Araki-Haag-Kastler axioms

For every bounded open subset $\mathcal{O}$ of Minkowski space we have W^*-algebra $\mathcal{A}(\mathcal{O})$ (weakly closed subalgebra of the algebra of bounded operators in Hilbert space $\mathcal{H}$ that is invariant with respect to involution *). We assume that $\mathcal{H}$ is a representation space of unitary representation of Poincaré group $\mathcal{P}$.

If $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$.

If $g \in \mathcal{P}$ then $\mathcal{A}(g\mathcal{O}) = g\mathcal{A}(\mathcal{O})g^{-1}$.

If the interval between any point of $\mathcal{O}_1$ and any point of $\mathcal{O}_2$ is space-like the algebras $\mathcal{A}(\mathcal{O}_1)$ and $\mathcal{A}(\mathcal{O}_2)$ commute.

The ground state θ of the Hamiltonian H (of the generator of time translations) is Poincaré invariant.
The set of vectors of the form $A\theta$ where $A \in \mathcal{A}(\mathcal{O})$ is dense in $\mathcal{H}$ (i.e. θ is a cyclic vector with respect to the union $\mathcal{A}$ of all algebras $\mathcal{A}(\mathcal{O})$).
A particle is an irreducible subrepresentation of the representation of Poincaré group in $\mathcal{H}$.

Scattering in algebraic approach

$\mathcal{A}$ is an associative algebra with involution. Space and time translations act as automorphisms of $\mathcal{A}$ and of the cone C of non-normalized states.

$\omega \in \mathcal{C}$ = translation-invariant stationary state.

Excitations of ω= elements of pre Hilbert space $\mathcal{H}$ obtained by means of GNS construction, $\hat{A}$-an operator in $\mathcal{H}$ corresponding to $A \in \mathcal{A}$. The closure of the algebra of operators of the form $\hat{A}$ in norm topology is denoted by $\mathcal{A}(\omega)$.

θ is the cyclic vector obeying $\omega(A) = \langle \hat{A}\theta, \theta \rangle$.

$(\tilde{M}N\nu)(X) = \nu(M^* X N)$

Let $\alpha(\tau, \mathbf{x})$ denote a smooth fast decreasing function. An operator of the form $B = \int d\tau d\mathbf{x} \alpha(\tau, \mathbf{x}) \hat{A}(\tau, \mathbf{x})$ is called a smooth operator. These operators belong to $\mathcal{A}(\omega)$ for all $A \in \mathcal{A}$.

$B(\tau, \mathbf{x}) = \int d\tau', d\mathbf{x}' \alpha(\tau - \tau', \mathbf{x} - \mathbf{x}') A(\tau', \mathbf{x}')$ is a smooth function .

Asymptotic commutativity

$||[B_1(\tau, \mathbf{x}), B_2]||$ and $||[\dot{B}_1(\tau, \mathbf{x}), B_2]||$

tend to zero faster than any power of $||\mathbf{x}||$ as $\mathbf{x} \to \infty$ for smooth operators B_1,B_2 and fixed τ.

Another definition

$||[B_1(\tau, \mathbf{x}), B_2]|| \leq \frac{C_n(\tau)}{1+||\mathbf{x}||^n}$

where $C_n(\tau)$ has at most polynomial growth and n is arbitrary.

Can be relaxed.

True in Araki-Haag-Kastler setting.

In what follows all operators are smooth.

The weakest form of cluster property is the following condition
$\omega(A(\mathbf{x},t)B) = \omega(A)\omega(B) + \rho(\mathbf{x},t)$
where $A, B \in \mathcal{A}$ and ρ is small in some sense for $\mathbf{x} \to \infty$.
To formulate a more general cluster property, we introduce the notion of correlation functions in the state ω :
$w_n(\mathbf{x}_1, t_1, \ldots \mathbf{x}_n, t_n) =$
$\omega(A_1(\mathbf{x}_1, t_1) \cdots A_n(\mathbf{x}_n, t_n)) =$
$\langle A_1(\mathbf{x}_1, t_1) \cdots A_n(\mathbf{x}_n, t_n) \rangle$,
where $A_i \in \mathcal{A}$.
These functions generalize Wightman functions of relativistic quantum field theory.

