! means that the problem is used in other problems or in lectures

* - a problem for extra credit

** - I do not know a solution of this problem (this does not mean that it is difficult, I did not try).

0!. A complex vector space $E$ is equipped with non-negative scalar product. Prove that we can obtain pre Hilbert space factorizing $E$ with respect to vectors with$\langle x, x\rangle = 0$.

Hint. Check that these vectors constitute a linear subspace of $E$.

Density matrices are defined as positive-definite self-adjoint operators having unit trace and acting in complex Hilbert space .

1. Prove that the set of density matrices is convex. Check that extreme points of this set are one-dimensional projectors $K_\Psi(x) = \langle x, \Psi\rangle \Psi$ where $||\Psi|| = 1$ (they are in one-to-one correspondence with non-zero vectors of Hilbert space with identification $\Psi \sim \lambda\Psi$).

2. Prove that the set of density matrices in two-dimensional Hilbert space is a three-dimensional ball and set of of its extreme points is a two-dimensional sphere.

3**.Prove that automorphisms of the set of density matrices are in one-to-one correspondence with unitary operators.

The linear envelope $\mathcal{T}$ of the set of density matrices is the space of all self-adjoint operators belonging to trace class. ( A self-adjoint operator belongs to trace class if it has discrete spectrum and the the series of its eigenvalues is absolutely convergent.) We consider $\mathcal{T}$ as a normed space with the norm $||T|| = \sum |\lambda_k|$ where $\lambda_k$ are eigenvalues of $T$. By definition an automorphism of the set of density matrices is a bicontinuous linear operator in $\mathcal{T}$ generating a bicontinuous map of the set density matrices. (One says that a map is bicontinuous if it is continuous and has continuous inverse.). It is obvious that a unitary operator specifies an automorphism of the set of density matrices by the formula $T \to UTU^{-1}$. One should prove that every automorphism has this form. In the next problems the term "operator" means "linear operator". It is convenient to define $e^{tA}$ where $A$ is an operator as a solution of the equation

$$\frac{dA(t)}{dt} = A \cdot A(t)$$

with initial condition $A(0) = 1$. If $(A_1, ..., A_n)$ is a family of commuting operators we define $e^{\sum t_i A_i}$ as a product $e^{t_1 A_1} ... e^{t_n A_n}$.

4. Prove that $e^{i\mathbf{aP}} = T_{\hbar\mathbf{a}}$, where $\mathbf{P}$ denotes the momentum operator $\mathbf{P} = \frac{\hbar}{i}\nabla$ and $T_{\mathbf{a}}$ stands for translation operator transforming the function $f(\mathbf{x})$ into a function $f(\mathbf{x} + \mathbf{a})$.

5. Let us define an operator $\mathcal{C}_A$ acting in the space of operators by the formula $\mathcal{C}_A(X) = [A, X]$. ( For definiteness one can assume that $A$ and $X$ are bounded operators in Hilbert space, but this assumption is not important.) Prove that

$$e^{t\mathcal{C}_A}(X) = e^{tA}Xe^{-tA},$$

or equivalently

$$e^{tA}Xe^{-tA} = ? + t[A, X] + \frac{t^2}{2!}[A, [A, X]] + \frac{t^3}{3!}[A, [A, [A, X]]] + ...$$

Hint. Differentiate these equalities.

6!*. Let us assume that the commutator of operators $X$ and $Y$ is a number $C$ (or, more generally, an operator $C$, commuting with $X$ and $Y$). Prove that

$$e^X e^Y = e^{X+Y} e^{\frac{1}{2}C},$$

$$e^X e^Y = e^C e^Y e^X.$$

7. Let us assume that for an operator $A$ acting in Banach space the norms of operators $e^{tA}$ where $t \in \mathbb{R}$ are uniformly bounded (i.e. $\sup_{-\infty<t<+\infty} ||e^{tA}||$ is finite). Prove that all eigenvalues of$A$ are purely imaginary.

8*. Let $A$ denote an operator acting in finite-dimensional complex vector space and the norms of operators $e^{tA}$ where $t \in \mathbb{R}$ are uniformly bounded. Prove that the operator $A$ is diagonalizable (i.e. there exists a basis consisting of eigenvectors of $A$).

Hint. Use Jordan normal form. Prove that all Jordan cells are one-dimensional.

