

# Scattering in geometric approach to quantum theory.

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First five sections of this paper were used in Lecture 7

## Abstract

We define inclusive scattering matrix in the framework of geometric approach to quantum field theory. We review the definitions of scattering theory in the algebraic approach and relate them to the definitions in geometric approach.

## 1 Introduction

Geometric approach to quantum theory where the starting point is the set of states was suggested in [1],[2]. In this approach one can work with convex set  $\mathcal{C}_0$  of normalized states or with convex cone  $\mathcal{C}$  of not necessarily normalized states ( proportional points of the cone  $\mathcal{C}$  specify the same state). In present paper we discuss scattering theory in geometric approach. Our starting point is a cone  $\mathcal{C}$  and a group  $\mathcal{V}$  consisting of automorphisms of the cone  $\mathcal{C}$ . Sometimes we will use also a semiring  $\mathcal{W}$  consisting of endomorphisms of the cone.

We review geometric and algebraic approaches to quantum theory and the relation between these approaches. We give definitions of scattering matrix and inclusive scattering in algebraic approach. This makes the present paper independent of papers [1],[2] and of the papers [?], [3] devoted to the scattering in algebraic approach.

Let us recall the relation of the geometric approach to the algebraic approach to quantum theory [2]. In algebraic approach a starting point is an associative

algebra  $\mathcal{A}$  with involution  $*$ . The cone  $\mathcal{C} \subset \mathcal{A}^*$  of not necessarily normalized states is defined as a set of linear functionals on  $\mathcal{A}$  obeying  $f(A^*A) \geq 0$ . Every element  $B \in \mathcal{A}$  specifies two operators on  $\mathcal{A}^*$ ; one of them, denoted by the same symbol  $B$ , transforms a functional  $f(A)$  into the functional  $f(AB)$ , another, denoted by the symbol  $\tilde{B}$ , transforms  $f(A)$  into the functional  $f(B^*A)$ . The operator  $\tilde{B}B$  is an endomorphism of the cone  $\mathcal{C}$ . The set of endomorphisms of a cone is a semiring (it is closed with respect to addition and composition of operators). We define  $\mathcal{V}$  as a group of involution preserving automorphisms of  $\mathcal{A}$  acting in natural way on  $\mathcal{C}$ . The semiring  $\mathcal{W}$  is defined as the minimal set of endomorphisms of  $\mathcal{C}$  containing all endomorphisms of the form  $\tilde{B}B$  and closed with respect to addition and composition (it is closed also with respect to multiplication on positive number as all semirings we consider).

To define scattering in any approach to quantum theory we need the notions of time and spatial translations. In algebraic approach translations (as any symmetries) are automorphisms of the algebra  $\mathcal{A}$ ; these automorphisms induce the automorphisms of the cone  $\mathcal{C}$  and other objects related to the algebra  $\mathcal{A}$ . In geometric approach translations should be regarded as elements of the group  $\mathcal{V}$  consisting of automorphisms of the cone  $\mathcal{C}$ ; their action on the cone induces an action on  $\mathcal{W}$ . Particles and quasiparticles are defined as elementary excitations of stationary translation-invariant state  $\omega$ .

In algebraic approach one can define the notion of scattering matrix of elementary excitations. It seems that it is impossible to generalize this notion to geometric approach, however, there exists a very natural definition of inclusive scattering matrix of elementary excitations of stationary translation-invariant state  $\omega$  in geometric approach. It is easy to show that this notion agrees with analogous notion in algebraic approach.

Notice that our constructions can be applied also to scattering of quasiparticles in equilibrium and non-equilibrium statistical physics. ( The conventional scattering matrix does not make sense in this situation, but the inclusive scattering matrix does; see [5],[3]).

In [4] we apply the notions of present paper to define scattering in the framework of Jordan algebras.

## 2 Geometric approach

In geometric approach to quantum theory we start with a convex closed cone  $\mathcal{C}$  of (non-normalized) states in Banach space  $\mathcal{L}$  (or, more generally, in complete topological linear space  $\mathcal{L}$ ). We fix a subgroup  $\mathcal{V}$  consisting of automorphisms of the cone  $\mathcal{C}$ .

In some cases it is useful to add to this data a subsemiring  $\mathcal{W}$  of the semiring  $\text{End}\mathcal{C}$  of endomorphisms of the cone. ( By definition an endomorphism is a continuous linear operator in  $\mathcal{L}$  transforming the cone into itself. An automorphism is an invertible endomorphism.)

The dynamics in quantum theory is governed by one-parameter group of time translations  $T_\tau$  acting on the cone  $\mathcal{C}$ . We assume that  $T_\tau \in \mathcal{V}$ . Time translations can be considered also as transformations of  $\mathcal{W}$  denoted by the same symbol  $T_\tau$ . If  $A \in \mathcal{W}$  the time translation acts as conjugation:  $T_\tau(A) = T_\tau A T_{-\tau}$ ; we will use the notation  $T_\tau(A) = A(\tau)$ .

Quantum field theory in geometric approach is specified by a cone  $\mathcal{C}$  with the action of spatial translations  $T_{\mathbf{x}}$  and time translations  $T_\tau$  (these translations should constitute a commutative subgroup of the group  $\mathcal{V}$ .) The same data specify statistical physics in the space  $\mathbb{R}^d$  where  $d$  stands for the dimension of the group of spatial translations. We use the notations

$$T_\tau T_{\mathbf{x}}(A) = T_\tau T_{\mathbf{x}} A T_{-\tau} T_{-\mathbf{x}} = A(\tau, \mathbf{x})$$

for an operator  $A$  acting in  $\mathcal{L}$ .