We consider the corresponding truncated correlation functions $w_n^T(\mathbf{x}_1, t_1, \ldots \mathbf{x}_n, t_n)$ denoted also by $\langle A_1(\mathbf{x}_1, t_1) \cdots A_n(\mathbf{x}_n, t_n)\rangle^T$.

Truncated correlation functions w^T can be defined recursively by the formula

$w_n, \mathbf{x}_1, \tau_1, k_1 \ldots, \mathbf{x}_n, \tau_n, k_n) =$
$\sum_{s=1}^n \sum_{\rho \in R_s} w_{\alpha_1}^T(\pi_1) \ldots w_{\alpha_s}^T(\pi_s)$.

Here R_s denotes the collection of all partitions of the set $\{1, ..., n\}$ into s subsets denoted $\pi_1, ..., \pi_s$, the number of elements in the subset π_i is denoted by α_i and $w_{\alpha_i}^T(\pi_i)$ stands for the truncated correlation function with arguments $\mathbf{x}_a, \tau_a, k_a$ where $a \in \pi_i$.

We have assumed that the state ω is translation-invariant; it follows that both correlation functions and truncated correlation functions depend on the differences $\mathbf{x}_i - \mathbf{x}_j, t_i - t_j$. We say that the state ω has the cluster property if the truncated correlation functions are small for $\mathbf{x}_i - \mathbf{x}_j \to \infty$. A strong version of the cluster property is the assumption that the truncated correlation functions tend to zero faster than any power of $\min \|\mathbf{x}_i - \mathbf{x}_j\|$. Then its Fourier transform with respect to variables $\mathbf{x}_i$ has the form $\nu_n(\mathbf{p}_2, \dots, \mathbf{p}_n, t_1, \dots, t_n)\delta(\mathbf{p}_1 + \dots + \mathbf{p}_n)$, where the function ν_n is smooth.

The Green function in translation-invariant stationary state ω is defined by the formula

$$G_n = \omega(T(A_1(\mathbf{x}_1, t_1) \dots A_r(\mathbf{x}_r, t_r))) =$$

$$\langle \Phi | T(\hat{A}_1(\mathbf{x}_1, t_1) \dots \hat{A}_r(\mathbf{x}_r, t_r)) | \Phi \rangle$$

where $A_i \in \mathcal{A}$ and T stands for time ordering. More precisely, this is a definition of Green function in $(\mathbf{x}, t)$-representation, taking Fourier transform with respect to $\mathbf{x}$ we obtain Green functions in $(\mathbf{p}, t)$- representation, taking in these functions inverse Fourier transform with respect to t we obtain Green functions in in $(\mathbf{p}, \epsilon)$ -representation.

Due to translation-invariance of ω we obtain that in $(\mathbf{x}, t)$-representation the Green function depends on differences $\mathbf{x}_i - \mathbf{x}_j, t_i - t_j$, in $(\mathbf{p}, t)$-representation it contains a factor $\delta(\mathbf{p}_1 + \cdots + \mathbf{p}_r)$. Similarly in $(\mathbf{p}, \epsilon)$ we have the same factor and the factor $\delta(\epsilon_1 + \cdots + \epsilon_r)$. We omit both factors talking about poles of Green functions.

For $n = 2$ in $(\mathbf{p}, \epsilon)$-representation G_2 has the form

$$G(\mathbf{p}_1, \epsilon_1 | A, A')\delta(\mathbf{p}_1 + \mathbf{p}_2)\delta(\epsilon_1 + \epsilon_2).$$

Poles of function $G(\mathbf{p}, \epsilon | A, A')$ correspond to particles, the dependence of the position of the pole on $\mathbf{p}$ specifies the dispersion law $\varepsilon(\mathbf{p})$ (we consider poles with respect to the variable ϵ for fixed $\mathbf{p}$).

