$\Omega = \sum \Omega^n$ -algebra of differential forms on domain $G \subset \mathbb{R}^m$
$\Omega = C^\infty(G) \otimes \Lambda^m$ (Grassmann algebra with coefficients depending on $x \in G$)
Differential n-form $\omega = \frac{1}{n!}\omega_{i_1,\ldots,i_n}(x)\xi^{i_1}\ldots\xi^{i_n}$
The coefficients are antisymmetric
$\xi^i = dx^i$, $\xi^i\xi^j = -\xi^j\xi^i$
$d = \xi^i \frac{\partial}{\partial x^i} = dx^i \frac{\partial}{\partial x^i}$
$d\omega = 0$-form is closed$=\omega \in Kerd$
$d^2 = 0$ hence $Imd \subset Kerd$,
Cohomology $H = Kerd/Imd$
Smooth manifold is pasted together from domains in $\mathbb{R}^n$

Symplectic manifold is specified by closed non-degenerate two-form $\omega = \frac{1}{2}\omega_{ij}(x)dx^i dx^j$
$d\omega = 0, \det \omega_{ij} \neq 0$
Phase space is a symplectic manifold with
$\omega = dp_i dq^i$
Locally for appropriate change of coordinates
$\omega = dp_i dq^i$ (Darboux coordinates)
Symplectomorphisms (canonical transformations) preserve the form ω
If $f : M' \to M$ is an embedding of symplectic manifold M' with two-form ω' into symplectic manifold M with two-form ω and $f^*\omega = \omega'$ we say that f is a symplectic embedding.

Solitons

Let us consider a translation-invariant Hamiltonian on a phase space consisting of vector-valued functions $f(\mathbf{x})$. We assume that corresponding equations of motion have the form

$$\frac{\partial f}{\partial t} = Af + B(f).$$

(Here A is a linear operator, B is a non-linear operator. Spatial translations act on functions as shifts of arguments.)

We assume B is at least quadratic, hence for small f the linear part dominates. In particular, $f \equiv 0$ is a solution. We say that a soliton (or solitary wave) is a finite energy solution of our equation having the form $s(\mathbf{x} - \mathbf{v}t)$.

Soliton is a bump moving with constant speed without changing the form. Generalized soliton is a bump that pulsates moving with constant average speed
For almost all initial data asymptotically we get several solitons and a tail obeying a linear equation
Grand conjecture (Soffer), Soliton resolution conjecture (Tao)

$D^+(t) : \mathcal{R} \to \mathcal{R}_{as}$
Initial data at the moment t to asymptotic data at $t \to +\infty$ (solitons, solution of linear equation)
$D^-(t) : \mathcal{R} \to \mathcal{R}_{as}$
Data at the moment t to asymptotic data at
$t \to -\infty$
$(D^+(t))^{-1} = S(t, +\infty), (D^-(t))^{-1} = S(t, -\infty)$
Asymptotic data $\to$ initial data at the moment t
Non-linear scattering matrix

$$S = S(0, +\infty)^{-1} S(0, -\infty) : \mathcal{R}_{as} \to \mathcal{R}_{as}$$

If the theory is Lorenz-invariant then applying Lorenz transformations to a soliton we get a family of solitons.
Similar statement for Galilean invariance.

Let us consider a symplectic manifold $\mathcal{M}$ with action of commutative group $\mathcal{T}$ of space and time translations. Let us fix a a $\mathcal{T}$-invariant (= stationary translation-invariant) point $m \in \mathcal{M}$.
A point $n \in \mathcal{M}$ is an excitation of m if it has finite energy (we assume that the energy of m is equal to zero).
We say that $\mathcal{E}$ is an elementary symplectic manifold if $\mathcal{T}$ acts on $\mathcal{E}$ by symplectomorphisms and in Darboux coordinates $\mathbf{p}, \mathbf{x}$ the spatial translations act by the formula $\mathbf{x} \to \mathbf{x} + \mathbf{a}$, $\mathbf{p} \to \mathbf{p}$. (Then the time translations act by the formula $\mathbf{x} \to \mathbf{x} + \mathbf{v}(\mathbf{p})t$,$\mathbf{p} \to \mathbf{p}$. Here $\mathbf{v}(\mathbf{p}) = \nabla \epsilon(\mathbf{p})$, where $\epsilon(\mathbf{p})$ stands for the Hamiltonian.)