Let us discuss the relation of the above definitions to the quantum theory in the algebraic approach. In this approach as in geometric one we need time and spatial translations to define elementary excitations and scattering. The time translations  $T_\tau$  and spatial translations  $T_{\mathbf{x}}$  act as automorphisms of  $\mathcal{A}$ ; these automorphisms induce automorphisms of the cone  $\mathcal{C}$  and semiring  $\mathcal{W}$  denoted by the same symbols. If  $\omega \in \mathcal{C}$  is a translation-invariant stationary state we can consider a representation of  $\mathcal{A}$  on a pre Hilbert space  $\mathcal{H}$  such that there exists a cyclic vector  $\theta \in \mathcal{H}$  obeying  $\omega(A) = \langle \theta, A\theta \rangle$ . (This representation is called GNS representation. We denote an operator in this representation corresponding to  $A \in \mathcal{A}$  by the same symbol  $A$ .) We can consider also the representation of  $\mathcal{A}$  in the Hilbert space  $\bar{\mathcal{H}}$  (in the completion of  $\mathcal{H}$ ). Time and spatial translations descend to  $\mathcal{H}$  and to  $\bar{\mathcal{H}}$ .

For every vector  $\Psi$  in the Hilbert space  $\bar{\mathcal{H}}$  we define the corresponding state  $\sigma$  by the formula  $\sigma(A) = \langle \Psi, A\Psi \rangle$ . If  $\Psi = \theta$  we have  $\sigma = \omega$ , if  $\Psi = B\theta$  we have  $\sigma = \tilde{B}B\omega$ .

### 3 Elementary excitations

Let us repeat the definitions and statements from [2] with small modifications.

We consider a translation-invariant stationary state  $\omega \in \mathcal{C}$ . We start with the definition of excitation of  $\omega$  in geometric approach. *We say that  $\sigma \in \mathcal{C}$  is an excitation of  $\omega$  if  $T_{\mathbf{x}}\sigma$  tends to  $C\omega$  as  $\mathbf{x}$  tends to  $\infty$  for some constant  $C$ .* ( We have in mind

weak convergence in this definition. Recall that  $u$  is a weak limit of  $u_\alpha$  if for every continuous linear functional  $f$  on  $\mathcal{L}$  the limit of  $f(u_\alpha)$  is equal to  $f(u)$ .) We say that proportional elements of a cone specify the same state, hence this condition means that for large  $\mathbf{x}$  the state  $T_{\mathbf{x}}\sigma$  coincides with  $\omega$ .

*An elementary excitation of  $\omega$  is defined as a map  $\sigma : \mathfrak{h} \rightarrow \mathcal{C}$  of an elementary space  $\mathfrak{h}$  into the set of excitations of  $\omega$ . This map should commute with translations and satisfy the following additional requirement: one can define a map  $L : \mathfrak{h} \rightarrow \text{End}(\mathcal{L})$  obeying  $\sigma(\phi) = L(\phi)\omega$ .*

Here  $\text{End}(\mathcal{L})$  denotes the space of continuous linear operators acting in  $\mathcal{L}$ .

Later we impose some extra conditions on the operators  $L(\phi)$ . Not very precisely one can say that the operators  $L(\phi)$  and  $L(\psi)$  should almost commute if supports of  $\phi$  and  $\psi$  are far away (see (4) and (7) for precise formulation).

Recall that *elementary space*  $\mathfrak{h}$  is defined as a space of smooth real-valued or complex-valued functions on  $\mathbb{R}^d \times \mathcal{I}$  with all derivatives decreasing faster than any power; it is equipped with  $L^2$  metric (here  $\mathcal{I}$  denotes a finite set consisting of  $m$  elements). The spatial translations act naturally on this space; we assume that the time translations also act on  $\mathfrak{h}$  and commute with spatial translations. In momentum representation an element  $\phi$  of  $\mathfrak{h}$  should be considered as a complex function of  $\mathbf{k} \in \mathbb{R}^d$  and discrete variable  $i \in \mathcal{I}$ . If  $\mathfrak{h}$  consists of real-valued functions then in momentum representation we should impose the condition  $\bar{\phi}(-\mathbf{k}) = \phi(\mathbf{k})$ . The spatial translation  $T_{\mathbf{x}}$  is represented as multiplication by  $e^{i\mathbf{x}\mathbf{k}}$  and the time translation  $T_\tau$  is represented as a multiplication by a matrix  $e^{-i\tau E(\mathbf{k})}$  where  $E(\mathbf{k})$  is a non-degenerate Hermitian matrix obeying  $E(-\mathbf{k}) = -E(\mathbf{k})$  if  $\mathfrak{h}$  consists of real-valued functions. We assume that  $E(\mathbf{k})$  a smooth function of at most polynomial growth, then the multiplication by  $E(\mathbf{k})$  is an operator acting in  $\mathfrak{h}$ . The eigenvalues of  $E(\mathbf{k})$  are denoted by  $\epsilon_s(\mathbf{k})$ .

Let us denote by  $U_\phi$  where  $\phi \in \mathfrak{h}$  an open subset of  $\mathbb{R}^d$  containing all points having the form  $\nabla\epsilon_s(\mathbf{k})$  where  $\mathbf{k}$  belongs to  $\text{supp}(\phi) = \cup_j \text{supp}(\phi_j)$  (to the union of supports of the functions  $\phi(\mathbf{k}, j)$ ).

**Lemma 1.** *Let us assume that  $\text{supp}(\phi)$  is a compact subset of  $\mathcal{R}$ . Then for large  $|\tau|$  we have*

$$|(T_\tau\phi)(\mathbf{x}, j)| < C_n(1 + |\mathbf{x}|^2 + \tau^2)^{-n}$$

where  $\frac{\mathbf{x}}{\tau} \notin U_\phi$ , the initial data  $\phi = \phi(\mathbf{x}, j)$  is the Fourier transform of  $\phi(\mathbf{k}, j)$ , and  $n$  is an arbitrary integer.