We prove that to find scattering amplitudes we should consider asymptotic behavior of Green functions in $\mathbf{p}, t$ representation as $t \to \pm\infty$.e. The asymptotic behavior of Green function in $(\mathbf{p}, t)$ representation is governed by the poles of Green function in $(\mathbf{p}, \epsilon)$-representation and by residues at these poles (by on-shell values of Green function).

If a function $\rho(t)$ has asymptotic behavior $e^{-itE_\pm}A_\pm$ as $t \to \pm\infty$ (in other words there exist finite limits $\lim_{t\to\pm\infty} e^{itE_\pm}\rho(t) = A_\pm$). Then the (inverse) Fourier transform $\rho(\epsilon)$ has poles at the points $E_\pm \pm i0$ with residues $\mp 2\pi i A_\pm$.

Hence the scattering amplitudes can be expressed in terms of on -shell values of Green functions (LSZ formula).

The proof of LSZ formula will be given in case when the theory has particle interpretation (i.e.there exist Møller matrices $S_{\pm}$ specifying unitary equivalence with free Hamiltonian in $\mathcal{H}_{as}$ and the Hamiltonian in the GNS Hilbert space $\mathcal{H}$).

To simplify notations we consider the case when we have only one single-particle state $\Phi(\mathbf{p})$ with dispersion law $\varepsilon(\mathbf{p})$, i.e.

$\hat{H}\Phi(\mathbf{p}) = \varepsilon(\mathbf{p})\Phi(\mathbf{p}), \hat{\mathbf{P}}\Phi(\mathbf{p}) = \mathbf{p}\Phi(\mathbf{p})$.

Let us suppose that one-particle spectrum does not overlap with multi-particle spectrum.
If the theory has particle interpretation then our condition means that $\varepsilon(\mathbf{p}_1 + \mathbf{p}_2) < \varepsilon(\mathbf{p}_1) + \varepsilon(\mathbf{p}_2)$.
The physical meaning of this condition: the energy-momentum conservation law forbids the decay of a particle. This condition is not always satisfied, however, stability of a particle is always guaranteed by some conservation laws. Our considerations can be applied in this more general situation.

Let us formulate LSZ formula more precisely and give more detailed proofs.
We assume that the elements $A_i \in \mathcal{A}$ are chosen in such a way that the projection of $\hat{A}_i\Phi$ on the one-particle space has the form
$\Phi(\phi_i) = \int \phi_i(\mathbf{p})\Phi(\mathbf{p})d\mathbf{p}$ where $\phi_i(\mathbf{p})$ is a non-vanishing function.We introduce the notation $\Lambda_i(\mathbf{p}) = \phi_i(\mathbf{p})^{-1}$.

Let us consider Green function
$G_{mn} = \omega(T(A_1^*(\mathbf{x}_1, t_1) \dots A_m^*(\mathbf{x}_m, t_m) \times$
$A_{m+1}(\mathbf{x}_{m+1}, t_{m+1}) \dots A_{m+n}(\mathbf{x}_{m+n}, t_{m+n}))$
in $(\mathbf{p}, \epsilon)$-representation. It is convenient to change slightly the definition of $(\mathbf{p}, \epsilon)$-representation changing the signs of variables $\mathbf{p}_i$ and ϵ_i for $1 \le i \le m$ (for variables corresponding to the operators A_i^*). Multiplying the Green function in $(\mathbf{p}, \epsilon)$-representation by

$$\prod_{1 \le i \le m} \overline{\Lambda_i(\mathbf{p}_i)}(\epsilon_i + \varepsilon(\mathbf{p}_i)) \prod_{m < j \le m+n} \Lambda_j(\mathbf{p}_j)(\epsilon_j - \varepsilon(\mathbf{p}_j)).$$

and taking the limit $\epsilon_i \to -\varepsilon(\mathbf{p}_i)$ for $1 \le i \le m$ and the limit $\epsilon_j \to \varepsilon(\mathbf{p}_j)$ for $m < j \le m+n$ we obtain normalized on-shell Green function denoted by σ_{mn}.

