BOSONIC STRING ACTION

$$S = -\frac{T}{2}\int_M d^2\sigma\sqrt{h}h^{ab}(\sigma)\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$$

h^{ab} -inverse to metric tensor h_{ab} on surface M, $h = |\det h_{ab}|$, $\eta_{\mu\nu}$ stands for Minkowski metric in D -dimensional space

Reparametrization invariance

Weyl invariance $h_{ab}(\sigma) \to \Lambda(\sigma)h_{ab}(\sigma)$

Poincaré invariance $X^\mu \to a^\mu_\nu X^\nu + b^\mu$

Excluding the metric we get Nambu-Goto action (the area of embedded surface).

In flat two-dimensional metric $ds^2 = d\sigma^2 - d\tau^2$

$X^\mu = X_R^\mu(\tau - \sigma) + X_L^\mu(\tau + \sigma)$

For closed string $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + \pi)$,hence

$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau - \sigma) + \frac{i}{2}\sum_{n \neq 0} \frac{1}{n}\alpha_n^\mu e^{-2in(\tau - \sigma)}$

$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu(\tau + \sigma) + \frac{i}{2}\sum_{n \neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau + \sigma)}$

For open string

$X^\mu = x^\mu + p^\mu\tau + i\sum_{n \neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\tau} \cos n\sigma$

After quantization

$[\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m+n}\eta^{\mu\nu}$

$[\alpha_m^\mu, \tilde{\alpha}_n^\nu] = 0$

$\alpha_{-n}^\mu = (\alpha_n^\mu)^*, \tilde{\alpha}_{-n}^\mu = (\tilde{\alpha}_n^\mu)^*$

Oscillators. Fock representation

$\alpha_m^\mu|0, p\rangle = \tilde{\alpha}_m^\mu|0, p\rangle = 0$ for $m > 0$

Indefinite norm because $[\alpha_m^0, \alpha_m^{0*}] = -1$

Notation: $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2}p^\mu$.

Energy-momentum tensor

$T_{ab} = -\frac{2}{T\sqrt{h}} \frac{\delta S}{\delta h^{ab}}$

Equation of motion $T_{ab} = 0$

L_m-Fourier components of energy-momentum for open strings

$L_m = \frac{1}{2} \sum_{-\infty}^{+\infty} \alpha_{m-n} \cdot \alpha_n$

Normal ordering for $L_0 =: \frac{1}{2} \sum_{-\infty}^{+\infty} \alpha_{-n} \cdot \alpha_n :$

$L_{-m} = L_m^*$

$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12} D(m^3 - m)\delta_{m+n}$

Virasoro algebra

For closed strings L_m and $\tilde{L}_m$ (two copies of Virasoro)

D-dimension, $D = 26$-critical dimension

In complex variables for closed string

$$X^\mu(z,\bar z) = x^\mu - ip^\mu \ln|z|^2 + i\sum_{-\infty}^{\infty}\left(\frac{\alpha_m^\mu}{z^m} + \frac{\tilde\alpha_m^\mu}{\bar z^m}\right)$$

$l_m = z^{m+1}\frac{\partial}{\partial z}, \tilde l_m = \bar z^{m+1}\frac{\partial}{\partial \bar z}$ correspond to $L_m, \tilde L_m$,
$\delta_m \alpha_r = r\alpha_{m+r}$, $\delta_m \tilde\alpha_r = r\tilde\alpha_{m+r}$

left $\rightarrow$ holomorphic, right $\rightarrow$ antiholomorphic

For open string

$$X^\mu(z,\bar z) = x^\mu - 2ip^\mu \ln|z|^2 + i\sum_{-\infty}^{\infty}\frac{\alpha_m^\mu}{m}\left(z^{-m} + \bar z^{-m}\right)$$

Classically $L_m = 0$

QM physical state

$L_m \psi = 0$ for $m > 0$, $(L_0 - 1)\psi = 0$

It follows that $\langle \psi | L_m | \psi \rangle = 0$ for all $m \neq 0$

For closed strings $(L_0 - \tilde{L}_0)|\psi\rangle = 0$

$|0, k\rangle$ -physical state if $k^2 = 2$ (tachyon)

$\zeta \alpha_{-1} |0, k\rangle$ -physical state if $k^2 = 0$ and $\zeta k = 0$

$T_a |0, k\rangle = e^{iak} |0, k\rangle$, $T_a \zeta a_{-1} |0, k\rangle = e^{iak} \zeta \alpha_{-1} |0, k\rangle$

where $T_a = (T_{a^0}, T_{\mathbf{a}})$ are time and space translations.

