We define elementary space $\mathfrak{h}$ as a space of test functions $\phi_a(\mathbf{x})$ where spatial translations act shifting the argument. Test functions take values in $\mathbb{C}^r$. For definiteness we assume that test functions belong to the space $\mathcal{S}$ of smooth fast decreasing functions.

In momentum representation spatial translations act as multiplication by $e^{i\mathbf{ka}}$ and time translations as multiplication by $e^{-iE(\mathbf{k})\tau}$. (This follows from the assumption that that time translations are unitary operators commuting with spatial translations.) Here $E(\mathbf{k})$ stands for Hermitian $r \times r$ matrix. Diagonalizing the matrix $E(\mathbf{k})$ we can reduce the general case to the case $r = 1$.

An elementary excitation of translation-invariant stationary state ω (a quasiparticle) is specified by a map σ of $\mathfrak{h}$ into the set of excitations. This map should commute with translations.
In algebraic approach excitations elements of pre Hilbert space $\mathcal{H}$ where the state ω is represented by cyclic vector θ, $\sigma(\phi) = \Phi(\phi)$ is a linear map,

$$\Phi(\phi) = B(\phi)\theta$$

The map σ induces a map $\sigma' : \mathfrak{h} \to \mathcal{C}$
$(\sigma'(\phi))(A) = \langle A\sigma(\phi), \sigma(\phi)\rangle$
If $\Phi(\phi) = B(\phi)\theta$ then $\sigma'(\phi) = L(\phi)\omega$ where

$$L(\phi) = \tilde{B}(\phi)B(\phi)$$

In geometric approach it is natural to assume that the map σ' specifying an elementary excitation is a Hermitian map $\sigma' : \mathfrak{h} \to \mathcal{C}$ and that $\sigma'(\phi) = L(\phi)\omega$ where $L : \mathfrak{h} \to \mathcal{L}$

A map f is Hermitian if there exists a function $F(x, y)$ linear with respect first argument and antilinear with respect to second argument obeying $F(x, x) = f(x)$

For every linear space E we construct a cone $C(E)$ as a minimal cone in $E \otimes \bar{E}$ containg all elements of the form $e \otimes \bar{e}$.

The cone $C(\mathfrak{h})$ is elementary cone.

σ' can be regarded as a linear map $\sigma' : C(\mathfrak{h}) \to \mathcal{C}$

Let us assume that $\text{supp}(\phi)$ is a compact set. Then for large $|\tau|$ we have

$$|(T_\tau \phi)(\mathbf{x})| < C_n(1 + |\mathbf{x}|^2 + \tau^2)^{-n}$$

where $\frac{\mathbf{x}}{\tau} \notin U_\phi$, the initial data $\phi = \phi(\mathbf{x})$ is the Fourier transform of $\phi(\mathbf{k})$, and n is an integer. Here $\text{supp}(\phi)$ is the closure of the set of points were $\phi(\mathbf{k}) \neq 0$, U_ϕ is a set of all points of the form $\nabla\epsilon(\mathbf{k})$ where $\mathbf{k}$ belongs to a neighborhood of $\text{supp}(\phi)$, the function $\epsilon(\mathbf{k})$ is smooth.
$(T_\tau \phi)(\mathbf{x}) = \int d\mathbf{k} e^{i\mathbf{k}\mathbf{x} - i\epsilon(\mathbf{k})\tau} \phi(\mathbf{k})$,
We say that τU_ϕ is an essential support of the function $(T_\tau \phi)(\mathbf{x})$. We say that functions do not overlap if their essential supports do not overlap.

What is a two-particle state?
Let us assume that for ϕ and ϕ' with distant essential supports (in coordinate representation) $B(\phi)$ almost (anti)commutes with $B(\phi')$ and $\dot{B}(\phi)$ almost (anti)commutes with $B(\phi')$. Boson and fermions.
The vector $B(\phi)B(\phi')\theta$ corresponds to a state of two distant particles.
This state can be written in the form $L(\phi)L(\phi')\omega$ where $L(\phi)=\tilde{B}(\phi)B(\phi)$, $L(\phi')=\tilde{B}(\phi')B(\phi')$
Notice that $L(\phi)$ commutes with $L(\phi')$ also in the case when $B(\phi)$ anticommutes with $B(\phi')$.