We prove that the normalized on-shell Green function coincides with scattering amplitudes. First, we give a proof for the case when A_i are smooth operators obeying $A_i\Phi = \Phi(\phi_i)$ (good operators). The general case can be reduced to the case when operators A_i are good. This statement is based on the remark that in the definition of Green function we can replace the operators A_i by operators $A'_i = \int \alpha(\mathbf{x}, t)A_i(\mathbf{x}, t)d\mathbf{x}dt$ without changing the normalized on-shell Green function and on the remark that we can assume that A'_i is a good operator.

To justify the second remark we should assume that the support of $\hat{\alpha}(\mathbf{p}, \omega)$ (of the Fourier transform of α) does not intersect the multi-particle spectrum and does not contain 0.

To justify the first remark we notice first of all that $A_i(\mathbf{x}, t) = \int \alpha(\mathbf{x}' - \mathbf{x}, t' - t) A_i(\mathbf{x}', t') d\mathbf{x}' t'$ is a convolution of $\alpha(-\mathbf{x}, -t)$ with $A_i(\mathbf{x}, t)$. It follows that the correlation function of operators $A_i'(\mathbf{x}, t)$ in $(\mathbf{x}, t)$-representation can be represented as a convolution of the correlation function of operators $A_i(\mathbf{x}, t)$ with the product of functions $\alpha(-\mathbf{x}_i, -t_i)$.This implies that in $(\mathbf{p}, \epsilon)$-representation correlation functions of A_i' can be obtained from correlation functions of A_i by means of multiplication by the product of functions $\hat{\alpha}(\mathbf{p}_i, \epsilon_i)$. Corresponding formulas remain correct for non-normalized on- shell Green functions.

Let us assume that we have several types of (quasi) particles defined as generalized functions $\Phi_k(\mathbf{p})$ obeying
$\mathbf{P}\Phi_k = \mathbf{p}\Phi_k(\mathbf{p}), H\Phi_k(\mathbf{p}) = \varepsilon_k(\mathbf{p})\Phi_k(\mathbf{p})$, where the functions $\varepsilon_k(\mathbf{p})$ are smooth . For definiteness we assume that the test functions belong to the space $\mathcal{S}$ of smooth fast decreasing functions. To guarantee that time translations act in the space $\mathcal{S}$ we should assume that the functions $\varepsilon_k(\mathbf{p})$ have at most polynomial growth.

Take some good operators $B_k \in \mathcal{A}$, obeying $\hat{B}_k\Phi = \Phi_k(\phi_k)$. Define $\hat{B}_k(f,t)$, where f is a function of $\mathbf{p}$ as $\int \tilde{f}(\mathbf{x},t)\hat{B}_k(\mathbf{x},t)d\mathbf{x}$ and $\tilde{f}(\mathbf{x},t)$ is a Fourier transform of $f(\mathbf{p})e^{-i\varepsilon_k(\mathbf{p})t}$ with respect to $\mathbf{p}$. Notice that

$$\hat{B}_k(f,t)\Phi = \Phi_k(f\phi_k)$$

does not depend on t.

Sometimes it is convenient to represent $\hat{B}_k(f,t)$ in the form

$$\hat{B}_k(f,t) = \int d\mathbf{p} f(\mathbf{p}) e^{i\varepsilon_k(\mathbf{p})t}\hat{B}_k(\mathbf{p},t)$$

where $\hat{B}_k(\mathbf{p},t) = \int d\mathbf{x} e^{-i\mathbf{p}\mathbf{x}}\hat{B}_k(\mathbf{x},t)$. (We understand $\hat{B}_k(\mathbf{p},t)$ as a a generalized function of $\mathbf{p}$.)

Let us consider vectors
$\Psi(k_1, f_1, \ldots, k_n, f_n | t_1, \ldots, t_n) =$
$\hat{B}_{k_1}(f_1, t_1) \cdots \hat{B}_{k_n}(f_n, t_n)\Phi$
We assume that $f_1, \ldots, f_n$ have compact support.
Let us introduce the notation $\mathbf{v}_i(\mathbf{p}) = \nabla \varepsilon_{k_i}(\mathbf{p})$.
The set of all $\mathbf{v}_i(\mathbf{p})$ such that $f_i(\mathbf{p}) \neq 0$ will be denoted U_i. *We assume that the sets* $\overline{U_i}$ *(the closures of* U_i*) do not overlap.* This assumption will be called *NO* condition in what follows.

