In quantum mechanics one can express the evolution operator and other quantities in terms of functional integrals. Today I prove this fact as well as corresponding results in geometric approach to quantum theory and in the formalism of L-functionals.

$L(q,\dot{q}) = \frac{m\dot{q}^2}{2} - V(q), H(p,q) = \frac{p^2}{2m} + V(q),$
$S[q(\tau)] = \int_0^t d\tau L(q(\tau), \dot{q}(\tau))$-action functional
$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}),\ \hat{U}(t) = e^{-\frac{it\hat{H}}{\hbar}}$
$\langle q_2|\hat{U}(t)|q_1\rangle$ *(matrix element of evolution operator) is represented by functional integral with integrand*

$$e^{i\frac{S[q(\tau)]}{\hbar}}$$

and integration domain consisting of functions $q(\tau)$ *obeying* $q(0) = q_1, q(t) = q_2$

To define the functional integral we approximate it with finite-dimensional integral (for example replacing integrals by integral sums and taking the limit.

Problems:

1) depends on the choice of approximation

2) the limit usually does not exist and to get a finite answer we should disregard infinite contributions

Gaussian integrals

$\int \exp +\langle b, x\rangle)dx = (\det A)^{-\frac{1}{2}} \exp(-\frac{1}{2}\langle A^{-1}b, b\rangle,$

$\int \exp(\frac{i}{2}\langle Ax, x\rangle + i\langle b, x\rangle)dx =$

$(\det A)^{-\frac{1}{2}} \exp(-\frac{i}{2}\langle A^{-1}b, b\rangle,$

Differentiating with respect to b we can calculate

$\int P(x) \exp(-\frac{1}{2}\langle Ax, x\rangle)dx$

where $P(x)$ is a polynomial

Perturbation theory

If $W(x) = Q(x) + gV(x)$ where Q is quadratic then

$$\frac{\int e^{W(x)dx}}{\int e^{Q(x)dx}}$$

can be represented as a series with respect to g as a sum of Feynman diagrams

More generally we can decompose $W(x)$ in a neighborhood of non-degenerate critical point x_0 as a sum of quadratic part and a part containing only monomials with respect to $x - x_0$ having degrees ≥ 3 and use the same techniques.

In geometric approach the evolution operator of physical system obeys the equation of motion

$$\frac{d\sigma}{dt} = H(t)\sigma(t)$$

where $H(t)$ is a linear operator acting in Banach space (or, more, generally, in complete topological vector space) $\mathcal{L}$. We say that $H(t)$ is the "Hamiltonian" of the physical system. In what follows we assume that $H(t) = H$ does not depend on time t. This condition is imposed only to simplify notations; all results can be proved also for time- dependent "Hamiltonian".

In the standard approach to QM the evolution operator acts in Hilbert space and $H(t)$ is a self-adjoint operator multiplied by i.

Let us consider linear operators acting in the space $\mathcal{L}$. A symbol of an operator A is a function $\underline{A}$ defined on some measure space. It should depend linearly of A. We assume that the symbol of the identity operator 1 is equal to 1 and the composition of operators corresponds to the operation on symbols denoted by $*$: if $C = AB$ then $\underline{C} = \underline{A} * \underline{B}$.

The evolution operator can be represented in the form

$$\sigma(t) = e^{tH} = \lim_{N\to\infty}(1 + \frac{tH}{N})^N.$$

For $N \to \infty$ the symbol of the operator $1 + \frac{tH}{N}$ can be approximated by $\exp \frac{t}{N}\underline{H}$:

$$\underline{1 + \frac{tH}{N}} = e^{\frac{t}{N}\underline{H}} + O(N^{-2}).$$

Using this relation we obtain an expression for the symbol of the evolution operator;

$$\underline{\sigma(t)} = \lim_{N\to\infty} I_N(t)$$

$$I_N(t) = e^{\frac{t}{N}\underline{H}} * \ldots * e^{\frac{t}{N}\underline{H}}$$

(N factors)

In many cases $I_N(t)$ can be interpreted as an approximation to a functional integral. Notice, however, that even without this interpretation we can apply Laplace or stationary phase method to calculate $I_N(t)$. This allows us to obtain some results that often are obtained in the language of functional integrals without using this language.