The proof of this lemma ( Lemma 2 in [2]) can be given by means of the stationary phase method.

We can express Lemma 1 saying that  $\tau U_\phi$  is an essential support of  $(T_\tau\phi)(\mathbf{x}, j)$  for large  $|\tau|$ .

In algebraic approach we define an elementary excitation of  $\omega$  as an isometric map  $\Phi$  of elementary space  $\mathfrak{h}$  into the space of GNS representation  $\mathcal{H}$  commuting with time and spatial translations. This definition agrees with the definition in geometric approach. To verify this fact we notice that the assumption that  $\theta$  is a cyclic vector implies the existence of operators  $B(\phi)$  obeying  $\Phi(\phi) = B(\phi)\theta$ . (Here  $\phi \in \mathfrak{h}$ .) We define a map  $\sigma : \mathfrak{h} \rightarrow \mathcal{C}$  saying that  $\sigma(\phi)$  is a linear functional on  $\mathcal{A}$  assigning a number  $\langle \Phi(\phi), A\Phi(\phi) \rangle$  to  $A \in \mathcal{A}$ . The map  $\sigma$  is quadratic if we are working over  $\mathbb{R}$ , it is Hermitian if we are working over  $\mathbb{C}$ . It commutes with time and spatial translations. Representing  $\sigma(\phi)$  in the form  $\sigma(\phi) = L(\phi)\omega$  where  $L(\phi) = \tilde{B}(\phi)B(\phi) \in \mathcal{W}$  we obtain that this map specifies an elementary excitation in geometric approach.

*We assume that  $B(\phi)$  is linear with respect to  $\phi$ ;* then  $L(\phi)$  is quadratic or Hermitian.

We say that a map  $\sigma$  of real vector spaces is quadratic if the expression  $\sigma(u + v) - \sigma(u) - \sigma(v)$  is linear with respect to  $u$  and  $v$ . A map  $\sigma$  of complex vector spaces is Hermitian if  $\sigma(u + v) - \sigma(u) - \sigma(v)$  is linear with respect to  $u$  and antilinear with respect to  $v$ . If  $V$  is a real vector space then the corresponding cone  $C(V)$  is defined as a convex envelope of the set of vectors of the form  $v \otimes v$  in the tensor square  $V \otimes V$ . (If we are dealing with topological vector spaces we should consider the closure of convex envelope.) A quadratic map  $V \rightarrow V'$  induces a linear map of the cone  $C(V) \rightarrow V'$ , a quadratic map of  $V$  into a cone  $C' \subset V'$  induces a linear map of cones  $C(V) \rightarrow C'$ . Similar statements are true for complex vector spaces and Hermitian maps. (The cone corresponding to complex vector space is defined as a convex envelope of the set of vectors of the form  $f \otimes \bar{f}$  in the tensor product  $V \otimes \bar{V}$ .) If  $V$  is a Hilbert space the corresponding cone can be identified with the cone of positive definite self-adjoint operators.

It is natural to assume that in geometric approach the maps  $\sigma$  and  $L$  are quadratic or Hermitian, but this assumption is not used in most of our statements.

Elementary excitations should be identified with particles or quasiparticles. Notice that particles and quasiparticles can be unstable; this means that we should consider also objects that only approximately obey the conditions we imposed on elementary excitations. The definition of inclusive scattering matrix given in the next section works also for such objects, but instead of the time  $\tau$  tending to  $\pm\infty$  we should consider large, but finite  $\tau$ . (This is true also for the conventional scattering matrix in algebraic approach.)

## 4 Scattering. Møller matrices.

Let us define the operator  $L(f, \tau) \in \mathcal{W}$  where  $f \in \mathfrak{h}$  by the formula

$$L(f, \tau) = T_\tau(L(T_{-\tau}f)) = T_\tau L(T_{-\tau}f)T_{-\tau}.$$

(We are using the same notation for time translations in  $\mathcal{C}$  and in  $\mathfrak{h}$ . The time translation acts on operators as conjugation with  $T_\tau$ .) *We assume that  $\sup_{\tau \in \mathbb{R}} \|T_\tau\| < \infty$  and the operators  $L(f)$  are bounded, hence  $\sup_{\tau \in \mathbb{R}} \|L(f, \tau)\| < \infty$ .* (Here and in what follows we assume that  $\mathcal{L}$  is a Banach space. If  $\mathcal{L}$  is a topological vector space specified by a system of seminorms we should impose the above conditions for every seminorm.)

Notice that  $L(f, \tau)\omega$  does not depend on  $\tau$ . (Using the fact that the map  $\sigma$  commutes with translations we obtain that  $L(f, \tau)\omega = T_\tau\sigma(T_{-\tau}f) = \sigma(f)$ .) This means that

$$\dot{L}(f, \tau)\omega = 0 \tag{1}$$

where the dot stands for the derivative with respect to  $\tau$ .

Let us introduce the notation

$$\Lambda(f_1, \dots, f_n | -\infty) = \lim_{\tau_1 \rightarrow -\infty, \dots, \tau_n \rightarrow -\infty} \Lambda(f_1, \tau_1, \dots, f_n, \tau_n) \tag{2}$$

where

$$\Lambda(f_1, \tau_1, \dots, f_n, \tau_n) = L(f_1, \tau_1), \dots, L(f_n, \tau_n)\omega.$$

We say that (2) is an *in-state*.