Scattering

The axiomatic method has many advantages over honest work

Bertrand Russell

In algebraic approach $\sigma(\phi) = B(\phi)\theta$
In geometric approach we require the existence of a map $L : \mathfrak{h} \to End(\mathcal{L})$ obeying $\sigma'(\phi) = L(\phi)\omega$.
We introduce notations

$$B(f, \tau) = T_\tau B(T_{-\tau} f))T_{-\tau}.$$

$$L(f, \tau) = T_\tau L(T_{-\tau} f)T_{-\tau},$$

$L(f, \tau)\omega = T_\tau L(T_{-\tau} f)\omega = T_\tau \sigma'(T_{-\tau} f)\omega = \sigma'(f)\omega$ does not depend on τ, hence

$$\dot{L}(f, \tau)\omega = 0$$

Similarly,

$$\dot{B}(f, \tau)\theta = 0$$

Let us define *in*-state by the formula

$\Psi(f_1, \cdots, f_n| -\infty) = \lim_{\tau\to-\infty} \Psi(f_1, \cdots, f_n|\tau)$

where

$\Psi(\tau) = B(f_1, \tau)...B(f_n, \tau)\theta$

in algebraic approach and by the formula

$\Lambda(f_1, \cdots, f_n| -\infty) = \lim_{\tau\to-\infty} \Lambda(f_1, \cdots, f_n|\tau)$

where

$\Lambda(f_1, ..., f_n|\tau) = L(f_1, \tau), ...L(f_n, \tau)\omega$

in geometric approach.

For $\tau \to -\infty$ the state

$$T_\tau \Lambda(f_1, \cdots, f_n | -\infty)$$

can be described as a collection of particles with wave functions $T_\tau f_i$. To prove this fact we use the formulas

$$T_\tau(L(f, \tau')) = T_{\tau+\tau'} L(T_{-\tau'} f) T_{-\tau-\tau'} = L(T_\tau f, \tau+\tau'),$$

$$T_\tau \Lambda(f_1, \cdots, f_n | -\infty) = \Lambda(T_\tau f_1, \cdots, T_\tau f_n | -\infty).$$

For $f_1, \cdots, f_n$ in a dense open subset of $\mathfrak{h} \times \cdots \times \mathfrak{h}$ the distance between essential supports of wave functions $T_\tau f_i$ tends to ∞ as $\tau \to -\infty$. The state $T_\tau \Lambda(f_1, \cdots, f_n | -\infty)$ describes a collision of particles with wave functions $(f_1, \cdots, f_n)$ if these functions do not overlap.

It is obvious that *the in-state is symmetric with respect to* $f_1, ..., f_n$ *if*

$$\lim_{\tau \to -\infty} ||[L(f_i, \tau), L(f_j, \tau)]|| = 0.$$

One can replace this condition by

$$||[L(\phi), L(\psi)]|| \leq \int d\mathbf{x} d\mathbf{x}' D^{ab}(\mathbf{x}-\mathbf{x}') |\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')|$$

where $D^{ab}(\mathbf{x})$ tends to zero faster than any power as $\mathbf{x} \to \infty$.
Then the *in*-state is symmetric if the sets U_{f_i} do not overlap.

Let us give conditions for the existence of the limit defining the *in*-state.
Let us assume that for $\tau \to -\infty$ the commutators $[\dot{L}(f_i, \tau), L(f_j, \tau)]$ are small. More precisely, the norms of these commutators should be bounded from above by a summable function of τ :

$$||[\dot{L}(f_i, \tau), L(f_j, \tau)]|| \leq c(\tau),$$

$$\int |c(\tau)| d\tau < \infty.$$

Then $\Lambda(\tau) = \Lambda(f_1, \cdots, f_n | \tau)$ *has a limit as* $\tau \to -\infty$.

It is sufficient to check that the norm of the derivative of this vector with respect to τ is a summable function of τ. (Then $\Lambda(\tau_2)-\Lambda(\tau_1)=\int_{\tau_1}^{\tau_2}\dot{\Lambda}(\tau)d\tau$ tends to zero as $\tau_1,\tau_2\to-\infty$.)
Calculating $\dot{\Lambda}(\tau)$ by means of Leibniz rule we obtain n summands; each summand has one factor with $\dot{L}$. The assumption about the behavior of commutators allows us to move the factor with derivative to the right if we neglect the terms tending to zero faster than a summable function of τ. It remains to notice that the expression with the derivative in the rightmost position vanishes.

Similar statements in algebraic approach
Let us assume that

$$|||[\dot{B}(f_i,\tau),B(f_j,\tau)]||| \leq c(\tau)$$

where $c(\tau)$ is a summable function. Then the vector

$$\Psi(\tau)=B(f_1,\tau)...B(f_n,\tau)\theta$$

has a limit in $\bar{\mathcal{H}}$ as τ tends to $-\infty$.