Let us consider a class of symbols generalizing $q - p$ symbols and Wick symbols in quantum mechanics. We assume that the symbol of an operator A acting in $\mathcal{L}$ is a function $\underline{A}(\alpha, \beta)$ of two variables (a function on $\mathcal{M} \times \mathcal{M}'$) and that the symbol of the product C of operators A and B can be expressed in terms of the symbols of operators A and B by the formula

$$\underline{C}(\alpha, \beta) = \int d\gamma d\gamma' \underline{A}(\alpha, \gamma)\underline{B}(\gamma', \beta)e^{c(\alpha,\gamma)+c(\gamma',\beta)-c(\alpha,\beta)-r(\gamma',\gamma)}$$

where $c(\alpha, \beta)$ and $r(\alpha, \beta)$ are functions on $\mathcal{M} \times \mathcal{M}'$.

For $q - p$-symbols $c(\mathbf{q}, \mathbf{p}) = r(\mathbf{q}, \mathbf{p}) = -i\mathbf{pq}$.

It follows that the symbol $\underline{C}(\alpha,\beta)$ of the product C of n operators $A_1, ..., A_N$ is given by the formula
$\underline{C}(\alpha,\beta) = \int d\gamma_1 d\gamma'_1 ... d\gamma_{N-1} d\gamma'_{N-1} \times$
$\underline{A}_1(\alpha,\gamma_1)\underline{A}_2(\gamma'_1,\gamma_2)...\underline{A}_n(\gamma'_{N-1},\beta)e^{\rho_N}$
where
$\rho_N = c(\alpha,\gamma_1) + c(\gamma'_1,\gamma_2) + ... + c(\gamma'_{N-1},\beta) - c(\alpha,\beta)) - r(\gamma'_1,\gamma_1) - ... - r(\gamma_{N-1},\gamma_{N-1})$

We see that in our case
$I_N(t) = \int d\gamma_1 d\gamma_1' ... d\gamma_{N-1} d\gamma_{N-1} e^{\rho_N} \times$
$\exp(\frac{t}{N}(\underline{H}(\alpha, \gamma_1) + \underline{H}(\gamma_1, \gamma_2) + ... + \underline{H}(\gamma_{N-1}, \beta))$

The simplest way to construct symbols of operators in quantum mechanics is to use the fact that the Fourier transform of delta-function is a constant. The matrix (the kernel in the language of mathematics) of unit operator is $\langle \mathbf{q}_2|1|\mathbf{q}_1\rangle = \delta(\mathbf{q}_1 - \mathbf{q}_2)$ in coordinate representation and $\langle \mathbf{p}_2|1|\mathbf{p}_1\rangle = \delta(\mathbf{p}_1 - \mathbf{p}_2)$ in momentum representation. Taking Fourier transform of matrix $\langle \mathbf{q}_2|A|\mathbf{q}_1\rangle$ of the operator A with respect to the variable $\mathbf{q}_1 - \mathbf{q}_2$ we obtain $q - p$ symbol:

$$\underline{A}^{q-p}(\mathbf{q}, \mathbf{p}) = \int d\mathbf{y} \langle \mathbf{y}|A|\mathbf{q}\rangle e^{i\mathbf{p}(q-\mathbf{y})}$$

Similarly taking Fourier transform of $\langle \mathbf{p}_2|A|\mathbf{p}_1\rangle$ with respect to variable $\mathbf{p}_1 - \mathbf{p}_2$ we obtain $p - q$-symbol.

