Lie group G , g , $g_1 g_2$

Lie algebra $\mathfrak{g}$, l_i $[l_1, l_2]$

$A(G)$ $f(g)$

$\mathcal{U}(\mathfrak{g})$ $P(l)$

$l \to K(l)$ vector field on G

$P(l) \to P(K(l))$ diff. operator

Pairing: $\langle f(g), P(l) \rangle =$

$= P(K(l)) f(g) \big|_{g=e}$

Comultiplication, bi-algebra

$\mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \qquad f(g) \to f(g_1 g_2)$

$\mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \qquad P(e) \to P(e \otimes I + I \otimes e)$

$\mathcal{A}$ commutative, noncocommutative

$\mathcal{U}$ noncommutative, cocommutative

„Quantization" – Deformation

lifts this degeneracy.

variables on G – noncommutative

commutation relations in $\mathfrak{g}$ –

– nonlinear

Interpretation of the quantum version

of Inverse Scattering Method

L3

Leningrad group, 1979 - 1984

P. Kulish, V. Korepin, L. Takhtajan, E. Sklyanin, N. Reshetikhin, A. Izergin, M. Semenov - Tian - Shansky, A. Reiman and later F. Smirnov, A. Volkov, A. Alexeev, S. Shatashvili, R. Kashaev, A. Bytsko ...