Then assuming asymptotic commutativity *one can prove that the vector*
$\Psi(k_1, f_1, \dots, k_n, f_n | t_1, \dots, t_n)$
has a limit denoted by
$\Psi(k_1, f_1, \dots, k_n, f_n | \pm\infty)$
as $t_i \to \infty$ *or* $t_i \to -\infty$. The set spanned by these limits will be denoted $\mathcal{D}_+$ or $\mathcal{D}_-$.
Notice that the assumption that the sets $\overline{U_i}$ do not intersect (NO condition) can be omitted if the space-time dimension is ≥ 4. In these dimensions we can drop the NO condition defining the sets $\mathcal{D}_\pm$.

The existence of the limit of the vectors Ψ allows us to define Møller matrices. We introduce the Hilbert space $\mathcal{H}_{as}$ as a Fock representation for the operators $a_k^+(f), a_k(f)$ obeying canonical commutation relations. (Let us emphasize that in our notations the operator $a_k(f)$ is linear with respect to f and the operator $a_k^+(f)$ is is conjugate to $a_k(\bar{f})$, hence it is also linear with respect to f.)
We define Møller matrices S_- and S_+ as operators defined on $\mathcal{H}_{as}$ and taking values in $\bar{\mathcal{H}}$ by the formula

$$\Psi(k_1, f_1, \dots, k_n, f_n|\pm\infty) = S_\pm(a^+_{k_1}(f_1\phi_{k_1}) \dots a^+_{k_n}(f_n\phi_{k_n})\theta$$

This formula specifies $S_\pm$ on a dense subset of the Hilbert space $\mathcal{H}_{as}$. These operators are isometric,

One can say that the vector
$e^{-iHt}\Psi(k_1, f_1, \dots, k_n, f_n| \pm \infty) =$
$\Psi(k_1, f_1 e^{-i\varepsilon_{k_1} t}, \dots, k_n, f_n e^{-i\varepsilon_{k_n} t}| \pm \infty)$
describes the evolution of a state corresponding to a collection of n particles with wave functions $f_1\phi_1 e^{-i\varepsilon_{k_1} t}$,…,$f_n\phi_n e^{-i\varepsilon_{k_n} t}$ as $t \to \pm\infty$.

The definition of $S_\pm$ that we gave specifies these operators as multi-valued maps (for example we can use different good operators in the construction and it is not clear whether we get the same answer). However, we can check that this map is isometric and every multi-valued isometric map is really single-valued. In particular, this means that the definition does not depend on the choice of good operators. It follows also that the vector $\Psi(k_1, f_1, ..., k_n, f_n | \pm\infty)$ does not change when we permute the arguments (k_i, f_i) and (k_j, f_j).

To prove that the map is isometric we express the inner product of two vectors of the form $\Psi(t)$ in terms of truncated correlation functions. Only two-point truncated correlation functions survive in the limit $t \to \pm\infty$. This allows us to say that the map is isometric.

Let us define in- and out -operators by the formulas

$$a_{in}(f)S_- = S_-a(f), a_{in}^+(f)S_- = S_-a^+(f),$$

$$a_{out}(f)S_+ = S_+a(f), a_{out}^+(f)S_+ = S_+a^+(f).$$

(For simplicity of notations we consider the case when we have only one type of particles. If we have several types of particles, in- and out-operators as well as the operators a^+, a are labelled by a pair (k, f) where f is a test function and k characterizes the type of particle.) These operators are defined on the image of S_- and S_+ correspondingly.