Let us assume that $|||[\dot{B}(\phi), B(\psi)]||| \leq \int d\mathbf{x}d\mathbf{x}' D^{ab}(\mathbf{x}-\mathbf{x}')|\phi_a(\mathbf{x})| \cdot |\psi_b(\mathbf{x}')|$ *where* $D^{ab}(\mathbf{x})$ *tends to zero faster than any power as* $\mathbf{x} \to \infty$. *Then for* $f_1, ..., f_n$ *in dense open subset of* $\mathfrak{h} \times ... \times \mathfrak{h}$ *the vector*

$$\Psi(f_1, \tau_1, ..., f_n, \tau_n) = B(f_1, \tau_1)...B(f_n, \tau_n)\theta$$

has a limit in $\bar{\mathcal{H}}$ *denoted by*

$$\Psi(f_1, ..., f_n| - \infty)$$

as τ_j *tend to* $-\infty$

Let us introduce the asymptotic bosonic Fock space $\mathcal{H}_{as}$ as a Fock representation of canonical commutation relations

$$[b(\rho), b(\rho')] = [b^+(\rho), b^+(\rho')] = 0, [b(\rho), b^+(\rho')] = \langle \rho, \rho' \rangle$$

where $\rho, \rho' \in \mathfrak{h}$.
We define Møller matrix S_- as a linear map of $\mathcal{H}_{as}$ into $\bar{H}$ that transforms $b^+(f_1)...b^+(f_n)|0\rangle$ into $\Psi(f_1, ..., f_n| - \infty)$. (Here $|0\rangle$ stands for the Fock vacuum.)
Notice that spatial and time translations act naturally in $\mathcal{H}_{as}$. The Møller matrix commutes with translations.

Imposing some additional conditions one can prove that the operator S_- can be extended to isometric embedding of $\mathcal{H}_{as}$ into $\bar{H}$.
Replacing $-\infty$ by $+\infty$ in the definition of S_- we obtain the definition of the Møller matrix S_+. If both Møller matrices are unitary (are surjective maps) we say that the theory has particle interpretation. In this case we can define the scattering matrix of elementary excitations (particles) by the formula

$$S = S_+^{-1} S_-$$

Let us define the *in*-operators a_{in}^{+} by the formula $a_{in}^{+}(f)=\lim_{\tau\to-\infty}B(f,\tau)$.
This limit exists as strong limit on vectors $\Psi(f_1,...,f_n|-\infty)$ if there exists the limit $\Psi(f,f_1,...,f_n|-\infty)$ (in particular, if all these functions do not overlap).
Operators a_{out}^{+} (*out*-operators) are defined by the formula
$a_{out}^{+}(f)=\lim_{\tau\to+\infty}B(f,\tau)$. We introduce the notation $a_{out}^{+}(f)=\int d\mathbf{p}f^k(\mathbf{p})a_{out,k}^{+}(\mathbf{p})$

Equivalently Møller matrix S_- can be defined as a map $\mathcal{H}_{as} \to \overline{\mathcal{H}}$ obeying

$$a_{in}^+(\rho) S_- = S_- b^+(\rho), S_- |0\rangle = \theta.$$

The operators $a_{in}(\rho), a_{out}(\rho)$ (Hermitian conjugate to $a_{in}^+(\rho)$ and $a_{out}^+(\rho)$) obey

$$a_{in}(\rho) S_- = S_- b(\rho), a_{out}(\rho) S_+ = S_+ b(\rho).$$

All above statements remain correct if commutators are replaced by anticommutators. Then instead of bosonic Fock space one should consider fermionic Fock space.

There exists an obvious relation between our considerations in geometric and algebraic approach. It is clear that the operator $L(f,\tau)$ in the space of states corresponds to the operator $B(f,\tau)$ in $\bar{\mathcal{H}}$ (i.e $L(f,\tau)=\tilde{B}(f,\tau)B(f,\tau)$.) It follows that the state $\Lambda(f_1,\dots,f_n|\tau)$ corresponds to vector $\Psi(f_1,\dots,f_n|\tau)$, the state $\Lambda(f_1,\dots,f_n|-\infty)$ (the *in*-state) corresponds to the vector $\Psi(f_1,\dots,f_n|-\infty)$.