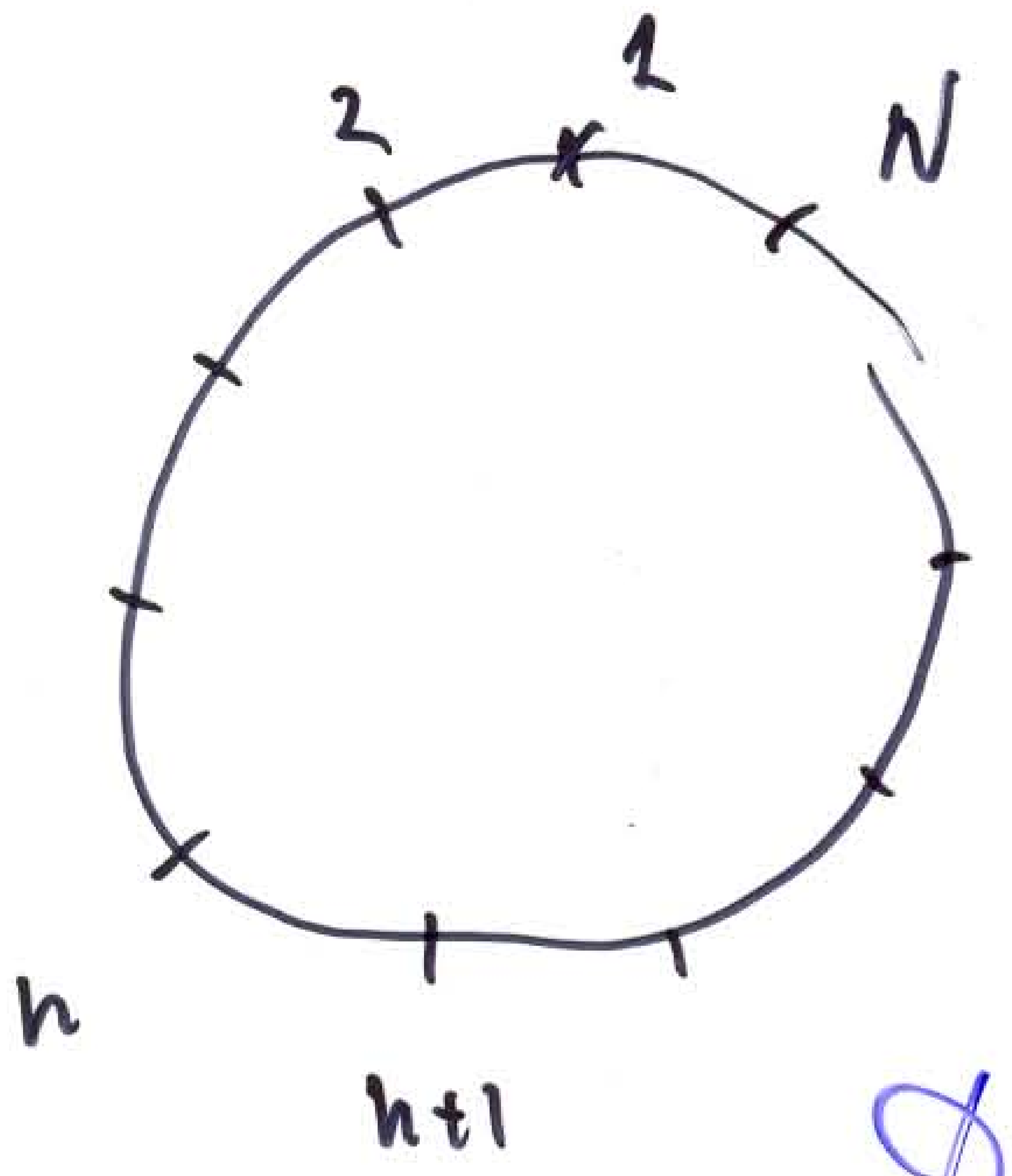
Spin chains Bethe, Hulthen

Lattice Models Lieb, Baxter

Factorizable Scattering Yang, Berezin, Mc. Guire, Brezin, Zinn - Justin

Inverse Scattering Method, Gardner - Green - Kruskal - Miura, Lax, Zakharov - Shabat, Zakharov - Faddeev

XXX spin chain



$$\mathcal{H} = \prod_{n=1}^{N} \otimes h_n$$

h_n - rep of $su(2)$

Dynamical variables $S_n^a = id \otimes id \dots \underset{n}{S^a} \dots id$

$a = 1, 2, 3$

$$[S_n^a, S_m^b] = i \varepsilon_{abc} S_n^c \delta_{mn}$$

spin $1/2$, $h_n = \mathbb{C}^2$

$$H = \sum_n H_{n,n+1}$$

$$H_{n,n+1} = S_n^1 S_{n+1}^1 + S_n^2 S_{n+1}^2 + S_n^3 S_{n+1}^3$$

$$S_{N+1}^a \equiv S_1^a$$

Casimir in $h_n \otimes h_{n+1}$

Change of variables, generating object – – Lax operator

Auxiliary space $V = \mathbb{C}^2$, $V \otimes h_n$

$$L_n(\lambda) = \begin{pmatrix} \lambda + i S_n^3 & i S_n^- \\ i S_n^+ & \lambda - i S_n^3 \end{pmatrix}$$

$$S_n^{\pm} = S_n^1 \pm i S_n^2$$

λ – „spectral parameter"

parallel transport along chain. Monodromy

$$M(\lambda) = \overrightarrow{\prod} L_n(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

„Spectral transform"

$$\{S_n^a\} \to M(\lambda)$$

FCR

$$R(\lambda-\mu)\, L_n^1(\lambda)\, L_n^2(\mu) = L_n^2(\mu)\, L_n^1(\lambda)\, R(\lambda-\mu)$$

$$V_1 \otimes V_2 \otimes h$$

$R \curvearrowright V \otimes V$, $L_n^1 \curvearrowright V_1 \otimes h$, $L_n^2 \curvearrowright V_2 \otimes h$

$$R(\lambda) = \begin{pmatrix} a(\lambda) & 0 & 0 & 0 \\ 0 & b(\lambda) & c(\lambda) & 0 \\ 0 & c(\lambda) & b(\lambda) & 0 \\ 0 & 0 & 0 & a(\lambda) \end{pmatrix}$$

$a(\lambda) = \lambda + i$, $b(\lambda) = \lambda$, $c(\lambda) = i$

$$R\, L_n^1 L_{n+1}^1 L_n^2 L_{n+1}^2 = R\, L_n^1 L_n^2 L_{n+1}^1 L_{n+1}^2 =$$

$$= L_n^2 L_n^1 L_{n+1}^2 L_{n+1}^1 R = L_n^2 L_{n+1}^2 L_n^1 L_{n+1}^1 R$$

$$R(\lambda-\mu)\, M_1(\lambda)\, M_2(\mu) = M_2(\mu)\, M_1(\lambda)\, R(\lambda-\mu)$$

$$\mathrm{tr}_V\, M(\lambda) = T(\lambda)$$

$$[T(\lambda), T(\mu)] = 0$$

$$H = \frac{d}{d\lambda} \ln T(\lambda) \Big|_{\lambda = -i/2}$$

Any spin, more complicated H.