$a^+_{in}(f\phi) = \lim_{t\to-\infty} \hat{B}(f,t)$,
$a^+_{out}(f\phi) = \lim_{t\to\infty} \hat{B}(f,t)$,
The limit is understood as a strong limit. It exists on the set of vectors of the form $\Psi(k_1, f_1, \dots, k_n, f_n|\pm)$ (with *NO* assumption for $d < 3$) The proof follows immediately from the fact that taking the limit $t_i \to \infty$ in the vectors $\Psi(k_1, f_1, \dots, k_n, f_n|t_1, \dots, t_n)$ we can first take the limit for $i > 1$ and then the limit $t_1 \to \infty$.

$$a^+_{in}(f\phi)\Psi(f_1,\ldots,f_n|-\infty) = \Psi(f,f_1,\ldots,f_n|-\infty),$$

$$a^+_{out}(f\phi)\Psi(f_1,\ldots,f_n|\infty) = \Psi(f,f_1,\ldots,f_n|\infty).$$

Similarly,

$$a_{in}(f)\Psi(\phi^{-1}f,f_1,\ldots,f_n|-\infty) = \Psi(f_1,\ldots,f_n|-\infty),$$

$$a_{out}(f)\Psi(\phi^{-1}f,f_1,\ldots,f_n|\infty) = \Psi(f_1,\ldots,f_n|\infty).$$

If the operators S_+ and S_- are unitary, we say that the theory has a particle interpretation. In this case (and also in the more general case when the image of S_- coincides with the image of S_+), we can define the scattering matrix

$$S = S_+^* S_-.$$

The scattering matrix is a unitary operator in $\mathcal{H}_{as}$. Its matrix elements in the basis $|\mathbf{p}_1, \dots, \mathbf{p}_n\rangle = \frac{1}{n!} a^+(\mathbf{p}_1) \dots a^+(\mathbf{p}_n)\theta$ (scattering amplitudes) can be expressed in terms of in- and out-operators:

$S_{mn}(\mathbf{p}_1, \dots, \mathbf{p}_m | \mathbf{q}_1, \dots, \mathbf{q}_n) =$
$\langle a_{in}^+(\mathbf{q}_1) \dots a_{in}^+(\mathbf{q}_n)\Phi, a_{out}^+(\mathbf{p}_1) \dots a_{out}^+(\mathbf{p}_m)\Phi\rangle$

In this formula and in what follows we omit numerical factors $(m!)^{-1}(n!)^{-1}$.

Notice that only when ω is a ground state can one hope that the particle interpretations exists. In other cases instead of a scattering matrix and cross-sections one should consider inclusive scattering matrix and inclusive cross-sections. The above formula is proved only for the case when all values of momenta $\mathbf{p}_i, \mathbf{q}_j$ are distinct . (More precisely, we should assume that all vectors $\mathbf{v}(\mathbf{p}_i) = \nabla\varepsilon(\mathbf{p}_i), \mathbf{v}(\mathbf{q}_j) = \nabla\varepsilon(\mathbf{q}_j)$ are distinct, but in the case when the function $\varepsilon(\mathbf{p})$ is strongly convex it is sufficient to assume that $\mathbf{p}_i, \mathbf{q}_j$ are distinct.) This formula should be understood in the sense of generalized functions and as test functions we should take collections of functions $f_i(\mathbf{p}_i), g_j(\mathbf{q}_j)$ with non-overlapping $\overline{U(f_i)}, \overline{U(g_j)}$.

Let us write this in more detail:
$S_{mn}(f_1,\ldots,f_m|g_1,\ldots,g_n)=$
$\int d^m\mathbf{p}d^n\mathbf{q}\prod f_i(\mathbf{p}_i)\prod g_j(\mathbf{q}_j)S_{mn}(\mathbf{p}_1,\ldots,\mathbf{p}_m|\mathbf{q}_1,\ldots,\mathbf{q}_n)$
$\langle a^+_{in}(g_1)\ldots a^+_{in}(g_n)\Phi, a^+_{out}(\bar{f}_1)\ldots a^+_{out}(\bar{f}_m)\Phi\rangle$
We obtain