If A is a differential operator with polynomial coefficients we can express it as a polynomial of operators $\hat{q}^j$ (operators corresponding to the coordinates q^j) and $\hat{p}_j = \frac{1}{i}\frac{\partial}{\partial q^j}$ (momentum operators) Representing A in $q-p$ form (coordinate operators from the left of momentum operators) and "removing hats" we obtain $q-p$-symbol. Notice that in our notations $\hbar = 1$. Sometimes it it is convenient to consider families of symbols $\underline{A}_{\hbar}^{q-p}(\mathbf{q},\mathbf{p})$ and $\underline{A}_{\hbar}^{p-q}(\mathbf{q},\mathbf{p})$ depending on parameter $\hbar$.

Let us illustrate general considerations above on the example of $q - p$-symbols in conventional quantum mechanics. For $q - p$-symbols we obtain that the symbol of the evolution operator can be calculated as $\lim_{N\to\infty} I_N(\mathbf{q}, \mathbf{p}, t)$ where
$I_N(\mathbf{q}, \mathbf{p}, t) = \int \prod_1^{N-1} d\mathbf{q}_\alpha d\mathbf{p}_\alpha \times$
$\exp(i \sum_1^N (\mathbf{p}_\alpha(\mathbf{q}_\alpha - \mathbf{q}_{\alpha-1}) - \frac{it}{N} \sum_1^N \underline{H}(\mathbf{p}_\alpha, \mathbf{q}_{\alpha-1}))$
with $\mathbf{p}_N = \mathbf{p}, \mathbf{q}_0 = \mathbf{q}_N = \mathbf{q}$.
The integrand can be represented as e^{iS_N} where S_N is an integral sum for the integral
$S[\mathbf{p}(\tau), \mathbf{q}(\tau)] = \int_0^t (\mathbf{p}(\tau)\dot{q}(\tau) - \underline{H}(\mathbf{p}(\tau), \mathbf{q}(\tau))d\tau.$
This integral can be interpreted as action functional.

We can say that the $q-p$-symbol of evolution operator can be represented as functional integral with the integrand

$$e^{iS[\mathbf{p}(\tau),\mathbf{q}(\tau)]}.$$

The integration domain is the set of functions $(\mathbf{p}(\tau),\mathbf{q}(\tau))$ obeying conditions

$$\mathbf{p}(t)=\mathbf{p},\mathbf{q}(0)=\mathbf{q}(t)=\mathbf{q}.$$

For matrix elements of the evolution operator we obtain a functional integral with the same integrand and with integration domain consisting of functions obeying conditions
$\mathbf{q}_(0)=\mathbf{q}_1,\mathbf{q}_(t)=\mathbf{q}_2$

If $\underline{H}(\mathbf{p}, \mathbf{q})$ is a sum of quadratic function of $\mathbf{p}$ (kinetic energy) and a function $V(\mathbf{q})$ (potential energy) we can integrate over $\mathbf{p}(\tau)$ and obtain a representation of matrix elements of evolution operator in the form of functional integral with the integrand

$$e^{iS[\mathbf{q}(\tau)]} = e^{i(\int_0^t d\tau (T(\dot{\mathbf{q}}(\tau)) - V(\mathbf{q}(\tau)))}$$

and the integration domain consists of functions $\mathbf{q}(\tau)$ obeying conditions
$\mathbf{q}_(0) = \mathbf{q}_1, \mathbf{q}_(t) = \mathbf{q}_2.$
Here T stands for kinetic energy expressed in terms of $\dot{\mathbf{q}}(\tau)$.

Covariant symbols

Let us consider two Banach spaces $\mathcal{L}$ and $\mathcal{L}'$ and non-degenerate scalar product $\langle l, l' \rangle$ that is linear with respect to $l \in \mathcal{L}$ and antilinear with respect to $l' \in \mathcal{L}'$. Let us fix two systems of vectors $e_\alpha \in \mathcal{L}$ and $e'_\beta \in \mathcal{L}'$ such that

$$\langle l, l' \rangle = \int \langle l, e'_\mu \rangle \langle e_\lambda, l' \rangle e^{-r(\lambda,\mu)} d\lambda d\mu$$

(we assume that $r(\lambda, \mu)$ is a function on measure space $\mathcal{M} \times \mathcal{M}'$).