9. Let us consider Grassmann algebra with generators $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$. Calculate

$$(\epsilon_1 + \epsilon_2)(\epsilon_2 + \epsilon_3)(\epsilon_3 + \epsilon_4),$$

$$\int (\epsilon_1 + \epsilon_2)(\epsilon_2 + \epsilon_3)(\epsilon_3 + \epsilon_4)(\epsilon_4 + \epsilon_1) d^4\epsilon,$$

$$\frac{\partial}{\partial \epsilon_2}(\epsilon_1 \epsilon_2 \epsilon_3),$$

$$\cos(\epsilon_1 \epsilon_2 + \epsilon_3 \epsilon_4).$$

10.Let us consider a unital associative algebra with commuting generators $x_1, ..., x_n$ and anticommuting generators $\xi_1, ..., \xi_n$ (tensor product of polynomial algebra and Grassmann algebra). Prove that $d^2 = 0$ and $\Delta^2 = 0$ where

$$d = \sum \xi_i \frac{\partial}{\partial x_i},$$

$$\Delta = \sum \frac{\partial}{\partial x_i} \frac{\partial}{\partial \xi_i}.$$

Weyl algebra is defined as unital associative algebra with generators $a_k^*, a_k$ obeying CCR( $[a_k, a_l^*] = \delta_{k,l}, [a_k, a_l] = [a_k^*, a_l^*] = 0$). An involution $^*$ in Weyl algebra transforms $a_k$ into $a_k^*$.

Fock representation of Weyl algebra=representation with cyclic vector $\theta$ obeying $\hat{a}_k \theta = 0$. Scalar product is defined by the condition that $\hat{a}_k^*$ is adjoint to $\hat{a}^k$

Fock space= completion of the space of Fock representation.

11. Poisson vector is defined by the formula$\Psi_\lambda = e^{\lambda \hat{a}^*} \theta$ where $\lambda \hat{a}^* = \sum \lambda_k \hat{a}_k^*$.

a) Check that Poisson vector is an eigenvector of the operator $\hat{a}_k$

b) Find scalar product of two Poisson vectors.

Hint. One can use the integral formula for scalar product given in the lecture in this problem and in Problem 12.

12. Prove that

$$\langle x, y \rangle = \int d\lambda^* d\lambda e^{-\lambda^* \lambda} \langle x, \Psi(\lambda^*) \rangle \langle \Psi(\lambda^*), y \rangle$$

13. Let us consider the Hamiltonian $\hat{H}(t) = \omega(t) \hat{a}^* \hat{a}$ (harmonic oscillator with time-dependent frequency).

a) Let us suppose that $\omega(t) = \omega_0 + \omega \exp(-\alpha t)$ for $t \geq 0$. (Here $\alpha$ is a small positive number.) Calculate the evolution of matrix entries of the density matrix in $\hat{H}$-representation (i.e. express matrix entries of $K(t)$ in terms of matrix entries of $K(0)$). Here $\hat{H} = \hat{H}(0)$.

b) Assuming that $\omega$ is a random variable with given mean value and dispersion uniformly distributed on some interval calculate the average of matrix entries of the density matrix $K(t)$ for $t = \frac{const}{\alpha}$.

14. A molecule is placed near a microwave oven. Give a rough estimate of decoherence time for this molecule.

Hint. Decoherence appears if the phase factors entering the expressions for non-diagonal matrix entries of density matrix are changed significantly by the electric field of the oven. You can calculate this change in perturbation theory; the first order contribution comes from dipole momentum. You can find the information about dipole momenta of ground state and excited states on the web; you need only the order of magnitude of these momenta and of electric field.

15. Let us define the $L$-functional corresponding to the density matrix $K$ by the the formula

$$L_K(\alpha) = tr e^{-\alpha \hat{a}^*} e^{\alpha^* \hat{a}} K.$$

(We consider the case when there is only one degree of freedom, $[\hat{a}, \hat{a}^*] = 1$.)

Calculate the $L$-functional corresponding to the coherent state ( to the normalized Poisson vector).

Reminder. Every normalized vector $\Phi$ defines a density matrix $K_\Phi$ and $trAK_\Phi = \langle A\Phi, \Phi\rangle$. Coherent state is a normalized eigenvector of $\hat{a}$.

16. Check the calculations on slides 21 and 22 of Lecture 4 and correct mistakes ( if there are mistakes).

17. Check the calculations on slides 18 and 19 of Lecture 5. Find relation between functions $c(\alpha, \beta)$ and $r(\alpha, \beta)$.

18**. Read Section 2A of the paper Takahashi, Kazutaka. "Quantum fluctuations in the transverse Ising spin glass model: A field theory of random quantum spin systems, Phys. Rev. B,76 (18), p.184422. Apply the techniques of Lecture 5 to obtain the functional integral derived in this section.

19** Lecture 5, slide 21