Pointwise and uniform convergence of Chernoff approximations to C_0-semigroups

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Introduction. This talk is devoted to the rate of convergence of Chernoff approximations [1, 2] to strongly continuous one-parameter semigroups [3, 4, 5]. We provide simple natural examples for which this convergence: is arbitrary high; is arbitrary slow; holds in the strong operator topology (so we have pointwise convergence) but does not hold in the norm operator topology (so we do not have uniform convergence on the unit ball). We also prove general theorem that gives estimate from above for the speed of decay of the norm of the residual appearing in Chernoff approximations. We provide also supplementary theorems which makes it easier to check the conditions of the main theorem. This talk is based on the results that are contained in the preprint [6] and the paper is submitted to a journal.

It is usually difficult to express the C_0-semigroup $\left(e^{tL}\right)_{t\geq 0}$ it terms of variable coefficients of operator L. However, if the so-called Chernoff function [1] is constructed, i.e. an operator-valued function G which satisfies the conditions of the Chernoff theorem (in particular, satisfies $G(t)f = f + tLf + o(t)$ as $t \to +0$ for all f from dense linear subspace of $\mathcal{F}$), then the semigroup can be given by the equality $e^{tL} = \lim_{n\to\infty} G(t/n)^n$. An advantage of this approach arises from the fact that often it is possible to construct G by an explicit and not very long formula which contains coefficients of operator L. Expressions $G(t/n)^n$ are called *Chernoff approximations* to e^{tL}. We have constructed [6] the following examples.

Examples of arbitrary high and arbitrary slow convergence

Proposition 1. There exists a Banach space $\mathcal{F}$, C_0-semigroup $(e^{tL})_{t\geq 0}$ in $\mathcal{F}$ with generator $(L, D(L))$, and Chernoff function G for operator $(L, D(L))$ such that Chernoff approximations converge on each vector (i.e. converge pointwise) but do not converge in operator norm (i.e. do not converge uniformly in the unit ball). More precisely:

1. $\lim_{n\to\infty} \|G(t/n)^n f - e^{tL} f\| = 0$ for all $f \in \mathcal{F}$ and all $t \geq 0$,
2. $\|e^{tL}\| = \|G(t)\| = 1$ for all $t \geq 0$,
3. for each $t > 0$ and each $n \in \mathbb{N}$ there exists $f_n \in \mathcal{F}$ such that $\|f_n\| = 1$ and the inequality $\|G(t/n)^n f_n - e^{tL} f_n\| \geq 1$ holds, so $\|G(t/n)^n - e^{tL}\| \geq 1 \not\to 0$ as $n \to \infty$.

Proposition 2. There exists C_0-semigroup $(e^{tL})_{t\geq 0}$ in Banach space $\mathcal{F}$, Chernoff function G and vector $f_0 \in \mathcal{F}$ such that $\|f_0\| = 1$ and $f_0 \notin D(L)$ but the speed of convergence on this vector is arbitrary high. More precisely: for arbitrary chosen non-increasing continuous function $v\colon [0, +\infty) \to [0, +\infty)$ vanishing at infinity at arbitrary high rate (e.g. $v(x) = (1+x)^{-k}$, $v(x) = e^{-x}$, $v(x) = e^{-e^x}$) and all $T > 0$ we have: $\sup_{t\in[0,T]} \|G(t/n)^n f_0 - e^{tL} f_0\| = Tv(n/T)$ for all $n = 1, 2, 3, \ldots$ such that $Tv(n/T) \leq 1$. Moreover, we have $\|e^{tL}\| = \|G(t)\| = 1$.

Proposition 3. There exists C_0-semigroup $(e^{tL})_{t\geq 0}$ in Banach space $\mathcal{F}$, Chernoff function G and vector $f_1 \in \mathcal{F}$ such that $\|f_1\| = 1$ and $f_1 \in \cap_{j=1}^{\infty} D(L^j)$ but the speed of convergence on this vector is arbitrary low, i.e. for arbitrary chosen non-increasing continuous function $v\colon [0, +\infty) \to [0, +\infty)$ vanishing at infinity at arbitrary low rate (e.g. $v(x) = (1 + x)^{-1/k}$, $v(x) = 1/\log(x + e)$, $v(x) = 1/\log(\log(x + e^e))$) and all $T > 0$ we have:
$\sup_{t\in[0,T]} \|G(t/n)^n f_1 - e^{tL} f_1\| = Tv(n/T)$ for all $n = 1, 2, 3, \ldots$ such that $Tv(n/T) \leq 1/2$. Moreover, we have $\|e^{tL}\| = \|G(t)\| = 1$.

