Error estimates for a splitting integrator for semilinear boundary coupled systems

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Introduction.

In the present talk we show the derivation of a numerical method, based on operator splitting, to abstract parabolic semilinear boundary coupled systems. The method decouples the linear components which describe the coupling and the dynamics in the bulk and on the surface, and treats the nonlinear terms by approximating the integral in the variation of constants formula. The convergence proof is based on estimates for a recursive formulation of the error, using the parabolic smoothing property of analytic semigroups and a careful comparison of the exact and approximate flows. Numerical experiments, including problems with dynamic boundary conditions, reporting on convergence rates are presented. The talk is based on the results in [1].

Setting of the problem.

We consider the abstract semilinear boundary coupled systems of the form:

$$\begin{cases} \dot u(t) = Au(t) + \mathcal{F}_1(u(t), v(t)) & \text{for } t \in (0,T], \quad u(0) = u_0 \in E, \\ \dot v(t) = Bv(t) + \mathcal{F}_2(u(t), v(t)) & \text{for } t \in (0,T], \quad v(0) = v_0 \in F, \\ Lu(t) = v(t) & \text{for } t \in [0,T], \end{cases}$$

where A, B are linear operators on the Banach spaces E and F, respectively, $\mathcal{F}_1$, $\mathcal{F}_2$ are suitable functions, and the two unknown functions u and v are related via the linear coupling operator L acting between (subspaces of) E and F. A typical setting would be that $L : E \to F$ is a *trace-type operator* between the space E (for the bulk dynamics) and the *boundary* space F (for the surface dynamics).

As a first step, following the idea described in [2], we rewrite the problem as an abstract Cauchy problem on the product space $E \times F$ with the notations $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)\colon \mathcal{D} \to E \times F$ and $\boldsymbol{u} := (u, v)\colon [0,T] \to E \times F$ as

$$\begin{cases} \dot{\boldsymbol{u}}(t) = \mathcal{A}\boldsymbol{u}(t) + \mathcal{F}\big(\boldsymbol{u}(t)\big)\big) & \text{for } t \in (0,T], \\ \boldsymbol{u}(0) = (u_0, v_0) \end{cases}$$

with

$$\mathcal{A} := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{with} \quad D(\mathcal{A}) := \big\{ \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in D(A) \times D(B) \colon Lx = y \big\}.$$

The numerical method.

We are now in the position to derive the numerical method. For a time step $\tau > 0$, for all $t_n = n\tau \in [0,T]$, we define the numerical approximation $\boldsymbol{u}_n = (u_n, v_n)$ to $\boldsymbol{u}(t_n) = (u(t_n), v(t_n))$ via the following steps.

Step 1. We approximate the integral in the variation of constants formula, giving the form of the exact solution $\boldsymbol{u}(t)$, by an appropriate quadrature rule.

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Step 2. We approximate the semigroup operators generated by $\mathcal{A}$ by using an operator splitting method. Due to its special form, this includes the approximation of a convolution. The choice of this approximate operator is determined by the used splitting method, see [3, Section 3].

We use first-order approximations in *Steps 1–2*, and show the first-order convergence of the combined method. We propose, as a first step into this direction, a Lie splitting scheme for abstract semilinear boundary coupled systems, where the semilinear term $\mathcal{F}$ is locally Lipschitz (and might include feedback). An important feature of our splitting method is that it *separates the flows on E and F*, i.e. separates the bulk and surface dynamics. This could prove to be a considerable computational advantage if the bulk and surface dynamics are fundamentally different (e.g. fast and slow reactions, linear–nonlinear coupling, etc.). The convergence result is based on studying stability and consistency, however, these two issues cannot be separated as in most convergence proofs, since this would lead to sub-optimal error estimates. Instead, the error is rewritten using recursion formula which, using the parabolic smoothing property of the analytic semigroup, leads to an induction process to ensure that the numerical solution stays within a strip around the exact solution.

A particular difficulty lies in the fact that the numerical method for the linear subproblems needs to approximate a convolution term in the exact flow as in [3], therefore the stability of these approximations cannot be merely established based on semigroup properties. Estimates from [3] together with new technical results yield an abstract first-order error estimate for semilinear problems (with a logarithmic factor in the time step), under suitable (local Lipschitz-type) conditions on the nonlinearities.

The main result.

The convergence result includes an error estimate as well.

Theorem 1. In the above setting, let $\boldsymbol{u} : [0,T] \to E \times F$ be the solution of the abstract Cauchy problem subject to certain conditions to be presented in the talk, and consider the approximations $\boldsymbol{u}_n$ at time t_n determined by the numerical method derived in the talk. Then there exists a $\tau_0 > 0$ and $C > 0$ such that for any time step $\tau \le \tau_0$ we have at time $t_n = n\tau \in [0,T]$ the error estimate

$$\|\boldsymbol{u}(t_n) - \boldsymbol{u}_n\| \le C\,\tau\,|\log(\tau)|.$$

The constant $C > 0$ is independent of n and $\tau > 0$, but depends on T, on constants related to the semigroups included, as well as on the exact solution $\boldsymbol{u}$.

We applied the novel numerical method to semilinear problems, even with dynamic boundary condition which is so far exluded from our assumptions. The numerical experiments illustrate the proved error estimates in all cases, even the dynamic boundary condition example complements our theoretical results.

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