Let us define covariant symbol $\underline{A}(\alpha, \beta)$ of operator A acting in $\mathcal{L}$ by the formula

$$\underline{A}(\alpha, \beta) = \frac{\langle Ae_\alpha, e_\beta \rangle}{\langle e_\alpha, e_\beta \rangle}$$

Then the symbol $\underline{C}$ of the product $C = AB$ of operators A and B is equal to
$\underline{C}(\alpha, \beta) = \int d\lambda d\mu B(\alpha, \mu) A(\lambda, \beta) \times$
$\exp(-r(\lambda, \mu) - c(\alpha, \beta) + c(\alpha, \mu) + c(\lambda, \beta))$
(We introduced notation $\langle e_\alpha, e'_\beta \rangle = e^{c(\alpha, \beta)}$)
$\langle ABe_\alpha, e_\beta \rangle = \langle Be_\alpha, A^* e_\beta \rangle$

If $\mathcal{L} = \mathcal{L}'$ is a space of Fock representation one can take e_α as Poisson vectors: $e_\alpha = e^{\alpha \hat{a}^*}\theta$. Then $c(\alpha, \beta) = r(\alpha, \beta) = \langle \alpha, \beta \rangle$.
Wick symbol.
More generally we can take as $\mathcal{L}$ (as $\mathcal{L}'$) the smallest linear space containing all Poisson vectors e_α with $\alpha \in L^p$ (with $\alpha \in L^q = (L^p)^*$)
Here $\frac{1}{p} + \frac{1}{q} = 1, L^q$ is dual to L^p.

L-functionals
$\mathcal{L}'$-Weyl algebra, $e'_\alpha = V_\alpha$-linear exponents
$\mathcal{L}$ - *L*-functionals, e_β -*L*-functionals corresponding to Poisson vectors in Fock space. ? Quadratic exponents?

L-functionals. Another definition

Take representation of Weyl algebra (of CCR) in Hilbert space $\mathcal{H}$. (We understand CCR as relations $[a_k, a_l^+] = \delta_{kl}, [a_k, a_l] = [a_k^+, a_l^+] = 0$, where k, l run over a discrete set M.)

To a density matrix K (or more generally to any trace class operator in $\mathcal{H}$) we can assign a functional $L_K(\alpha^*, \alpha)$ defined by the formula

$$L_K(\alpha^*, \alpha) = Tr e^{-\alpha a^+} e^{\alpha^* a} K$$

Here αa^+ stands for $\sum \alpha_k a_k^+$ and $\alpha^* a$ for $\sum \alpha_k^* a_k$.

One can say that L_K is a generating functional of correlation functions.
One can consider also a more general case when CCR are written in the form

$$[a(k), a^+(k')] = \hbar\delta(k, k'), [a(k), a(k')] = [a^+(k), a^+(k')] =$$

k, k' run over a measure space M. If α is square-integrable, the expression for L_K is well defined.

An action of Weyl algebra $\mathcal{A}$ on $\mathcal{L}$ (on the space of L-functionals) can be specified by operators

$$b^+(k) = \hbar c_1^+(k) - c_2(k), b(k) = c_1(k)$$

obeying CCR. Here $c_i^+(k)$ are multiplication operators by α_k^*, α_k and $c_i(k)$ are derivatives with respect to α_k^*, α_k. This definition is prompted by relations
$L_{a(k)K} = b(k)L_K, L_{a^+(k)K} = b^+(k)L_K,$
Another representation of $\mathcal{A}$ on $\mathcal{L}$ is specified by the operators

$$\tilde{b}^+(k) = -\hbar c_2^+(k) + c_1(k), \tilde{b}(k) = -c_2(k),$$

obeying CCR and satisfying
$L_{Ka^+(k)} = \tilde{b}(k)L_K, L_{Ka(k)} = \tilde{b}^+(k)L_K,$

Let us consider a Hamiltonian $\hat{H}$ in a space of representation of CCR. We will write $\hat{H}$ in the form

$$\hat{H} = \sum_{m,n} \sum_{k_i, l_j} H_{m,n}(k_1, ...k_m | l_1, ..., l_n) a^+_{k_1} ... a^+_{k_m} a_{l_1} ... a_{l_n}$$