$\Lambda(f_1, \cdots, f_n| -\infty)$ specifies a map of symmetric power of $\mathfrak{h}$ into the cone $\mathcal{C}$. This map (defined on a dense subset) will be denoted by $\tilde{S}_-$; it can be regarded as an analog of the Møller matrix S_-.
The above statements allow us to relate $\tilde{S}_-$ with S_- for theories that can be formulated algebraically. In this case S_- maps symmetric power of $\mathfrak{h}$ considered as a subspace of the Fock space into $\bar{\mathcal{H}}$. Composing this map with the natural map of $\bar{\mathcal{H}}$ into the cone of states $\mathcal{C}$ we obtain $\tilde{S}_-$.
The map $\tilde{S}_-$ is not linear, but in the case when L is quadratic or Hermitian it induces a multilinear map of the symmetric power of the cone $C(\mathfrak{h})$ corresponding to $\mathfrak{h}$ into the cone $\mathcal{C}$.

Inclusive cross-section of the process $(M, N) \to (Q_1 ..., Q_m)$ is defined as a sum (more precisely a sum of integrals) of effective cross-sections of the processes $(M, N) \to (Q_1, ..., Q_m, R_1, ..., R_n)$ over all possible $R_1, ..., R_n$. If the theory does not have particle interpretation this formal definition of inclusive cross-section does not work, but still the inclusive cross-section can be defined in terms of probability of the process $(M, N) \to (Q_1, ..., Q_n +$ something else$)$ and expressed in terms of inclusive S-matrix defined below.

To verify this statement we consider the expectation value

$$\nu(a^+_{out,k_1}(\mathbf{p}_1)a_{out,k_1}(\mathbf{p}_1)\dots a^+_{out,k_m}(\mathbf{p}_m)a_{out,k_m}(\mathbf{p}_m))$$

where ν is an arbitrary state. This quantity is the probability density in momentum space for finding m outgoing particles of the types $k_1,\dots,k_n$ with momenta $\mathbf{p}_1,\dots,\mathbf{p}_m$ plus other unspecified outgoing particles. It gives inclusive cross-section if ν is an *in*-state:

$$\nu=\Lambda(g_1,...,g_n|-\infty)=\lim_{\tau\to-\infty}L(g_1,\tau)...L(g_n,\tau)\omega$$

Let us consider the expression

$$\langle 1|L(g'_1,\tau')...L(g'_{n'},\tau')L(g_1,\tau)...L(g_n,\tau)|\omega\rangle$$

We assume that g'_i as well as g_j are not overlapping, then this expression has a limit as $\tau'\to+\infty, \tau\to-\infty$; we denote this limit by Q. It is clear that Q can be written in the form

$$Q=\lim_{\tau'\to+\infty}\langle 1|L(g'_1,\tau')...L(g'_{n'},\tau')\nu\rangle$$

where $\nu=\Lambda(g_1,...,g_n|-\infty)$ stands for *in*-state. Notice that Q does not change if we permute $g_1,...,g_n$ (in the limit $\tau\to-\infty$ the operators $L(g_j,\tau)$ commute). Similarly Q does not change if we permute $g'_1,...,g'_{n'}$.

Using formulas $L(g,\tau) = \tilde{B}(g,\tau)B(g,\tau)$ and $(\tilde{M}N\nu)(X) = \nu(M^*XN)$ and noticing that $\langle 1|\sigma\rangle = \sigma(1)$ we obtain that
$Q =$
$\lim_{\tau'\to+\infty} \nu(B^*(g'_{n'},\tau')...B^*(g'_1,\tau')B(g'_1,\tau')...B(g'_{n'},\tau'))$
Finally using $\lim_{\tau'\to+\infty} B(g,\tau') = a^+_{out}(g)$ we see that

$$Q = \nu(a_{out}(g'_{n'})...a_{out}(g'_1)a^+_{out}(g'_1)...a^+_{out}(g'_{n'})))$$

We say that

$$Q = Q(g'_1, ..., g'_{n'}, g_1, .., g_n)$$

is *inclusive scattering matrix.*
It is quadratic (more precisely Hermitian) with respect to its arguments, hence we can replace it with multilinear function having $2(n + n')$ arguments. It also can be called inclusive scattering matrix.
Inclusive cross-sections can be obtained from inclusive scattering matrix.

Inclusive scattering matrix in geometric approach

$$\lim_{\tau'\to+\infty,\tau\to-\infty}\langle\alpha|L(g_1,\tau')...L(g_m,\tau')\times$$

$$L(f_1,\tau)...L(f_n,\tau)|\omega\rangle$$

$\alpha\in\mathcal{L}^*$ and $\omega\in\mathcal{L}$ are translation-invariant stationary states
They are on equal footing. Interchanging α and ω we should get a kind of duality.