Part I: Spherical two-distance sets.

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Definitions

A set S in Euclidean space $\mathbf{R}^n$ is called a *two-distance set*, if there are two distances c and d, and the distances between pairs of points of S are either c or d. If a two-distance set S lies in the unit sphere $\mathbf{S}^{n-1}$, then S is called *spherical two-distance set.* In other words, S is a set of unit vectors, there are two real numbers a and b, $-1 \leq a, b < 1$, and inner products of distinct vectors of S are either a or b.

Restrictions on distances

The ratios of distances of two-distance sets are quite restrictive. Namely, Larman, Rogers, and Seidel (1977) have proved the following fact: if the cardinality of a two-distance set S in $\mathbf{R}^n$, with distances c and d, $c < d$, is greater than $2n+3$, then the ratio c^2/d^2 equals $(k-1)/k$ for an integer number k with

$$2 \leq k \leq \frac{1+\sqrt{2n}}{2}.$$

Finiteness

Einhorn and Schoenberg (1966) proved that there are finitely many two-distance sets S in $\mathbf{R}^n$ of the cardinality $|S| \geq n+2$.

Upper bounds

Delsarte, Goethals, and Seidel (1977) proved that the largest cardinality of spherical two-distance sets in $\mathbf{R}^n$ (we denote it by $g(n)$) is bounded by $n(n+3)/2$, i.e.,

$$g(n) \leq \frac{n(n+3)}{2} .$$

Moreover, they give examples of spherical two-distance sets with $n(n+3)/2$ points for $n = 2, 6, 22$. Therefore, in these dimensions we have equality $g(n) = n(n+3)/2$.

Upper bounds

Blockhuis (1984) showed that the cardinality of (Euclidean) two-distance sets in $\mathbf{R}^n$ does not exceed $(n+1)(n+2)/2$.

Lower bounds

Let unit vectors $e_1, \ldots, e_{n+1}$ form an orthogonal basis of $\mathbf{R}^{n+1}$. Denote by Δ_n the regular simplex with vertices $2e_1, \ldots, 2e_{n+1}$. Let Λ_n be the set of points $e_i + e_j$, $1 \leq i < j \leq n+1$. Since Λ_n lies in the hyperplane $\sum x_k = 2$, we see that Λ_n represents a spherical two-distance set in $\mathbf{R}^n$. On the other hand, Λ_n is the set of mid-points of the edges of Δ_n. Thus,

$$g(n) \geq |\Lambda_n| = \frac{n(n+1)}{2}.$$

Euclidean two-distance sets

For $n < 7$ the largest cardinality of Euclidean two-distance sets is $g(n)$, where $g(2) = 5,\ g(3) = 6,\ g(4) = 10,\ g(5) = 16$, and $g(6) = 27$. However, for $n = 7, 8$ Lisoněk (1997) discovered non-spherical maximal two-distance sets of the cardinality 29 and 45 respectively.

Polynomial method

The upper bound $n(n+3)/2$ for spherical two-distance sets, the bound $\binom{n+2}{2}$ for Euclidean two-distance sets, as well as the bound $\binom{n+s}{s}$ for $s-$distance sets (Bannai, Bannai, Stanton; Blokhuis) were obtained by the polynomial method. The main idea of this method is the following: to associate sets to polynomials and show that these polynomials are linearly independent as members of the corresponding vector space.

Upper bound

The polynomial method can be apply to improve upper bounds for spherical two-distance sets with $a + b \geq 0$.

Theorem

Let X be a spherical two-distance set in $\mathbf{R}^n$ with inner products a and b. If $a + b \geq 0$, then

$$|X| \leq \frac{n(n+1)}{2}.$$

Nozaki's theorem

Theorem (Hiroshi Nozaki, June 2008)

Let X be a spherical s-distance set in $\mathbf{R}^n$ with inner products $a_1, \ldots, a_s$. Let

$$\prod_{k=1}^{s}(t - a_k) = \sum_{k=1}^{s} f_k G_k^{(n)}(t).$$

Then

$$|X| \leq \sum_{k:f_k>0} h_k,$$

where

$$h_k = \binom{n+k-2}{k} + \binom{n+k-3}{k-1}.$$

Denote by $\rho(n)$ the largest possible cardinality of spherical two-distance sets in $\mathbf{R}^n$ with $a + b \geq 0$.

Theorem

If $n \geq 7$, then

$$\rho(n) = \frac{n(n+1)}{2}$$

Proof.