For  $\tau \rightarrow -\infty$  the state

$$T_\tau \Lambda(f_1, \dots, f_n | -\infty)$$

can be described as a collection of particles with wave functions  $T_\tau f_i$ . To prove this fact we use the formulas

$$T_\tau(L(f, \tau')) = T_{\tau+\tau'}L(T_{-\tau'}f)T_{-\tau-\tau'} = L(T_\tau f, \tau + \tau'),$$

$$T_\tau \Lambda(f_1, \dots, f_n | -\infty) = \Lambda(T_\tau f_1, \dots, T_\tau f_n | -\infty).$$

For  $f_1, \dots, f_n$  in a dense open subset of  $\mathfrak{h} \times \dots \times \mathfrak{h}$  the distance between essential supports of wave functions  $T_\tau f_i$  tends to  $\infty$  as  $\tau \rightarrow -\infty$ . This follows from Lemma 1.

This remark allows us to say that the state  $T_\tau \Lambda(f_1, \dots, f_n | -\infty)$  describes a collision of particles with wave functions  $(f_1, \dots, f_n)$ .

It is obvious that *the in-state (2) is symmetric with respect to  $f_1, \dots, f_n$  if*

$$\lim_{\tau \rightarrow -\infty} \|[L(f_i, \tau), L(f_j, \tau)]\| = 0. \quad (3)$$

One can replace (3) by

$$\|[L(\phi), L(\psi)]\| \leq \int d\mathbf{x} d\mathbf{x}' D^{ab}(\mathbf{x} - \mathbf{x}') |\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')| \quad (4)$$

where  $D^{ab}(\mathbf{x})$  tends to zero faster than any power as  $\mathbf{x} \rightarrow \infty$ .

Then the *in-state* is symmetric if the sets  $U_{f_i}$  do not overlap.

Let us give conditions for the existence of the limit

$$\lim_{\tau_1 \rightarrow -\infty, \dots, \tau_n \rightarrow -\infty} \Lambda(f_1, \tau_1, \dots, f_n, \tau_n). \quad (5)$$

For simplicity we consider the case when  $\tau_1 = \dots = \tau_n = \tau$ .

**Lemma 2.** *Let us assume that for  $\tau \rightarrow -\infty$  the commutators  $[\dot{L}(f_i, \tau), L(f_j, \tau)]$  are small. More precisely, the norms of these commutators should be bounded from above by a summable function of  $\tau$ :*

$$\|[L(f_i, \tau), L(f_j, \tau)]\| \leq c(\tau), \quad \int |c(\tau)| d\tau < \infty. \quad (6)$$

*Then the vector  $\Lambda(\tau) = \Lambda(f_1, \tau, \dots, f_n, \tau)$  has a limit as  $\tau \rightarrow -\infty$ .*

It is sufficient to check that the norm of the derivative of this vector with respect to  $\tau$  is a summable function of  $\tau$ . (Then  $\Lambda(\tau_2) - \Lambda(\tau_1) = \int_{\tau_1}^{\tau_2} \Lambda(\tau) d\tau$  tends to zero as  $\tau_1, \tau_2 \rightarrow -\infty$ .)

Calculating  $\dot{\Lambda}(\tau)$  by means of Leibniz rule we obtain  $n$  summands; each summand has one factor with  $\dot{L}$ . The assumption about the behavior of commutators allows us to move the factor with derivative to the right if we neglect the terms tending to zero faster than a summable function of  $\tau$ . It remains to notice that the expression with the derivative in the rightmost position vanishes due to (1).

If  $\mathcal{L}$  is a complete topological linear space with the topology specified by a system of seminorms we can generalize the above proof assuming an analog of (6) for every seminorm.

Notice that Lemma 1 implies that the distance between essential supports of functions  $T_\tau f_i$  grows linearly as  $\tau \rightarrow -\infty$  if the sets  $U_{f_i}$  do not overlap. This allows us to derive the existence of the limit for  $f_1, \dots, f_n$  in a dense open subset of  $\mathfrak{h} \times \dots \times \mathfrak{h}$  if we assume that the commutator  $[\dot{L}(T_{-\tau} f), L(T_{-\tau} g)]$  is small when the essential supports of  $T_\tau f$  and  $T_\tau g$  are far away for  $\tau \rightarrow \infty$ . One can make this statement precise in various ways. For example, one can prove the following statement:

**Lemma 3.** *Let us assume that*

$$||[\dot{L}(\phi), L(\psi)]|| \leq \int d\mathbf{x} d\mathbf{x}' D^{ab}(\mathbf{x} - \mathbf{x}') |\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')| \quad (7)$$

where  $D^{ab}(\mathbf{x})$  tends to zero faster than any power as  $\mathbf{x} \rightarrow \infty$ . Then the limit (2) exists if the sets  $U_{f_i}$  do not overlap (hence it exists for  $f_1, \dots, f_n$  in dense open subset of  $\mathfrak{h} \times \dots \times \mathfrak{h}$ ).

Instead of (6) we can assume that

$$||[L(f_i, \tau + \alpha) - L(f_i, \tau), L(f_j, \tau)]|| \leq c|\tau|^{-a}, a > 1 \quad (8)$$

In this formula  $c$  is a constant,  $\alpha$  belongs to a finite interval. The proof can be easily modified: instead of conventional Leibniz rule one should use difference Leibniz rule.

Slightly modifying the proof of Lemma 3 we can derive the following statement

**Lemma 4.** *Let us assume that*

$$||[T_\alpha(L(\phi)), L(\psi)]|| \leq \int d\mathbf{x} d\mathbf{x}' D^{ab}(\mathbf{x} - \mathbf{x}') |\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')| \quad (9)$$

where  $D^{ab}(\mathbf{x})$  tends to zero faster than any power as  $\mathbf{x} \rightarrow \infty$  and  $\alpha$  runs over a finite interval. Then the limit (2) exists if the sets  $U_{f_i}$  do not overlap (hence it exists for  $f_1, \dots, f_n$  in dense open subset of  $\mathfrak{h} \times \dots \times \mathfrak{h}$ ).