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Main result. Our main result is stated as follows.

Theorem 1. Suppose that

1. Number $T > 0$ is given, and C_0-semigroup $(e^{tA})_{t\geqslant 0}$ with generator $(A, D(A))$ in Banach space $\mathcal{F}$ satisfies for some $M_1 \geq 1$ and $w \geq 0$ the condition $\|e^{tA}\| \leqslant M_1 e^{wt}$ for all $t \in [0, T]$.

2. There is a mapping $S\colon (0, T] \to \mathscr{L}(\mathcal{F})$, i.e. $S(t)\colon \mathcal{F} \to \mathcal{F}$ is a linear bounded operator for each $t \in (0, T]$. There exists some constant $M_2 \geq 1$ that $\|S(t)^k\| \leqslant M_2 e^{kwt}$ for all $t \in (0, T]$ and all $k = 1, 2, 3, \ldots$

3. Numbers $m \in \{0, 1, 2, \ldots\}$ and $p \in \{1, 2, 3, \ldots\}$ are fixed. There is a $(e^{tA})_{t\geqslant 0}$-invariant subspace $\mathcal{D} \subset D(A^{m+p}) \subset \mathcal{F}$, i.e. $(e^{tA})(\mathcal{D}) \subset \mathcal{D}$ for any $t \geq 0$ (for example $\mathcal{D}$ may be equal to $D(A^{m+p})$).

4. There exist such functions $K_j\colon (0, T] \to [0, +\infty)$, $j = 0, 1, \ldots, m+p$ that for all $t \in (0, T]$ and all $f \in \mathcal{D}$ we have

$$\left\| S(t)f - \sum_{k=0}^{m} \frac{t^k A^k f}{k!} \right\| \leqslant t^{m+1} \sum_{j=0}^{m+p} K_j(t) \|A^j f\|.$$

Then:

1. For all $t > 0$, all integer $n \geq t/T$ and all $f \in \mathcal{D}$ we have

$$\|S(t/n)^n f - e^{tA} f\| \leq \frac{M_1 M_2 t^{m+1} e^{wt}}{n^m} \sum_{j=0}^{m+p} C_j(t/n) \|A^j f\|,$$

where $C_{m+1}(t) = K_{m+1}(t)e^{-wt} + M_1/(m+1)!$ and $C_j(t) = K_j(t)e^{-wt}$ for $j \neq m+1$.

2. If $\mathcal{D}$ is dense in $\mathcal{F}$ and for all $j = 0, 1, \ldots, m+p$ we have $K_j(t) = o(t^{-m})$ when $t \to +0$, then for all $g \in \mathcal{F}$ and $\mathcal{T} > 0$ the following equality is true:

$$\lim_{\mathcal{T}/T \leq n \to \infty} \sup_{t \in (0, \mathcal{T}]} \left\| S(t/n)^n g - e^{tA} g \right\| = 0.$$

Example 1. Let us consider particular modeling example. Suppose $\|e^{tA}\| \leq e^t$, $\|S(t)\| \leq e^t$, $\|S(t)f - f - tAf - \frac{1}{2}t^2A^2 f\| \leq t^2\sqrt{t}\|A^3 f\|$ for all $f \in D(A^3)$ and $t \in (0; 1]$. Then $\mathcal{D} = D(A^3)$, $m = 2$, $M_1 = M_2 = w = 1$, $K_0(t) = K_1(t) = 0$, $K_2(t) = 1/\sqrt{t}$ for any $t \in (0; 1]$. So estimate in the item of theorem 1 states that for any fixed $t > 0$ the following estimate is true for all $f \in D(A^3)$ and integer $n \geq t$, having the following asymptotic behaviour as $n \to \infty$:

$$\|S(t/n)^n f - e^{tA} f\| \leq \frac{t^3 e^t}{n^2} \left(\frac{1}{\sqrt{t/n}} + \frac{e^{t/n}}{3!} \right) \|A^3 f\| =$$

$$= e^t \left(\frac{t^2\sqrt{t}}{n\sqrt{n}} + \frac{e^{t/n} t^3}{6n^2} \right) \|A^3 f\| = \frac{t^2 \sqrt{t} e^t}{n\sqrt{n}} \|A^3 f\| + O\left(\frac{1}{n^2}\right).$$

Application of the main result. Let us use the symbol $UC_b(\mathbb{R})$ for the space of all bounded, uniformly continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ with the norm $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$. Let us use symbol $HC_b(\mathbb{R})$ for the space of all bounded, Hölder continuous functions $u\colon \mathbb{R} \to \mathbb{R}$, and for each $n \in \{1, 2, 3, \ldots\}$ let us denote by $HC_b^n(\mathbb{R})$ the space of all such functions $u \in HC_b(\mathbb{R})$, that $u', \ldots, u^{(n)} \in HC_b(\mathbb{R})$. It is clear that $HC_b^n(\mathbb{R}) \subset UC_b(\mathbb{R})$ and $HC_b^n(\mathbb{R})$ is dense in $UC_b(\mathbb{R})$ for all $n \in \{1, 2, 3, \ldots\}$. Similarly, the space $UC_b^n(\mathbb{R})$ is defined. The theorem 1 implies the following proposition.