There are two operators in $\mathcal{L}$ corresponding to $\hat{H}$:

$$\hat{H} = \sum_{m,n} \sum_{k_i, l_j} H_{m,n}(k_1, ...k_m | l_1, ..., l_n) b^+_{k_1} ... b^+_{k_m} b_{l_1} ... b_{l_n}$$

(we denote it by the same symbol) and

$$\tilde{H} = \sum_{m,n} \sum_{k_i, l_j} H_{m,n}(k_1, ...k_m | l_1, ..., l_n) \tilde{b}^+_{k_1} ... \tilde{b}^+_{k_m} \tilde{b}_{l_1} ... \tilde{b}_{l_n}$$

The equation of motion for the L-functional $L(\alpha^*, \alpha)$ has the form

$$i\hbar \frac{dL}{dt} = HL = \hat{H}L - \tilde{H}L$$

(We introduced the notation $H = \hat{H} - \tilde{H}$.)
It corresponds to the equation for density matrices.

For translation-invariant Hamiltonians $H_{m,n}$ should contain $\delta(k_1 + ... + k_m - l_1 - ... - l_n)$ (momentum conservation)

The equations of motion for L-functionals make sense even in the situation when the equations of motion in the Fock space are ill-defined (but there are no ultraviolet divergences). This is related to the fact that vectors and density matrices from all representations of CCR are described by L-functionals. This means that applying the formalism of L-functionals we can avoid the problems related to the existence of inequivalent representations of CCR.

In perturbation theory for translation-invariant Hamiltonians these problems appear as divergences related to infinite volume. Therefore in the standard formalism it is necessary to consider at first a Hamiltonian in finite volume V (to make volume cutoff or, in another terminology, infrared cutoff) and to take the limit $V \to \infty$ in physical quantities. In the formalism of L-functionals we can work directly in infinite volume.

Adiabatic approximation in the formalism of L-functionals is simpler. Let us consider a family $H(g)$ of "Hamiltonians" and a smooth family of stationary states $\omega(g)$. (For example we can take $H(g) = H_0 + gV$.) Then $\omega(g(t))$ is a solution of the equations of motion for non-stationary "Hamiltonian" $H(g(t))$ if we can disregard $\dot{g}(t)$: $\frac{d\omega(g(t))}{dt} = 0 = H(g(t))\omega(g(t))$.

Consider the operator $\sigma_\alpha(t, t_0)$ describing the evolution from the time t_0 until the time t under the "Hamiltonian" $H_0 + ge^{-\alpha|t|}V$. Then

$$\omega(g) = \lim_{\alpha \to 0} \sigma_\alpha(0, -\infty)\omega(0).$$

We can define adiabatic S-matrix as
$\sigma_\alpha(+\infty, -\infty)$.
If the adiabatic parameter α tends to zero then the adiabatic S-matrix multiplied by some factors tends to inclusive scattering matrix.

Let us define generalized Green functions (GGreen functions) in the translation-invariant state ω by the following formula where $B_i \in \mathcal{A}$:

$$G_n = \omega(NM)$$

where

$$N = T(B_1(\mathbf{x}_1, t_1) \dots B_n(\mathbf{x}_n, t_n))$$

stands for chronological product (times decreasing) and

$$M = T^{opp}(B_1^*(\mathbf{x}_1', t_1') \dots B_n^*(\mathbf{x}_n', t_n'))$$

stands for antichronological product (times increasing).

Introduce notations $M = T^{opp}(B'^*), N = T(B)$.

In the formalism of L-functionals

$$(T(B\tilde{B}')\omega)(x) = \omega(T(B)xT^{opp}(B'^{*})) == \omega(NxM)$$

To get GGreen function take $x = 1$.

Hence we can apply the formalism of L-functionals to calculate GGreen functions. This gives a very simple derivation of the diagram techniques for the calculation of GGreen functions.