Some definable counterexamples in models of set theory

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Σ^0_∞ = arithmetical definability, Σ^1_∞ = analytical definability.

Theorem (Shoenfield, see Jech, *Millenium*, Cor 25.21)

Every Σ^1_2 *or* Π^1_2 *set* $X \subseteq \omega$ *is Gödel-constructible, that is,* $X \in \mathbf{L}$*.*

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There is a generic extension $\mathbf{L}[a]$ *of* $\mathbf{L}$ *by a real* $a \in 2^\omega$, *in which it is true that* a *is* Δ^1_3, *but* $a \notin \mathbf{L}$.

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- If $T \subseteq 2^{<\omega}$ is a perfect tree then

$$[T] = \{x \in 2^\omega : \forall\, n\, (x \restriction n \in T)\} \quad \text{(all branches of } T\text{);}$$

this is a perfect subset of 2^ω, the **Cantor space**.

The construction of Jensen's forcing $\mathbb{P}$ for the Δ^1_3 real theorem goes on in $\mathbf{L}$ in the form $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{P}_\xi$, where each $\mathbb{P}_\xi$ is a countable arboreal forcing defined by transfinite induction on ξ so that:

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3 in the universe: a pair of reals $a \neq b$ in 2^ω is $(\mathbb{P} \times \mathbb{P})$-generic over $\mathbf{L}$ iff for each $\xi < \omega_1^{\mathbf{L}}$ there are trees $S, T \in \mathbb{P}_\xi$ such that $a \in [S]$ and $b \in [T]$.

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The main tool of the construction is a $\diamond$-sequence in **L**, that is, a $\Delta_1^{\mathbf{L}_{\omega_1}}$-definable sequence $\langle S_\alpha \rangle_{\alpha<\omega_1}$ of sets $S_\alpha \subseteq \alpha$, such that for each $X \subseteq \omega_1$, the set $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ is a **stationary** subset of ω_1.

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The existence of such a $\diamond$-sequence is a corollary of $\mathbf{V} = \mathbf{L}$ (see Jech, *Millenium*, Theorem 13.21), but not a theorem of **ZFC**.

Corollary (Jech *Millenium* Lemma 28.8 and Cor 28.9)

Assume that a forcing notion $\mathbb{P} \in \mathbf{L}$ *satisfies* 1, 2, 3, *and a real* $a \in 2^\omega$ *is* $\mathbb{P}$*-generic over* $\mathbf{L}$. *Then*

- A *a is the only* $\mathbb{P}$*-generic real in* $\mathbf{L}[a]$;
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Thus by Jensen's theorem (1970),

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Theorem (generalization of the above theorem)

Let $n \geq 2$. *There is a generic extension* $\mathbf{L}[a]$ *of* $\mathbf{L}$ *by a real* $a \in 2^\omega$, *in which it is true that*

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Digression: when all definable reals are constructible? TOC

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Lemma

Sacks-forcing is cone-homogeneous, in the sense that, for any two perfect trees $S, T \subseteq 2^{<\omega}$*, the cones* $\mathbb{SF}_S = \{S' \in \mathbb{SF} : S' \subseteq S\}$ *and* $\mathbb{SF}_T$ *are* $\subseteq$*-isomorphic.*

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- *Hence if* $\varphi(k)$ *is a parameter-free formula then it holds in any Sacks-generic extension of* $\mathbf{L}$ *that* $\{k < \omega : \varphi(k)\}$ *belongs to* $\mathbf{L}$.
- *In particular all* Σ^1_∞ *reals in such an extension belong to* $\mathbf{L}$.

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Our goal: Given $n \geq 3$, find a generic real $a \in 2^\omega$ s. t. it is true in $\mathbf{L}[a]$ that $a \in \Delta^1_{n+1}$, $a \notin \mathbf{L}$, but all Σ^1_n reals belong to $\mathbf{L}$.

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The idea: Let $n \geq 3$. Find a forcing $\mathbb{P}(n)$ such that

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Recall Jensen's forcing construction original

TOC

The construction of Jensen's forcing notion $\mathbb{P}$ for the $\varDelta^1_3$ real theorem goes on in $\mathbf{L}$ so that $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{P}_\xi$, where each $\mathbb{P}_\xi$ is a countable arboreal forcing defined by transfinite induction on ξ, and:

1. in $\mathbf{L}$: the sequence $\langle \mathbb{P}_\xi \rangle_{\xi<\omega_1}$ is $\varDelta_1^{\mathbf{L}_{\omega_1}}$ $(= \varDelta^1_2$ in-the-codes).