Applying Lemma 1 and the assumption (9) we obtain estimates for commutators  $[T_\alpha(L(T_{-\tau}f)), L(T_{-\tau}g)]$  that are sufficient to prove the inequality (8), hence the existence of the limit (5). ( We are using the relation

$$||[L(f, \tau + \alpha), L(g, \tau)]|| = ||[T_{\tau+\alpha}(L(T_{-(\tau+\alpha)}f), T_\tau(L(T_{-\tau}g)))]|| \leq \quad (10)$$

$$C ||[T_\alpha(L(T_{-(\tau+\alpha)}f), L(T_{-\tau}g))]||$$

and its particular case for  $\alpha = 0$ .) To prove the existence of more general limit (2) we should slightly modify our arguments.

Let us review shortly the scattering theory in the algebraic approach (for more details see [3])<sup>1</sup>. Recall that in this approach an elementary excitation of translation-invariant stationary state  $\omega$  is specified by an isometric map  $\Phi : \mathfrak{h} \rightarrow \mathcal{H}$  commuting

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<sup>1</sup>Notice that the operators  $B(f, \tau)$  of present paper correspond to the operators  $B(f\phi^{-1}, \tau)$  of [3]. The properties of operators  $B(f, \tau)$  that are taken for granted in the present paper are derived in [3] from asymptotic commutativity of the algebra  $\mathcal{A}$  and some properties of energy-momentum spectrum.

with translations and obeying  $\Phi(f) = B(f)\theta$  where  $B(f) \in \mathcal{A}$ . ( Here  $\theta$  stands for a a vector corresponding to  $\omega$  in the space  $\mathcal{H}$  of GNS representation.)

Let us define the operator  $B(f, \tau)$  by the formula

$$B(f, \tau) = T_\tau(B(T_{-\tau}f)) = T_\tau B(T_{-\tau}f)T_{-\tau}.$$

Notice that  $B(f, \tau)\theta$  does not depend on  $\tau$ . This follows from the remark that  $\omega$  is stationary, hence  $T_{-\tau}\theta = \theta$  and  $B(f, \tau)\theta = T_\tau\Phi(T_{-\tau}f) = \Phi(f)$ .

**Lemma 5.** *Let us assume that*

$$\|[\dot{B}(f_i, \tau), B(f_j, \tau)]\| \leq c(\tau)$$

where  $c(\tau)$  is a summable function. Then the vector

$$\Psi(\tau) = B(f_1, \tau) \dots B(f_n, \tau)\theta$$

has a limit in  $\bar{\mathcal{H}}$  as  $\tau$  tends to  $-\infty$ .

**Lemma 6.** *Let us assume that*

$$\|[\dot{B}(\phi), B(\psi)]\| \leq \int d\mathbf{x} d\mathbf{x}' D^{ab}(\mathbf{x} - \mathbf{x}') |\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')| \quad (11)$$

where  $D^{ab}(\mathbf{x})$  tends to zero faster than any power as  $\mathbf{x} \rightarrow \infty$ . Then for  $f_1, \dots, f_n$  in dense open subset of  $\mathfrak{h} \times \dots \times \mathfrak{h}$  the vector

$$\Psi(f_1, \tau_1, \dots, f_n, \tau_n) = B(f_1, \tau_1) \dots B(f_n, \tau_n)\theta$$

has a limit in  $\bar{\mathcal{H}}$  as  $\tau_j$  tend to  $-\infty$ ; this limit will be denoted by

$$\Psi(f_1, \dots, f_n| - \infty)$$

The proof of Lemma 5 is very similar to the proof of Lemma 2 and the proof of Lemma 6 repeats the proof of Lemma 3.

Let us introduce the asymptotic bosonic Fock space  $\mathcal{H}_{as}$  as a Fock representation of canonical commutation relations

$$[b(\rho), b(\rho')] = [b^+(\rho), b^+(\rho')] = 0, [b(\rho), b^+(\rho')] = \langle \rho, \rho' \rangle$$

where  $\rho, \rho' \in \mathfrak{h}$ .

We define Møller matrix  $S_-$  as a linear map of  $\mathcal{H}_{as}$  into  $\bar{\mathcal{H}}$  that transforms  $b^+(f_1) \dots b^+(f_n)|0\rangle$  into  $\Psi(f_1, \dots, f_n| - \infty)$ . ( Here  $|0\rangle$  stands for the Fock vacuum.)

Imposing some additional conditions one can prove that the operator  $S_-$  can be extended to isometric embedding of  $\mathcal{H}_{as}$  into  $\bar{\mathcal{H}}$  (see [3]).

Replacing  $-\infty$  by  $+\infty$  in the definition of  $S_-$  we obtain the definition of the Møller matrix  $S_+$ . If both Møller matrices are surjective maps we say that the theory has particle interpretation. In this case we can define the scattering matrix of elementary excitations (particles) by the formula  $S = S_+^{-1}S_-$ .

Let us define the *in*-operators  $a_{in}^+$  by the formula

$$a_{in}^+(f) = \lim_{\tau \rightarrow -\infty} B(f, \tau). \quad (12)$$

This limit exists as a strong limit on vectors  $\Psi(f_1, \dots, f_n | -\infty)$  if there exists the limit  $\Psi(f, f_1, \dots, f_n | -\infty)$ .

Operators  $a_{out}^+$  (*out*-operators) are defined by the formula

$$a_{out}^+(f) = \lim_{\tau \rightarrow +\infty} B(f, \tau). \quad (13)$$

Equivalently Møller matrix  $S_-$  can be defined as a map  $\mathcal{H}_{as} \rightarrow \bar{\mathcal{H}}$  obeying

$$a_{in}^+(\rho)S_- = S_-b^+(\rho), S_-|0\rangle = \theta.$$

The operators  $a_{in}(\rho), a_{out}(\rho)$  (Hermitian conjugate to  $a_{in}^+(\rho)$  and  $a_{out}^+(\rho)$ ) obey

$$a_{in}(\rho)S_- = S_-b(\rho), a_{out}(\rho)S_+ = S_+b(\rho).$$

Notice that spatial and time translations act naturally in  $\mathcal{H}_{as}$ . The Møller matrix commutes with translations.

