}$ on $\mathcal{L}$ (on the space of L-functionals) can be specified by operators

$$b^+(k) = \hbar c_1^+(k) - c_2(k), b(k) = c_1(k)$$

obeying CCR. Here $c_i^+(k)$ are multiplication operators by α_k^*, α_k and $c_i(k)$ are derivatives with respect to α_k^*, α_k. This definition is prompted by relations

$$L_{a(k)K} = b(k)L_K, L_{a^+(k)K} = b^+(k)L_K,$$

Another representation of $\mathcal{A}$ on $\mathcal{L}$. is specified by the operators

$$\tilde{b}^+(k) = -\hbar c_2^+(k) + c_1(k), \tilde{b}(k) = -c_2(k),$$

obeying CCR and satisfying

$$L_{Ka^+(k)} = \tilde{b}(k)L_K, L_{Ka(k)} = \tilde{b}^+(k)L_K,$$

Let us consider a Hamiltonian $\hat{H}$ in a space of representation of CCR. We will write $\hat{H}$ in the form

$$\hat{H} = \sum_{m,n} \sum_{k_i, l_j} H_{m,n}(k_1, ...k_m | l_1, ..., l_n) a^+_{k_1} ... a^+_{k_m} a_{l_1} ... a_{l_n} \tag{1}$$

There are two operators in $\mathcal{L}$ corresponding to $\hat{H}$:

$$\hat{H} = \sum_{m,n} \sum_{k_i, l_j} H_{m,n}(k_1, ...k_m | l_1, ..., l_n) b^+_{k_1} ... b^+_{k_m} b_{l_1} ... b_{l_n} \tag{2}$$

(we denote it by the same symbol) and

$$\tilde{H} = \sum_{m,n} \sum_{k_i, l_j} H_{m,n}(k_1, ...k_m | l_1, ..., l_n) \tilde{b}^+_{k_1} ... \tilde{b}^+_{k_m} \tilde{b}_{l_1} ... \tilde{b}_{l_n} \tag{3}$$

The equation of motion for the L-functional $L(\alpha^*, \alpha)$ has the form

$$i\hbar \frac{dL}{dt} = HL = \hat{H}L - \tilde{H}L$$

(We introduced the notation $H = \hat{H} - \tilde{H}$.) It corresponds to the equation for density matrices.

The equations of motion for L-functionals make sense even in the situation when the equations of motion in the Fock space are ill-defined (but there are no ultraviolet divergences). This is related to the fact that vectors and density matrices from all representations of CCR are described by L-functionals. This means that applying the formalism of L-functionals we can avoid the problems related to the existence of inequivalent representations of CCR.

In perturbation theory for translation-invariant Hamiltonians these problems appear as divergences related to infinite volume. Therefore in the standard formalism it is necessary to consider at first a Hamiltonian in finite volume V (to make volume cutoff or, in another terminology, infrared cutoff) and to take the limit $V \to \infty$ in physical quantities.

In the formalism of L-functionals we can work directly in infinite volume. We can define adiabatic S-matrix and adiabatic generalized Green functions repeating the standard definitions. If the adiabatic parameter a tends to zero then the adiabatic S-matrix multiplied by some factors tends to inclusive scattering matrix. The adiabatic Green functions in the formalism of L-functionals tend to GGreen functions. This gives a very simple derivation of the diagram techniques for the calculation of GGreen functions.