Symplectic manifold M equipped with non-degenerate two-form ω
Isotropic submanifold=submanifold where this two-form vanishes
Lagrangian submanifold $L =$ maximal isotropic submanifold
More precisely, the tangent space T_xL of Lagrangian submanifold at every point $x \in L$ should be a Lagrangian subspace of the tangent space of T_xM considered as linear symplectic space. This means that $e \in T_xM$ belongs to T_xL iff $\omega(e, \ell) = 0$ for every $\ell \in T_xL$.

Quantization
Symplectic manifold $\rightarrow$ Hilbert space
Lagrangian manifold $\rightarrow$ vector in Hilbert space (in semiclassical approximation)
Phase space with coordinates $(p, q) \rightarrow$ Hilbert space of functions $\psi(q)$ (or $\psi(p)$)
Lagrangian manifold $p_i = \frac{\partial \sigma(q)}{\partial q^i} \rightarrow$ vector $e^{\frac{i}{\hbar}\sigma(q)}$

One-dimensional local theory with action functional $S[q] = \int dt L(q, \dot{q})$

$\delta S = \int_{t_0}^{t_1} dt \text{EM} + \text{BT}$

δS -variation of S =de Rham differential of S on the space of fields

EM=$(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i})\delta q^i$

EM=0 -equations of motion

BT=$p_i \dot{q}^i|_{t_0}^{t_1}$ where $p_i = \frac{\partial L}{\partial \dot{q}^i}$

In D-dimensional local theory with action functional $S[q] = \int dV L(q, \partial_\mu q)$
$\delta S = \int_V dV$EM+BT
EM=0 -equations of motion
BT (boundary terms)=
$\int_v dv A_k \delta q^k = \sum \mathrm{BT}_i^{out} - \sum \mathrm{BT}_j^{in}$
Here $v = \partial V$ stands for the boundary of bounded domain V,
BT can be interpreted as one -form on the space of fields.

On the space $\mathcal{E}$ of fields satisfying equations of motions $\delta\alpha_v$ where $\alpha_v = \int_v dv A_k \delta q^k$ and v is a cycle is a closed (even exact) two- form that depends only on homology class of v. (Notice that the integrand in the definition of α_v is a $(D-1)$-form in D-dimensional space that is not necessarily exact or even closed).
The form $\delta\alpha_v$ specifies a presymplectic structure on $\mathcal{E}$

Consider as an example $\beta\gamma$-model on complex one-dimensional manifold (more details about this model in the next lecture). The action functional of this model is defined by the formula $S = \int_V \beta\bar{\partial}\gamma$ where β is a J-differential (transforms like $(dz)^J$), γ is a $(1-J)-$differential, and $\bar{\partial} = d\bar{z}\frac{\partial}{\partial\bar{z}}$.

$\delta S = \int_V (\delta\beta\bar{\partial}\gamma - \delta\gamma\bar{\partial}\beta) + \int_v \beta\delta\gamma$

v is the boundary of V. Equations of motion are $\bar{\partial}\gamma = 0, \bar{\partial}\beta = 0$ (the space $\mathcal{E}$ consists of holomorphic fields). We see that in the case when v is a boundary the expression $\alpha_v = \int_v \beta\delta\gamma$ is an exact one-form on $\mathcal{E}$ (it is equal to δS.)

It follows that for homologous cycles v_1, v_2 (i.e. in the case when $v_2 - v_1 = \partial V$) we have $\delta\alpha_{v_1} = \delta\alpha_{v_2}$; in other words $\delta\alpha_v = \int_v \delta\beta\delta\gamma$ is a closed (even exact) two-form on $\mathcal{E}$ that depends only of homology class of the cycle v.
The same formulas work for *bc*-model where the action functional is given by the formula $S = \int_V b\bar{\partial}c$ where b and c are anticommuting fields.

Two-dimensional CFT (conformal field theory). In classical theory action functional depends on some fields ϕ and Riemannian metric on orientable 2D surface, it should be reparametrization-invariant and Weyl invariant. In other words it depends only on conformal structure. (Conformal structure+orientation=complex structure.) Phase space =the space of solutions of EM on a disc with deleted origin (sometimes it is more convenient to consider $\mathbb{C}^1$ with deleted origin= infinite cylinder= a sphere with two deleted points instead the disc with deleted point) Quantized phase space denoted by $\mathcal{H}$

Segal's axioms (for $c = 0$)

• Space $\mathcal{P}_{g,n}$ of compact complex curves of genus g with n embedded non-overlapping disks (with n marked points and coordinates $|z| \leq 1$ around these points)

• Vector space of quantum states $\mathcal{H}$ with bilinear inner product $\mathcal{H} \otimes \mathcal{H} \to \mathbb{C}$

• Map $\mathcal{P}_{g,n} \to \mathcal{H}^n$
If M is an element of $\mathcal{P}_{g,n}$ we denote by $\phi(M)$ the corresponding element of $\mathcal{H}^n$
In classical theory the space of solutions to EM on M with deleted centers of discs is a Lagrangian submanifold of (phase space)^n.
Quantizing it we obtain $\phi(M)$

$\sigma_n : \mathcal{P}_{g,n} \to \mathcal{P}_{g+1,n-2}$
(Remove two last discs and paste together the boundaries.)
$\rho_{n,n'} : \mathcal{P}_{g,n} \times \mathcal{P}_{g',n'} \to \mathcal{P}_{g+g',n+n'-2}$
(Take $M \in \mathcal{P}_{g,n}$ and $M' \in \mathcal{P}_{g',n'}$. Remove the last disc in M and the first disc in M'. Paste the boundaries.)
$S_n : \mathcal{H}^n = \mathcal{H}^{n-2} \otimes \mathcal{H}^2 \to \mathcal{H}^2$
$R_{n,n'} : \mathcal{H}^n \otimes \mathcal{H}^{n'} = \mathcal{H}^{n-1} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}^{n'-1} \to \mathcal{H}^{n+n'-2}$
MAIN AXIOMS
σ_n agrees with S_n,
$\rho_{n,n'}$ agrees with $R_{n,n'}$

$\mathcal{P}_{0,2} \times \mathcal{P}_{0,2} \to \mathcal{P}_{0,2}$
$\mathcal{P}_{0,2}$- Neretin semigroup
$\mathcal{P}_{g,n} \times \mathcal{P}_{0,2} \to \mathcal{P}_{g,n}$
Neretin semigroup acts on $\mathcal{P}_{g,n}$
Lie algebra of Neretin semigroup is an algebra generated by vector fields
$l_n = z^{n+1}\frac{d}{dz}$ (Witt algebra)
$[l_m, l_n] = (m-n)l_{m+n}$,
After quantization
$[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12}c(m^3-m)\delta_{m+n}$
c-central charge
Another construction of the action of Witt algebra: change the coordinates in the disc