There exists an obvious relation between our considerations in geometric and algebraic approach. It is clear that the operator  $L(f, \tau)$  in the space of states corresponds to the operator  $B(f, \tau)$  in  $\bar{\mathcal{H}}$  (i.e  $L(f, \tau) = \tilde{B}(f, \tau)B(f, \tau)$ .) It follows that the state  $\Lambda(f_1, \tau_1, \dots, f_n, \tau_n)$  corresponds to vector  $\Psi(f_1, \tau_1, \dots, f_n, \tau_n)$ , the state  $\Lambda(f_1, \dots, f_n | -\infty)$  (the *in*-state) corresponds to the vector  $\Psi(f_1, \dots, f_n | -\infty)$ .

The relation (9) implies that (5) specifies a map of symmetric power of  $\mathfrak{h}$  into the cone  $\mathcal{C}$ . This map (defined on a dense subset) will be denoted by  $\tilde{S}_-$ ; it can be regarded as an analog of the Møller matrix  $S_-$  in the geometric approach. The above statements allow us to relate  $\tilde{S}_-$  with  $S_-$  for theories that can be formulated algebraically. In this case  $S_-$  maps symmetric power of  $\mathfrak{h}$  considered as a subspace of the Fock space into  $\bar{\mathcal{H}}$ . Composing this map with the natural map of  $\bar{\mathcal{H}}$  into the cone of states  $\mathcal{C}$  we obtain  $\tilde{S}_-$ .

The map  $\tilde{S}_-$  is not linear, but in the case when  $L$  is quadratic or Hermitian it induces a multilinear map of the symmetric power of the cone  $C(\mathfrak{h})$  corresponding to  $\mathfrak{h}$  into the cone  $\mathcal{C}$ .

Notice that one can weaken the conditions on the map  $L$  assuming asymptotic commutativity of the semiring  $\mathcal{W}$ .

Constructing the scattering matrix in algebraic approach we imposed some conditions on commutators ( for example the condition (11) in Lemma 5). These conditions can be replaced by similar conditions on anticommutators, the above statements remain correct after slight modifications. ( In particular, we should consider the fermionic Fock space instead of bosonic one.)

## 5 Inclusive scattering matrix

Instead of the cone  $\mathcal{C}$  one can consider the dual cone  $\mathcal{C}^*$ . The semiring  $\mathcal{W}$  and the group of translations act on  $\mathcal{C}^*$ .

Let us consider a translation invariant element  $\alpha \in \mathcal{C}^*$  obeying the conditions similar to the conditions we imposed on  $\omega$ . ( In algebraic approach we can take  $\alpha(\sigma) = \sigma(1)$ , the value of  $\sigma$  on the unit of algebra.)

Taking

$$\lim_{\tau \rightarrow +\infty} \alpha(L(g_1, \tau) \dots L(g_m, \tau) \Lambda(f_1, \dots, f_n | -\infty))$$

we obtain a number characterizing the result of the collision. In bra-ket notation we can write this number as

$$\lim_{\tau' \rightarrow +\infty, \tau \rightarrow -\infty} \langle \alpha | L(g_1, \tau') \dots L(g_m, \tau') L(f_1, \tau) \dots L(f_n, \tau) | \omega \rangle \quad (14)$$

By definition of elementary excitation  $\sigma(\phi)$  is a quadratic (or Hermitian) map, hence it is natural to assume that the map  $L(\phi)$  is also quadratic (or Hermitian). Then it can be extended to a bilinear (or sesquilinear) map  $L(\tilde{\phi}, \phi)$ . (If we assume that the bilinear map is symmetric then the extension is unique, but in algebraic approach it is convenient to consider an extension that is not symmetric. Recall that in the algebraic approach we define  $L(\phi)$  as  $\tilde{B}(\phi)B(\phi)$ ; the extension can be defined by the formula  $L(\tilde{\phi}, \phi) = \tilde{B}(\tilde{\phi})B(\phi)$ .)

Using the extension we can define a functional

$$\begin{aligned} \sigma(\tilde{g}'_1, g'_1, \dots, \tilde{g}'_{n'}, g'_{n'}, \tilde{g}_1, g_1, \dots, \tilde{g}_n, g_n) = \\ \langle \alpha | \lim_{\tau'_i \rightarrow +\infty, \tau_j \rightarrow -\infty} L(\tilde{g}'_1, g'_1, \tau'_1) \dots L(\tilde{g}'_{n'}, g'_{n'}, \tau'_{n'}) L(\tilde{g}_1, g_1, \tau_1) \dots L(\tilde{g}_n, g_n, \tau_n) | \omega \rangle \end{aligned} \quad (15)$$

that is linear or antilinear with respect to all of its arguments.

We say that (15) is inclusive scattering matrix. ( If we do not assume that the map  $L(\phi)$  is quadratic or Hermitian the inclusive scattering matrix should be defined by the formula (14).)

The inclusive scattering matrix was expressed in [5],[6],[11] in terms of on-shell GGreen functions that appear in the formalism of  $L$ -functionals (used in [5], [6], [11]) and in Keldysh formalism [8],[9],[10].