26-dimensional particles

L-functionals=positive linear functionals $\mathbf{L}$ on Weyl algebra

$\mathbf{L}(\xi, \kappa, A_n, A_n^*) =$

$\mathbf{L}\Big(e^{i\xi x} e^{i\kappa p} \exp(\sum_{n=1}^{\infty} A_n \alpha_{-n}) \exp(\sum_{n=1}^{\infty} (-A_n^*) \alpha_n)\Big)$

Physical state conditions $\mathbf{L}(L_n) = 0$ for all $n \neq 0$,

$\mathbf{L}(L_0 - 1) = 0$

Vertex operators

$V(k,\tau) =: e^{ikX(,\tau)} := \exp\Big(k \cdot \sum_{n=1}^{\infty} \frac{\alpha_{-n}}{n} e^{in\tau}\Big) \times$
$\times e^{ik\cdot x(\tau)} \exp\Big(- k \cdot \sum_{n=1}^{\infty} \frac{\alpha_n}{n} e^{-in\tau}\Big)$

where $x^\mu(\tau) = x^\mu + p^\mu \tau$

Simple action on L-functionals

$(\mathbf{V}(k,\tau)\mathbf{L})(\Psi) = \mathbf{L}(V(k,\tau)\Psi - \Psi V^*(k,\tau))$

$[L_m, V(k,\tau)] = e^{im\tau}(-i\frac{d}{d\tau} + \frac{1}{2}mk^2)V(k,\tau)$

$[L_m, A(\tau)] = e^{im\tau}(-i\frac{d}{d\tau} + mJ)A(\tau)$

Conformal dimension $J = \frac{k^2}{2}$

If $J = 1$ then $[L_m, A_0] = 0$, hence A_0 transforms a physical state into physical state

More general vertex operators have the form

$$\zeta^{\mu\nu\ldots\rho} : \dot{X}^{\mu}\dot{X}^{\nu}\ldots\dot{X}^{\rho}e^{ikX} :$$

If this operator has conformal dimension it is equal to $N + \frac{k^2}{2}$ where N is the number of factors $\dot{X}$.
Vertex operators of conformal dimension 1 correspond to particles with the mass M where $M^2 = -2 + 2N$
Conformal dimension of $A(\tau)$ describes behavior of $A(\tau)$ by the change of variables $\tau \to \tau'(\tau)$:
$A'(\tau') = \left(\frac{d\tau}{d\tau'}\right)^J A(\tau)$
Scaling dimension describes the behavior of $A(\tau)$ by the change of variables $\tau \to \tau' = \lambda\tau$

R-module= "vector space" over a ring R
(Additive group V with multiplication by $r \in R$ from the left obeying $(rr')v = r(r'v)$ and $1 \cdot v = v$. If V is a vector space and R is an algebra then R-module =representation of algebra R)
e_k is a system of generators R-module V if V is a minimal submodule containing e_k
e_k is a free system of generators of V if every map $e_k \to E_k$ where $E_k \in V'$ can be extended to a homomorphism $V \to V'$.
Free module = a module that has a free system of generators=direct sum of several copies of R
$\mathbb{Z}$-graded R-module =direct sum of R-modules V_k where $k \in \mathbb{Z}$. Such a module can be regarded as a $\mathbb{Z}_2$-graded R-module

DIFFERENTIAL MODULE. HOMOLOGY

A $\mathbb{Z}_2$-graded R-module $E = E_0 + E_1$ is called a differential module if it is equipped with a parity reversing homomorphism d obeying $d^2 = 0$. We define the space of cycles $Z = Z_0 + Z_1$ as $Ker d$ and the space of boundaries $B = B_0 + B_1$ as the image of d. (More precisely, $Z_i = Ker d \bigcap E_i, B_1 = dE_0 = E_0/Z_0, B_0 = dE_1 = E_1/Z_1$.) Homology $H = H_0 + H_1$ is defined as $Ker d/Im d = Z_0/B_0 + Z_1/B_1$.

EULER CHARACTERISTIC

Let us consider $\mathbf{Z}_2$-graded vector space $E_0 + E_1$ equipped with a differential d.(Recall that d is a parity reversing linear operator obeying $d^2 = 0$.)

We see that

$dimH_0 = dimZ_0 - dimB_0,$

$dimH_1 = dimZ_1 - dimB_1,$

$dimB_1 = dimE_0 - dimZ_0,$

$dimB_0 = dimE_1 - dimZ_1.$

It follows immediately from these equations that

$dimH_0 - dimH_1 = dimE_0 - dimE_1$

This number is called Euler characteristic and denoted $\chi(E)$.

Notice that above considerations remain correct if E is a differential R-module and dimension is replaced by any functional ϕ on the class of R-modules obeying

$$\phi(A/B) = \phi(A) - \phi(B)$$

(Euler-Poincare functional). For example, in the case when R is a group algebra $\mathbf{F}G$ we can identify R-modules with representations of the group G; then we can define ϕ as the character of representation.
Euler characteristic of $\mathbb{Z}$-graded vector space $E = \sum E_k$ is equal to

$$\chi(E) = \sum (-1)^k dim E_k.$$

Lefschetz trace formula

Let us consider differential module $E = E_0 + E_1$ and a parity preserving linear operator $A : E \to E$ commuting with the differential d. This operator induces and operator $\hat{A} : H \to H$ acting on homology.It is easy to prove that the supertrace of the operator $\hat{A}$ is equal to the supertrace of A:

$Tr\hat{A}|_{H_0} - Tr\hat{A}|_{H_1} = TrA|_{E_0} - TrA|_{E_1}$

(Lefschetz trace formula).