A symplectic embedding of an elementary symplectic space into the set of excitations of m commuting with space and time translations specifies a family of solitons

Assume that $\mathcal{M}$ consists of vector-valued functions $f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^d$ and spatial translations act as shifts $\mathbf{x} \to \mathbf{x} + \mathbf{a}$. A translation-invariant point m is a constant function, we can take $m = 0$. A symplectic embedding $\mathcal{E} \to \mathcal{M}$ commuting with spatial translations sends $(\mathbf{p}, 0)$ into a function $s_{\mathbf{p}}(\mathbf{x})$ and $(\mathbf{p}, \mathbf{a})$ into the function $s_{\mathbf{p}}(\mathbf{x} + \mathbf{a})$. The assumption that this map commutes with time translations means that $s_{\mathbf{p}}(\mathbf{x} - \mathbf{v}(\mathbf{p})t)$ satisfies the equations of motion.

To define (quasi)particles and scattering in geometric approach we need spatial translation $T_{\mathbf{a}}$ and time translations T_τ of cone of states $\mathcal{C}$ (a homomorphism of the group $\mathcal{T}$ of space and time translations into the group of automorphisms of the cone).

In algebraic approach we need an action of $\mathcal{T}$ on the algebra $\mathcal{A}$ (a homomorphism of $\mathcal{T}$ into the group of automorphisms of $\mathcal{A}$). It induces an action on the cone.

We introduce notation $A(\tau, \mathbf{x})$ for $T_\tau T_{\mathbf{x}} A$ where $A \in \mathcal{A}$. A state ω is translation- invariant and stationary iff $\omega(A(\tau, \mathbf{x})) = \omega(A)$.

If $\mathcal{A}$ is Weyl algebra with generators $\hat{a}^*(\mathbf{x}), \hat{a}(\mathbf{x})$ obeying CCR (coordinate representation) then spatial translations act as shifts $\mathbf{x} \to \mathbf{x} + \mathbf{a}$ and the time translation is specified by formal Hamiltonian where the coefficient functions are fast decreasing functions of differences $\mathbf{x}_i - \mathbf{x}_j$

If $\mathcal{A}$ is Weyl algebra with generators $\hat{a}^*(\mathbf{k}), \hat{a}(\mathbf{k})$ obeying CCR (momentum representation) then spatial translations act as multiplication by $\exp(\pm i\mathbf{ka})$ and the time translation is specified by formal Hamiltonian where the coefficient functions are smooth functions multiplied by δ-functions coming from momentum conservation.

If we have translation symmetry we define an excitation of translation-invariant stationary state $\omega \in \mathcal{C}$ as a state σ obeying
$(T_{\mathbf{a}}\sigma)(A) \to \text{const} \cdot \omega(A)$
In algebraic approach we can apply GNS construction to define pre Hilbert space $\mathcal{H}$ with cyclic vector Φ corresponding to ω (obeying $\omega(A) = \langle \hat{A}\Phi, \Phi \rangle$). The translations $T_{\mathbf{a}}$ and T_τ descend to $\mathcal{H}$ as unitary operators.
Momentum and energy operators are defined by formulas
$T_{\mathbf{a}} = e^{i\hat{\mathbf{P}}\mathbf{a}}, T_\tau = e^{-i\hat{H}\tau}$

Cluster property

$\lim_{\mathbf{x}\to\infty} \omega(A(\tau,\mathbf{x})B) = \omega(A)\omega(B)$

$\lim_{\mathbf{x}\to\infty} \omega(B'A(\tau,\mathbf{x})B) = \omega(A)\omega(B'B)$

$\lim_{\mathbf{x}\to\infty} \dot{\omega}(A(\tau,\mathbf{x})B) = \dot{\omega}(A)\omega(B)$

It follows from cluster property that a state $\sigma(A) = \langle \hat{A}\hat{B}\Phi, \hat{B}\Phi\rangle = \omega(B^*AB)$ *corresponding to a vector* $B\Phi \in \mathcal{H}$ *is an excitation of* ω

$(T_{\mathbf{x}}\sigma)(A) = \sigma(A(0,\mathbf{x}) = \omega(B^*A(0,\mathbf{x})B) \to \omega(A)\omega(B^*B)$

as $\mathbf{x} \to \infty$.

In algebraic approach we identify excitations of ω *with elements of* $\mathcal{H}$.