$$S_{mn}(f_1,,f_m|g_1,\ldots,g_n)=$$

$\lim_{t\to\infty,\tau\to-\infty}\langle\Phi|\hat{B}(\bar{f}_m\phi^{-1},t)^*\ldots\hat{B}(\bar{f}_1\phi^{-1},t)^*\times$
$\hat{B}(g_1\phi^{-1},\tau)\ldots\hat{B}(g_n\phi^{-1},\tau))|\Phi\rangle=$
$\lim_{t\to\infty,\tau\to-\infty}\omega(B(\bar{f}_m\bar{\phi}^{-1},t)^*\ldots B(\bar{f}_1\bar{\phi}^{-1},t)^*\times$
$B(g_1\phi^{-1},\tau)\ldots B(g_n\phi^{-1},\tau))$
where $B(f,t)^*=\int d\mathbf{x}B^*(\mathbf{x},t)\overline{\tilde{f}(\mathbf{x},t)}$.

Notice that in the same way we can obtain a more general formula

$$S_{mn}(f_1,\dots,f_m|g_1,\dots,g_n) = \lim_{t_i\to\infty,\tau_j\to-\infty}\omega(B_m(\bar{f}_m\phi_m^{-1},t_m)^*\dots B_1(\bar{f}_1\phi_1^{-1},t_1)^*\times B_{m+1}(g_1\phi_{m+1}^{-1},\tau_1)\dots B_{m+n}(g_n\phi_{m+n}^{-1},\tau_n))$$

where B_i are different good operators and $B_i\Phi=\Phi(\phi_i)$. Due to NO condition the ordering of factors with t_i tending to infinity is irrelevant. The same is true for factors with $\tau_j\to-\infty$. This means that we can assume that the factors in this formula are time ordered.

One can represent this formula in the form

$$S_{mn}(f_1, \ldots, f_m | g_1, \ldots, g_n) =$$

$$\int d^{m+n}\mathbf{p} \lim_{t_i\to\infty, \tau_j\to-\infty} \langle\Phi| f_m \bar{\phi}_m^{-1} e^{i\varepsilon_m(\mathbf{p}_m)t_m} \hat{B}_m(\mathbf{p}_m, t_m)^* \times$$

$$\ldots f_1 \bar{\phi}_1^{-1} e^{i\varepsilon_1(\mathbf{p}_1)t_1} \hat{B}_1(\mathbf{p}_1, t_1)^* \times$$

$$g_1 \phi_{m+1}^{-1} e^{-i\varepsilon_{m+1}(\mathbf{p}_{m+1})\tau_1} \hat{B}_{m+1}(\mathbf{p}_{m+1}, \tau_n) \ldots \times$$

$$g_n \phi_{m+n}^{-1} e^{-i\varepsilon_{m+n}(\mathbf{p}_{m+n})\tau_n} \hat{B}_{m+n}(\mathbf{p}_{m+n}, \tau_n) |\Phi\rangle$$

Equivalently we can write
$S_{mn}(\mathbf{p}_1, \dots, \mathbf{p}_m | \mathbf{p}_{m+1} \dots, \mathbf{p}_{m+n}) =$
$\lim_{t_1,\dots,t_m \to \infty, t_{m+1},\dots,t_{m+n} \to -\infty} \langle \Phi | \bar{\phi}_m^{-1} e^{i\varepsilon_m(\mathbf{p}_m) t_m} \hat{B}_m(\mathbf{p}_m, t_m)^*$
$\dots \bar{\phi}_1^{-1} e^{i\varepsilon_1(\mathbf{p}_1) t_1} \hat{B}_1(\mathbf{p}_1, t_1)^* \times$
$\phi_{m+1}^{-1} e^{-i\varepsilon_{m+1}(\mathbf{p}_{m+1}) t_{m+1}} \hat{B}_{m+1}(\mathbf{p}_{m+1}, t_{m+1}) \dots \times$
$\phi_{m+n}^{-1} e^{-i\varepsilon_{m+n}(\mathbf{p}_{m+n}), t_{m+n}} \hat{B}_{m+n}(\mathbf{p}_{m+n}, t_{m+n}) | \Phi \rangle$
Notice that here we also can assume that the factors are time ordered.
Green function! LSZ is proved.