For R-modules the trace is an arbitrary functional on endomorphisms of R-modules such that the trace of endomorphism C of module X transforming a submodule Y into itself is equal to the trace of C restricted to Y plus the trace of the operator induced by C on the quotient X/Y.

BRST-formalism

We want to calculate a partition function $Tre^{-\beta\hat{A}}$. where $\hat{A}$ is an operator acting in vector space V. Represent V as homology of differential (of $BRST$-operator) Q acting in $E = E_0 + E_1$ (i.e. $V = H_0$, $H_1 = 0$. Lift $\hat{A}$ to a parity preserving operator A commuting with Q. Then the partition function is equal to the supertrace of $e^{-\beta A}$.

HOMOTOPY, QUASI-ISOMORPHISM

A homomorphism Φ of differential modules induces a homomorphism Φ_* of corresponding homology. (By definition homomorphism of differential modules commutes with the differential.)

We say that Φ is a quasi-isomorphism if Φ_* is an isomorphism.

Two homomorphisms $\Phi_i : E' \to E'', i = 1, 2$ are homotopic if there exists R-linear, parity reversing map h such that $\Phi_1 - \Phi_2 = hd + dh$. Two homotopic homomorphisms induce the same map on homology: $(\Phi_1)_* = (\Phi_2)_*$. To prove this we take a cycle x in E' and notice that $\Phi_1 x - \Phi_2 x = h(dx) + d(hx)$. The first term vanishes because x is a cycle, the second term vanishes in homology because it is a boundary. If we have two homomorphisms of differential modules $\Phi : E' \to E''$, $\Psi : E'' \to E'$ and the compositions $\Phi\Psi$, $\Psi\Phi$ both are homotopic to identity we say that modules are homotopy equivalent. Then homomorphisms induce isomorphisms of homology (because $\Phi_*\Psi_*$ and $\Psi_*\Phi_*$ are identities).

FREE RESOLUTIONS

Take R-module E. It can be represented as a quotient $E = E_0/V_0$ where E_0 is a free module. Further $V_0 = E_1/V_1, V_1 = E_2/V_2, V_2 = E_3/V_3, ...$ where E_i is free.

We obtain a sequence of free modules and homomorphisms

$... \to E_n \to E_{n-1} \to ... \to E_2 \to E_1 \to E_0.$

This sequence can be regarded as differential $\mathbb{Z}$-graded module $\mathcal{E}$. It is called free resolution of E. It is easy to check that all homology of $\mathcal{E}$ are trivial except H_0 and $H_0 = E$. Two free resolutions are homotopy equivalent.

$\mathcal{E}$ is quasi-isomorphic to E considered as a differential module with trivial differential and trivial grading (all elements have degree 0).

EXAMPLES OF DIFFERENTIALS
M-smooth manifold with coordinates x^i,
$\Omega(M) = \sum \Omega^k(M)$-differential forms.
$d = dx^i \frac{\partial}{\partial x^i}$- de Rahm differential
Homology $H^k(M)$ of de Rham differential-
cohomology of M

E-functions of commuting variables x^i and anticommuting variable c_k
$E = \sum E_r$ where $E_r = R \otimes \Lambda_r$ where R is a commutative ring
$d = f_k(x)\frac{\partial}{\partial c_k}$-Koszul differential $d : E_r \to E_{r-1}$
$H_0 = R/I$ where I is an ideal generated by $f_1, .., f_n$
H_0-ring of functions on variety singled out by equations $f_1(x) = 0, ..., f_n(x) = 0$
$H_r = 0$ for $r > 0$ generically

Constraints $T_a x = 0$ where $x \in E$ is an element of a module E
$[T_a, T_b] = f^k_{ab} T_k$ where f^k_{ab} are structure constants of Lie algebra $\mathfrak{g}$
T_a specify a representation of Lie algebra $\mathfrak{g}$
Differential in $E \otimes \Lambda[c^1, ..., c^n]$
$Q = T_a c^a - \frac{1}{2} f^k_{ab} c^a c^b \frac{\partial}{\partial c^k} = T_a c^a - \frac{1}{2} f^k_{ab} c^a c^b b_k$
where $[c^k, b_l]_+ = \delta^k_l$, $[c^k, c^l]_+ = [b_k, b_l]_+ = 0$
$H^k(\mathfrak{g}, E) = KerQ/ImQ$-cohomology of Lie algebra $\mathfrak{g}$ with coefficients in $\mathfrak{g}$-module E (grading=degree with respect to c)
$H^0 = \{x \in E | T_a x = 0\}$
Another form of the differential $Q = \left(T_a + \frac{1}{2} T^c_a\right) c^a$ where $T^c_a = f^k_{am} b_k c^m$ is adjoint representation.