XXZ spin $1/2$

$$H = S_n^1 S_{n+1}^1 + S_n^2 S_{n+1}^2 + \cos\gamma\, S_n^3 S_{n+1}^3$$

$$L_n(\lambda) = \frac{1}{\sin\gamma} \begin{pmatrix} \sinh(\lambda + i\gamma S_n^3) & i \sin\gamma\, S_n^- \\ i\sin\gamma\, S_n^+ & \sinh(\lambda - i\gamma S_n^3) \end{pmatrix}$$

Same FCR, $a = \sinh(\lambda + i\gamma)$, $b = \sinh\lambda$, $c = i\sin\gamma$

higher spin? Same FCR if

deform commutation relations

Kulish $[S^3, S^\pm] = \pm S^\pm$

Reshetikhin

1981 $[S^+, S^-] = \dfrac{\sinh 2\gamma S^3}{\sin\gamma}$

$[x]_q = \dfrac{\sinh \gamma x}{\sin\gamma}$, $q = e^\gamma$

1982 - Sklyanin , elliptic

1984 Takhtajan , L.F.

$$R_q T_1 T_2 = T_2 T_1 R_q$$

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \quad R_q = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

$ab = qba \quad ac = qca, \quad db = q^{-1} bd, \quad dc = q^{-1} cd$

$ad - da = (q - q^{-1}) bc \qquad bc = cb$

$\det_q T = ad - q bc$ - central

$$\Delta T = T \otimes T$$

$$\Delta T_{ik} = \sum T_{il} \otimes T_{lk}$$

Thus: $U_q(sl(2))$, $A_q(SL(2))$

q - deformation parameter

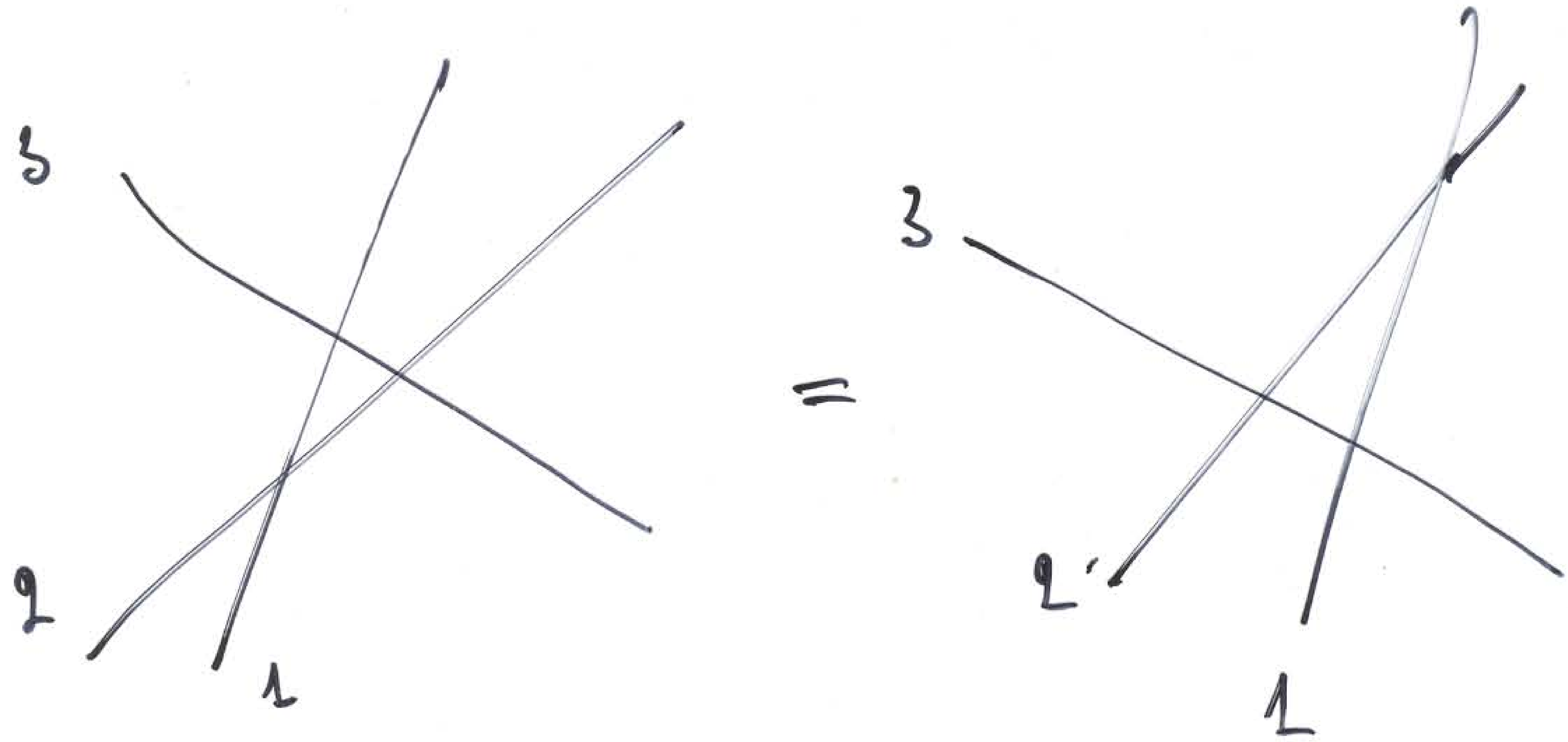
Including λ - affine groups

Consistency?

$$\begin{tikzcd}[column sep=small]
 & T_2T_1T_3 \arrow[r] & T_2T_3T_1 \arrow[dr] & \\
T_1T_2T_3 \arrow[ur] \arrow[dr] & & & T_3T_2T_1 \\
 & T_1T_3T_2 \arrow[r] & T_3T_1T_2 \arrow[ur] &
\end{tikzcd}$$

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

Yang - Baxter



$$P \; x \otimes y = y \otimes x$$

$$\hat{R} = P R$$

$$\hat{R}_{12} \hat{R}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{R}_{12} \hat{R}_{23}$$

Braid group.

Yang-Baxter relation

Bazhanov, Jimbo

Jimbo, Drinfeld

Chevalley-Serre

Woronovich

RTF (1987) treat A_q and U_q similarly