Quasi-particles=elementary excitations of translation-invariant stationary state ω
Particles=elementary excitations of ground state
Thermal quasi-particles=elementary excitations of equilibrium state

Elementary excitations in algebraic approach

$\hat{\mathbf{P}}\Phi(\mathbf{p}) = \mathbf{p}\Phi(\mathbf{p}) \sim T_{\mathbf{a}}\Phi(\mathbf{p}) = e^{i\mathbf{pa}}\Phi(\mathbf{p})$,

$\hat{H}\Phi(\mathbf{p}) = \epsilon(\mathbf{p})\Phi(\mathbf{p}) \sim T_{\tau}\Phi(\mathbf{p}) = e^{-i\epsilon\tau}\Phi(\mathbf{p})$

$\Phi(\mathbf{p})$- generalized function

$\Phi(\phi) = \int d\mathbf{p}\phi(\mathbf{p})\Phi(\mathbf{p})$ where $\phi(\mathbf{p})$ is a test function

Normalization condition

$\langle\Phi(\mathbf{p}), \Phi(\mathbf{p}')\rangle = \delta(\mathbf{p} - \mathbf{p}')$ or

$\langle\Phi(\phi), \Phi(\phi')\rangle = \langle\phi, \phi'\rangle$

If there are several types of excitations $\Phi(\mathbf{p})$ and $\epsilon(\mathbf{p})$ depend on discrete index.

If we have rotational invariance then in three-dimensional space this index takes $2s + 1$ values for a particle having spin s

We define elementary space $\mathfrak{h}$ as a space of test functions $\phi_a(\mathbf{x})$ where spatial translations act shifting the argument. The test functions take values in $\mathbb{C}^r$

In momentum representation spatial translations act as multiplication by $e^{i\mathbf{ka}}$ and time translations as multiplication by $e^{-iE(\mathbf{k})\tau}$. Here $E(\mathbf{k})$ stands for Hermitian $r \times r$ matrix. Diagonalizing the matrix $E(\mathbf{k})$ we can reduce the general case to the case $r = 1$.

For $r = 1$ elementary space is a quantization of elementary symplectic manifold.

An elementary excitation of translation-invariant stationary state ω (a quasiparticle) is specified by a map σ of $\mathfrak{h}$ into the set of excitations. This map should commute with translations.

In algebraic approach $\sigma(\phi) = \Phi(\phi)$ is a linear map.

In geometric approach $(\sigma(\phi))(A) = \langle A\Phi(\phi), \Phi(\phi)\rangle$

If $\Phi(\phi) = B(\phi)\Phi$ then $\sigma(\phi) = L(\phi)\omega$ where $L(\phi) = \widetilde{B^*}(\phi)B^*(\phi)$

Only translation symmetry is relevant in the definition of scattering, but if there are other symmetries we can impose additional conditions on the state ω and on the map σ.
In particular, in relativistic quantum field theory it is natural to assume that the ground state and the corresponding vector Φ are Poincaré-invariant, that Poincaré group acts on elementary space $\mathfrak{h}$ specifying an irreducible representation and the map σ commutes with this action.

Let us consider translation-invariant Hamiltonian of non-relativistic quantum mechanics in Fock space. In this case

$$\Phi(\mathbf{p}) = \hat{a}^*(\mathbf{p})\theta$$

is an elementary excitation (a particle).

If

$\int d\mathbf{p}_1 ... d\mathbf{p}_n \Psi(\mathbf{p}_1, ...\mathbf{p}_n)\delta(\mathbf{p}_1 + ... + \mathbf{p}_n) \times$
$\hat{a}^*(\mathbf{p}_1)...\hat{a}^*(\mathbf{p}_n)\theta$

is a bound state then

$\Phi(\mathbf{p}) = \int d\mathbf{p}_1 ... d\mathbf{p}_n \Psi(\mathbf{p}_1, ...\mathbf{p}_n) \times$
$\delta(\mathbf{p} - \mathbf{p}_1 - ... - \mathbf{p}_n)\hat{a}^*(\mathbf{p}_1)...\hat{a}^*(\mathbf{p}_n)\theta$

is an elementary excitation (composite particle).

One particle state $\Phi(\phi) = B(\phi)\Phi$
$T_\tau \Phi(\phi) = \Phi(T_\tau \phi)$
In momentum representation
$(T_\tau \phi)(\mathbf{k}) = e^{-i\epsilon(\mathbf{k})\tau} \phi(\mathbf{k})$
In coordinate representation
$(T_\tau \phi)(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{x}\mathbf{k} - i\epsilon(\mathbf{k})\tau} \phi(\mathbf{k})$
for large $|\tau|$ is small outside the set τU where U is defined as ϵ-neighborhood of the set of points $\mathbf{v}(\mathbf{k}) = \nabla \epsilon(\mathbf{k})$ where $\phi(\mathbf{k}) \neq 0$.
We say that τU is an essential support of $(T_\tau \phi)(\mathbf{x})$ for large $|\tau|$.