Theorem 1 implies that $\rho(n) \leq n(n+1)/2$. On the other hand, the set of mid-points of the edges of a regular simplex has $n(n+1)/2$ points and $a+b \geq 0$ for $n \geq 7$. Indeed, for this spherical two-distance set we have

$$a = \frac{n-3}{2(n-1)}, \quad b = \frac{-2}{n-1}.$$

Thus,

$$a+b = \frac{n-7}{2(n-1)} \geq 0.$$

□

Theorem (Delsarte et al; Kabatyanskiy & Levenshtein)

Let T be a subset of the interval $[-1,1]$. Let S be a set of unit vectors in $\mathbf{R}^n$ such that the set of inner products of distinct vectors of S lies in T. Suppose a polynomial f is a nonnegative linear combination of Gegenbauer polynomials $G_k^{(n)}(t)$, i.e.,

$$f(t) = \sum_k f_k G_k^{(n)}(t), \quad \textit{where} \quad f_k \geq 0.$$

If $f(t) \leq 0$ for all $t \in T$ and $f_0 > 0$, then

$$|S| \leq \frac{f(1)}{f_0}$$

Let S be a spherical two-distance set in $\mathbf{R}^n$ with inner products a and b, where $a > b$, $|S| > 2n+3$. Then the Larman-Rogers-Seidel theorem yields

$$b = b_k(a) := \frac{ka-1}{k-1},$$

where $k \in \left\{2, \dots, \lfloor \frac{1+\sqrt{2n}}{2} \rfloor \right\}$.
If $a + b = a + b_k(a) < 0$, then

$$a \in I_k := \left[\frac{2-k}{k}, \frac{1}{2k-1}\right).$$

For given $k \in \left\{2, \ldots, \lfloor \frac{1+\sqrt{2n}}{2} \rfloor \right\}$ and $a \in I_k$ we define

$$Q_k^{(n)}(a) := \inf_{f \in D_k(a)} f(1)/f_0,$$

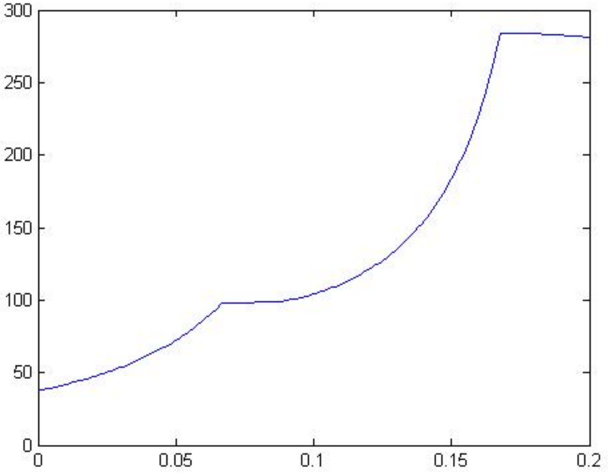
where $D_k(a) = \{f : \text{ all } f_i \geq 0, f_0 > 0, \; f(a) \leq 0, f(b_k(a)) \leq 0\}$.

Theorem

Let S be a spherical two-distance set in $\mathbf{R}^n$ with inner products a and $b_k(a)$. Then

$$|S| \leq Q_k^{(n)}(a).$$

The graph of the function $Q_3^{(25)}(a)$



n	$\widehat{\omega}$	ρ	k
7	28	28	2
8	31	36	2
9	34	45	2
10	37	55	2
11	40	66	2
12	44	78	2
13	47	91	2
14	52	105	2
15	56	120	2
16	61	136	2
17	66	153	2
18	76	171	3
19	96	190	3
20	126	210	3
21	176	231	3
22	275	253	3
23	277	276	3

n	$\widehat{\omega}$	ρ	k
24	280	300	3
25	284	325	3
26	288	351	3
27	294	378	3
28	299	406	3
29	305	435	3
30	312	465	3
31	319	496	3
32	327	528	3
33	334	561	3
34	342	595	3
35	360	630	2
36	416	666	2
37	488	703	2
38	584	741	2
39	721	780	2
40	928	820	2

Theorem

If $6 < n < 22$ *or* $23 < n < 40$, *then*

$$g(n) = \frac{n(n+1)}{2}.$$

For $n = 23$ *we have*

$$g(23) = 276 \text{ or } 277.$$