$$L^+ = \begin{pmatrix} q^{H/2} & (q-q^{-1})X_- \\ 0 & q^{-H/2} \end{pmatrix}$$

$$L^- = \begin{pmatrix} q^{-H/2} & 0 \\ -(q-q^{-1})X_+ & q^{H/2} \end{pmatrix}$$

$$R_\pm L_1^\pm L_2^\pm = L_2^\pm L_1^\pm R_\pm$$

$$R_+ L_1^+ L_2^- = L_2^- L_1^+ R_+$$

$$R_+ = P R_q P \,, \qquad R_- = R_q^{-1}$$

$$q^{H/2} X_\pm = q^{\pm 1} X_\pm q^{H/2}$$

$$X_+ X_- - X_- X_+ = \frac{q^H - q^{-H}}{q-q^{-1}}$$

$$\Delta L_+ = L_+ \otimes L_+ \; ; \qquad \Delta L_- = L_- \otimes L_-$$

$$\langle T, L^{\pm} \rangle = R_{\pm}$$

Drinfeld (1986)

Universal R-matrix $U_q \otimes U_q$

$$R \Delta R^{-1} = \Delta'$$

$$\Delta' = \sigma \cdot \Delta$$

$$R_q = (\rho \otimes \rho) R$$

$$R = q^{H \otimes H} \exp_q \{ (q - q^{-1})^2 X_+ \otimes X_- \}$$

$$\exp_q x = 1 + \sum \frac{q^{\frac{k(k-1)}{2}} x^k}{(q - q^{-1}) \cdots (q^k - q^{-k})} =$$

$$= \exp \sum \frac{(-1)^\ell x^\ell}{\ell (q^\ell - q^{-\ell})}$$

q- dilog

From 1986-87 lot of developments

Knots

CFT

q - special functions

Crystal basis

Correlation functions for integrable models

q - differential calculus

Noncommutative geometry

Modular double

$$\mathcal{B} = U_q \otimes U_{\tilde{q}}$$

$$q = e^{i\pi\tau}, \quad \tilde{q} = e^{i\pi/\tau}$$

$$S_q(x) = \frac{\exp_q(x)}{\exp_{\tilde{q}}(\tilde{x})}, \quad \tilde{x} = x^{1/\tau}$$

$\mathcal{B}$ $\quad q^H, X_+, X_-$

$\tilde{\mathcal{B}} = \tilde{q}^H, \quad X_+^{1/\tau}, \quad X_-^{1/\tau}$

$$[\mathcal{B}, \tilde{\mathcal{B}}] = 0$$

$$R = q^{H \otimes H} S_q\left((q-q^{-1})^2 X_+ \otimes X_-\right)$$

serves both $\mathcal{B}$ and $\tilde{\mathcal{B}}$.

Conclusion

Concrete problem in Mathematical Physics, leads to algebraic structure, which, when abstracted, gives a new mathematical object.

The best way to develope mathematics