Theorem 2. Suppose that

1. Numbers $m, q \in \{1, 2, 3, \dots\}$ are fixed, and $\hat{q} = 2\lfloor (q+1)/2 \rfloor$. Functions a, b, c from the class $HC_b^{2m+\hat{q}-2}(\mathbb{R})$ are given, and $\inf_{x\in\mathbb{R}} a(x) > 0$. Operator L on $UC_b(\mathbb{R})$ with domain $D(L) = HC_b^2(\mathbb{R})$ is defined by the formula

$$Lu = au'' + bu' + cu.$$

2. Numbers $T > 0$, $M \geq 1$ and $\sigma \geq 0$ are given. For any $t \in (0, T]$ a bounded linear operator $S(t)$ on $UC_b(\mathbb{R})$ is defined such that $\|S(t)^k\| \leqslant Me^{k\sigma t}$ for all $k = 1, 2, 3, \dots$.

3. There exist nonnegative constants $K_0, K_1, \dots, K_{2m+q}$ such that for all $t \in (0, T]$ and all $f \in UC_b^{2m+q}(\mathbb{R})$ we have

$$\left\| S(t)f - \sum_{k=0}^{m} \frac{t^k L^k f}{k!} \right\| \leqslant t^{m+1} \sum_{j=0}^{2m+q} K_j \|f^{(j)}\|.$$

Then:

1. The closure $\overline{L}$ of operator L is a generator of C_0-semigroup $(e^{t\overline{L}})_{t\geqslant 0}$ in Banach space $UC_b(\mathbb{R})$, and the condition $\|e^{t\overline{L}}\| \leqslant e^{\gamma t}$ for all $t \geq 0$ is satisfied, where $\gamma = \max(0, \sup_{x\in\mathbb{R}} c(x))$.

2. For all $t > 0$, all integer $n \geq t/T$ and all $f \in UC_b^{2m+\hat{q}}(\mathbb{R})$ we have

$$\|S(t/n)^n f - e^{t\overline{L}} f\| \leq \frac{Mt^{m+1}e^{wt}}{n^m} \sum_{j=0}^{2m+\hat{q}} C_j \|f^{(j)}\|,$$

where $w = \max(\sigma, \gamma)$, $\hat{q} = 2\lfloor (q+1)/2 \rfloor$ and $C_0, C_1, \dots, C_{2m+\hat{q}}$ are nonnegative constants that are independent of t and n.

3. For all $g \in UC_b(\mathbb{R})$ and all $\mathcal{T} > 0$ the following equality is true:

$$\lim_{\mathcal{T}/T \leq n \to \infty} \sup_{t\in(0,\mathcal{T}]} \left\| S(t/n)^n g - e^{t\overline{L}} g \right\| = 0.$$

The following result is an example of usage of the theorem 2.

Example 2. Suppose that functions $a, b, c \in HC_b^2(\mathbb{R})$ are given such that $\inf_{x\in\mathbb{R}} a(x) > 0$. For each $u \in UC_b^2(\mathbb{R})$ set $Au = au'' + bu' + cu$. For each $t \geq 0$, each $f \in UC_b(\mathbb{R})$ and each $x \in \mathbb{R}$ set

$$(S(t)f)(x) = \frac{1}{4} f\left(x + 2\sqrt{a(x)t}\right) + \frac{1}{4} f\left(x - 2\sqrt{a(x)t}\right) + \\ + \frac{1}{2} f\left(x + 2b(x)t\right) + tc(x)f(x).$$

Then there exist nonnegative constants $C_0, C_1, \dots, C_4$ such that for all $t > 0$, all $n \in \{1, 2, 3, \dots\}$ and all $f \in UC_b^4(\mathbb{R})$ the following inequality holds:

$$\|S(t/n)^n f - e^{t\overline{A}} f\| \leq \\ \leq \frac{t^2 e^{\|c\|t}}{n} \big(C_0\|f\| + C_1\|f'\| + C_2\|f''\| + C_3\|f'''\| + C_4\|f^{(IV)}\|\big).$$

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