The functional (15) can be considered as a generalized function

$$\sigma(\tilde{\mathbf{k}}'_1, \tilde{i}'_1, \mathbf{k}'_1, i'_1, \dots, \tilde{\mathbf{k}}'_{n'}, \tilde{i}'_{n'}, \mathbf{k}'_{n'}, i'_{n'}, \tilde{\mathbf{k}}_1, \tilde{i}_1, \mathbf{k}_1, i_1, \dots, \tilde{\mathbf{k}}_n, \tilde{i}_n, \mathbf{k}_n, i_n) \quad (16)$$

This generalized function is defined for open dense subset of its arguments. It is sufficient to require that  $\tilde{\mathbf{k}}'_i \neq \tilde{\mathbf{k}}'_j, \mathbf{k}'_i \neq \mathbf{k}'_j, \tilde{\mathbf{k}}_i \neq \tilde{\mathbf{k}}_j, \mathbf{k}_i \neq \mathbf{k}_j$ , for  $i \neq j$  if we assume that  $\mathbf{k} \neq \mathbf{k}'$  implies  $\nabla \epsilon_j(\mathbf{k}) \neq \nabla \epsilon'_j(\mathbf{k}')$  (Recall that we use the notation  $\epsilon_j(\mathbf{k})$  for eigenvalues of the matrix  $E(\mathbf{k})$ .) More generally we can consider the sets  $U(\mathbf{k})$  consisting of vectors  $\nabla \epsilon_j(\mathbf{k})$  and assume that the sets  $U(\mathbf{k})$  and  $U(\mathbf{k}')$  do not overlap. Then the essential support of a function  $T_{-\tau}(f)$  is far away from the essential support of a function  $T_{-\tau}(f')$  if the support of  $f$  lies in the neighborhood of  $\mathbf{k}$ , the support of  $f'$  lies in the neighborhood of  $\mathbf{k}' \neq \mathbf{k}$  and  $\tau \rightarrow \infty$ .

One can say that the function (16) gives matrix elements of inclusive scattering matrix.

Let us show that in the algebraic approach inclusive cross-sections can be expressed in terms of these matrix elements. Notice that in this approach

$$\begin{aligned} & \sigma(\tilde{g}'_1, g'_1, \dots, \tilde{g}'_{n'}, g'_{n'}, \tilde{g}_1, g_1, \dots, \tilde{g}_n, g_n) = \\ & \langle 1 | \lim_{\tau'_i \rightarrow +\infty, \tau_j \rightarrow -\infty} \tilde{B}(\tilde{g}'_1, \tau'_1) B(g'_1, \tau'_1) \dots \tilde{B}(\tilde{g}'_{n'}, \tau'_{n'}) B(g'_{n'}, \tau'_{n'}) \tilde{B}(\tilde{g}_1, \tau_1) B(g_1, \tau_1) \dots \tilde{B}(\tilde{g}_n, \tau_n) \\ & B(g_n, \tau_n) | \omega \rangle = \langle a_{out}^+(\tilde{g}'_1) \dots a_{out}^+(\tilde{g}'_{n'}) \Psi(\tilde{g}_1, \dots, \tilde{g}_n | -\infty, a_{out}^+(g'_1) \dots a_{out}^+(g'_{n'}) \Psi(g_1, \dots, g_n | -\infty) \rangle = \\ & \langle a_{out}(g'_{n'}) \dots, a_{out}(g'_1) a_{out}^+(\tilde{g}'_1) \dots a_{out}^+(\tilde{g}'_{n'}) \Psi(\tilde{g}_1, \dots, \tilde{g}_n | -\infty), \Psi(g_1, \dots, g_n | -\infty) \rangle = \end{aligned} \quad (17)$$

We have used (13) and relations  $(\tilde{B}_1 B_2 \omega)(A) = \omega(B_1^* A B_2) = \langle \theta, , B_1^* A B_2 \theta \rangle = \langle B_1 \theta, A B_2 \theta \rangle$ ,  $\langle 1 | \tilde{B}_1 B_2 | \omega \rangle = \langle B_1 \theta, B_2 \theta \rangle$  in this derivation.

In terms of generalized functions

$$\begin{aligned} & \sigma(\tilde{\mathbf{k}}'_1, \tilde{i}'_1, \mathbf{k}'_1, i'_1, \dots, \tilde{\mathbf{k}}'_{n'}, \tilde{i}'_{n'}, \mathbf{k}'_{n'}, i'_{n'}, \tilde{\mathbf{k}}_1, \tilde{i}_1, \mathbf{k}_1, i_1, \dots, \tilde{\mathbf{k}}_n, \tilde{i}_n, \mathbf{k}_n, i_n) = \\ & \langle a_{out}(\mathbf{k}'_{n'}, i'_{n'}) \dots, a_{out}(\mathbf{k}'_1, i'_1) a_{out}^+(\tilde{\mathbf{k}}'_1, \tilde{i}'_1) \dots a_{out}^+(\tilde{\mathbf{k}}'_{n'}, \tilde{i}'_{n'}) \Psi(\tilde{\mathbf{k}}_1, \tilde{i}_1, \dots, \tilde{\mathbf{k}}_n, \tilde{i}_n | -\infty), \Psi(\mathbf{k}_1, i_1, \dots, \mathbf{k}_n, i_n | -\infty) \rangle \end{aligned} \quad (18)$$

The inclusive cross-section of the process  $(M, N) \rightarrow (Q_1, \dots, Q_m)$  is defined as a sum (more precisely a sum of integrals) of effective cross-sections of the processes  $(M, N) \rightarrow (Q_1, \dots, Q_m, R_1, \dots, R_n)$  over all possible  $R_1, \dots, R_n$ . If the theory does not have particle interpretation this formal definition of inclusive cross-section does not work, but still the inclusive cross-section can be defined in terms of probability of the

process  $(M, N) \rightarrow (Q_1, \dots, Q_n + \text{something else})$  and expressed in terms of inclusive scattering matrix defined above. To verify this statement we consider the expectation value

$$\nu(a_{out}^+(\mathbf{p}_1, k_1) a_{out}(\mathbf{p}_1, k_1) \dots a_{out}^+(\mathbf{p}_m, k_m) a_{out}(\mathbf{p}_m, k_m)) \quad (19)$$

where  $\nu$  is an arbitrary state.