2. in the universe: a real $a \in 2^\omega$ is $\mathbb{P}$-generic over $\mathbf{L}$ iff: for each $\xi < \omega_1^{\mathbf{L}}$ there is a tree $T \in \mathbb{P}_\xi$ such that $a \in [T]$ — hence being $\mathbb{P}$-generic is $\varPi^1_2$ by 1.

3. in the universe: a pair of reals $a \neq b$ in 2^ω is $(\mathbb{P} \times \mathbb{P})$-generic iff, $\forall\, \xi < \omega_1^{\mathbf{L}}$, there are trees $S, T \in \mathbb{P}_\xi$ with $a \in [S]$, $b \in [T]$.

The construction of the modified forcing $\mathbb{P} = \mathbb{P}(n)$ for the Δ^1_n real theorem goes on in $\mathbf{L}$ so that $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{P}_\xi$, where each $\mathbb{P}_\xi$ is a countable arboreal forcing defined by transfinite induction on ξ, and:

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4 $\mathbb{P}$ is **ee** to $\mathbb{SF}$ w. r. t. the forcing of Σ^1_n formulas, hence all trees $T \in \mathbb{P}$ force the same parameter-free Σ^1_n sentences.

As a corollary we obtain:

Theorem (generalization of Jensen's theorem)

Let $n \geq 2$. *There is a generic extension* $\mathbf{L}[a]$ *of* $\mathbf{L}$ *by a real* $a \in 2^\omega$, *in which it is true that*

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Proof

Let $\mathbb{P}(n)$ satisfy **1**, **2**, **3**, **4** as above.
Take an arbitrary $\mathbb{P}(n)$-generic real $a \in 2^\omega$. □

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Application 4: DC does not follow from countable AC. TOC

DC (dependent choices) claims that if E is a binary relation on the reals with $\mathtt{ran}\,\mathsf{E} \subseteq \mathtt{dom}\,\mathsf{E} \neq \varnothing$, then there exists a chain of reals x_n satisfying $x_n \; \mathsf{E} \; x_{n+1}$ for all n.

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Further goal: given $n \geq 2$, show by the SDF-G-K method that $\mathbf{\Pi}^1_{n+1}$-**DC** is not provable in $\mathbf{ZF}$ + full $\mathbf{AC}_\omega + \mathbf{\Pi}^1_n$-**DC**.

- Let $\mathbf{A}_2$ be the 2nd order PA (no choice).
- Let $\mathbf{A}_2^{\text{nopar}}$ be the 2nd order PA (no choice), but with the Comprehension schema only for parameter-free formulas.

Application 5: parameters in the Comprehension schema TOC

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Conjecture (from a discussion with L. Beklemishev)

A $\mathbf{A}_2$ is equiconsistent with $\mathbf{A}_2^{\text{nopar}}$, but

B $\mathbf{A}_2$ is stronger than $\mathbf{A}_2^{\text{nopar}}$ and

C $\mathbf{A}_2$ is not even finitely axiomatizable over $\mathbf{A}_2^{\text{nopar}}$.

Let $\langle \mathbb{P}_j \rangle_{j<\omega} \in \mathbf{L}$ be a sequence of independent clones of the Jensen forcing $\mathbb{P}$ as above. Let $\mathbb{Q} = \prod_{j<\omega} \mathbb{P}_j$ be the finite-support product. Let $\langle a_j \rangle_{j<\omega}$ be a $\mathbb{Q}$-generic sequence of reals $a_j \in 2^\omega$.

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Let $m_k(a_0)$ be kth number m such that $a_0(m) = 1$. Define $A = \{m_k(a_0) + 2 : k < \omega\} \supsetneqq B = \{m_k(a_0) + 2 : k < \omega \wedge a_1(k) = 1\}$. Note that A depends on a_0, B depends on a_0, a_1.

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Then $a_1 \notin M$ by the independency, but on the other hand a_1 is definable in M with a_0 as a parameter, because $a_1(m) = 1$ iff there is a real b, $\mathbb{P}_{m_k(a_0)+2}$-generic over $\mathbf{L}$.

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Thus Comprehension fails for M with parameter a_0!

Appl 6: type-theoretic definability and Tarski problem

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Theorem (K and Lyubetsky)

Any recursive and formally non-contradictory conjunction of sentences $D_{1p} \in D_{2p}$ and $D_{1p} \notin D_{2p}$, $p \geq 1$, is consistent with **ZFC**.

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See Jech, *Millenium*, Thm 25.20 for details. back