This quantity is the probability density in momentum space for finding  $m$  outgoing particles of the types  $k_1, \dots, k_n$  with momenta  $\mathbf{p}_1, \dots, \mathbf{p}_m$  plus other unspecified outgoing particles. It gives inclusive cross-section if  $\nu$  is an *in*-state.

Comparing this statement with (18) we obtain that inclusive cross-section can be obtained from inclusive scattering matrix if  $\tilde{\mathbf{k}}_i$  tends to  $\mathbf{k}_i$  and  $\tilde{\mathbf{k}}'_i$  tends to  $\mathbf{k}'_i$ . ( We assume that the expression

$$\nu(a_{out}^+(\tilde{\mathbf{p}}_1, k_1) a_{out}(\mathbf{p}_1, k_1) \dots a_{out}^+(\tilde{\mathbf{p}}_m, k_m) a_{out}(\mathbf{p}_m, k_m)) \quad (20)$$

tends to (19) as  $\tilde{\mathbf{p}}_i$  tends to  $\mathbf{p}_i$ .)

## 6 Discussion

Let us discuss some properties of the above construction of *in*-state and of inclusive scattering matrix.

We start again with elementary excitation  $\sigma : \mathfrak{h} \rightarrow \mathcal{C}$  of state  $\omega$ . By definition of elementary excitation there exists a map  $L : \mathfrak{h} \rightarrow \mathcal{W}$  obeying  $\sigma(\phi) = L(\phi)\omega$ . The map  $L$  is not unique; let us prove that under some conditions the *in*-state does not change when we are changing  $L$ . More precisely we can prove the following statement:

*Let us assume that the maps  $L_i : \mathfrak{h} \rightarrow \mathcal{W}$  can be used to define in-state and*

$$\| [L_i(\phi), L_j(\psi)] \| \leq \int d\mathbf{x} d\mathbf{x}' D^{ab}(\mathbf{x} - \mathbf{x}') |\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')|.$$

*where  $D^{ab}$  tends to zero faster than any power. Then*

$$\Lambda(f_1, \dots, f_n | -\infty) = \lim_{\tau_1 \rightarrow -\infty, \dots, \tau_n \rightarrow -\infty} L_{i_1}(f_1, \tau_1), \dots, L_{i_n}(f_n, \tau_n) \omega.$$

*(We assume that the sets  $U_{f_i}$  do not overlap.)*

To prove this statement we notice first of all that  $L_i(f, \tau)\omega = L_j(f, \tau)\omega$  hence the choice of the operator  $L_i$  in the rightmost position does not matter. Then we use the fact that one can move every factor to the rightmost position without changing the limit (the commutators are small when  $\tau_j \rightarrow -\infty$ .)

Similar statement is true for inclusive scattering matrix.

Let us consider a Poincaré -invariant theory. Recall that in our definitions we started with the homomorphism of the translation group  $\mathcal{T}$  into a subgroup of  $\mathcal{W}$  consisting of invertible operators. We assume that this homomorphism can be extended to a homomorphism of the Poincaré group  $\mathcal{P}$ . The translation group acts also on the elementary space  $\mathfrak{h}$ ; we assume that this action also can be extended to the action of Poincaré group and that the elementary excitation of Poincaré invariant state  $\omega$  considered as a map  $\sigma : \mathfrak{h} \rightarrow \mathcal{C}$  commutes with the actions of Poincaré group on  $\mathfrak{h}$  and  $\mathcal{C}$  : for every  $P \in \mathcal{P}$  and  $f \in \mathfrak{h}$  we have

$$\sigma(Pf) = P\sigma(f) \quad (21)$$

Then we say that the theory is Poincaré-invariant.

By definition of elementary excitation there exists a map  $L : \mathfrak{h} \rightarrow \mathcal{W}$  obeying  $\sigma(f) = L(f)\omega$ . If  $L$  commutes with Poincaré transformations the scattering is obviously Poincaré-invariant. However, one can prove Poincaré invariance of scattering in much more general situation. Let us sketch a proof of this fact assuming that

$$\lim_{\tau \rightarrow -\infty} \|[L(Pf_i, \tau), L^P(f_j, \tau)]\| = 0 \quad (22)$$

(We introduced notation  $L^P(f, \tau) = PL(f, \tau)P^{-1}$ .)

The generalized Møller matrix  $\tilde{S}_-$  is a map of the symmetric power of  $\mathfrak{h}$  into  $\mathcal{C}$ . Let us check that this map commutes with actions of Poincaré group. (Similar proof can be applied to inclusive scattering matrix.)

We should identify

$$L(Pf_1, \tau), \dots L(Pf_n, \tau)\omega \quad (23)$$

with

$$PL(f_1, \tau), \dots L(f_n, \tau)\omega = L^P(f_1, \tau), \dots L^P(f_n, \tau)\omega$$

in the limit  $\tau \rightarrow -\infty$ . We will show that we can replace  $L(Pf_i, \tau)$  with  $L^P(f_i, \tau)$  in any number of factors of (23) without changing the limit. For the rightmost factor this statement is equivalent to (21). Let us assume that this statement is correct for the last  $k$  factors. Then it is true also for  $(k+1)$ -th factor from the right. (To prove this we interchange the  $(k+1)$ -th factor with  $k$ -th factor from the right using (22) and use the induction hypothesis.) We proved the statement by induction.

Modifying the the considerations in the proof of Lemma 3 we can give various conditions for Poincaré invariance of scattering theory on a dense subset of  $\mathfrak{h} \times \dots \times \mathfrak{h}$ .

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