Теорема Дженкинса 1966 года как обратный результат к теореме Шталя 1985 года

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Stahl's Theory

Stahl's Theory, 1985–1986.

Let f be a multi-valued analytic function, i.e., f analytic in $\widehat{\mathbb{C}} \setminus \Sigma$, $\Sigma \subset \mathbb{C}$, $\#\Sigma < \infty$, but f not single-valued in $\widehat{\mathbb{C}} \setminus \Sigma$.

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Let $f_\infty \in \mathcal{H}(\infty)$ be a germ of f, $\mathfrak{K}(f_\infty)$ be the family of all admisible sets for f_∞: $K \in \mathfrak{K}(f_\infty) \Leftrightarrow$

- $K \subset \mathbb{C}$ is a compact set and $D(K) := \widehat{\mathbb{C}} \setminus K$ is a domain;
- the germ f_∞ admits a meromorphic extention to $D(K)$.

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Stahl Theorem, 1985

There exist a compact set $S \in \mathfrak{K}(f_\infty)$ *such that*

$$\operatorname{cap}(S) = \min_{K \in \mathfrak{K}(f_\infty)} \operatorname{cap}(K).$$

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The properties of S*:*

• *for some finite set* e *the set* $S^\circ := S \setminus e$ *consists of a finite number of (open) analytic arcs;*

• *compact set* S *posseses the* S*-property, i.e., for the Green function* $g_S(z, \infty)$ *of the domain* $D := \widehat{\mathbb{C}} \setminus S$ *we have*

$$\frac{\partial g_S(z,\infty)}{\partial n^+} = \frac{\partial g_S(z,\infty)}{\partial n^-}, \quad z \in S^\circ.$$

Jenkins' Theorem

Let S be a set in the $\mathbb{C}$ plane consisting of a finite number of Jordan arcs such that $D := \widehat{\mathbb{C}} \setminus S$ is a domain.

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Let $g(z, \infty) = g_S(z, \infty)$ be the Green's function of D. It is well known that an orthogonal trajectory of the level curves of $g(z, \infty)$, apart from a finite number of exceptions, will be an open arc with limiting end points at $z = \infty$ and a point of S. Every point of S will be a limiting end point for two such orthogonal trajectories, with at most a finite number of exceptions.

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Let $\mathcal{S}^\perp$ be the set of orthogonal trajectories which occur in such pairs and let T be the involutory transformation defined on $\mathcal{S}^\perp$ by associating with an element of the other one with the same end point on S.

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There is a natural metric determined on $\mathcal{S}^{\perp}$ by the variation of the conjugate of the Green's function $h(z,\infty)=h_S(z,\infty)$. We will denote it by $d\mu$. In particular

$$\int_{\mathcal{S}^{\perp}} d\mu = 2\pi.$$

Jenkins' Theorem

Jenkins Theorem, 1966

Let S be a set consisting of a finite number of Jordan arcs in $\mathbb{C}$ and such that its complement D is connected.

(a) *Let the involutory transformation T be measure preserving in the metric $d\mu$.*

(b) *Let K be a compact set in the $\mathbb{C}$ such that if $L \in \mathcal{S}^{\perp}$ then K meets either L or TL.*

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(a) *Let the involutory transformation T be measure preserving in the metric $d\mu$.*
(b) *Let K be a compact set in the $\mathbb{C}$ such that if $L \in \mathcal{S}^{\perp}$ then K meets either L or TL.*
Then

$$\operatorname{cap}(K) \geqslant \operatorname{cap}(S),$$

where equality can occur only if K differs from S at most by a set of zero capacity.

Jenkins' Theorem and Stahl's Theorem

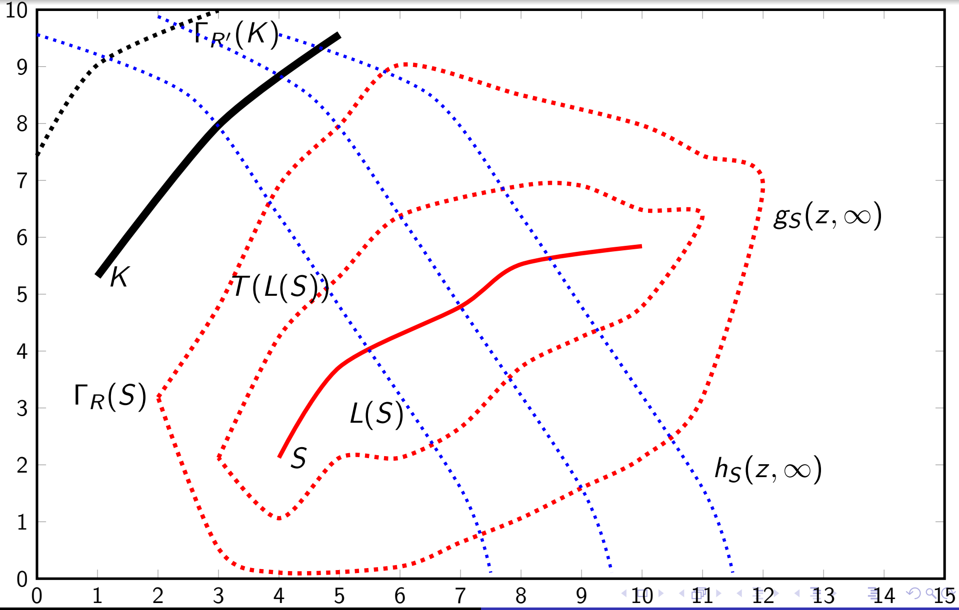
Since under conditions of Stahl's Theorem

$$\frac{\partial h(z,\infty)}{\partial s} = \frac{\partial g(z,\infty)}{\partial n}, \quad z \in S^\circ := S \setminus e,$$

$\Rightarrow$ the property of measure preserving of T is equivalent to Stahl S-property

$$\frac{\partial g(z,\infty)}{\partial n^+} = \frac{\partial g(z,\infty)}{\partial n^-}, \quad z \in S^\circ.$$

Jenkins' Theorem



Jenkins' Theorem, Proof

Let compact set K be regular and $g_K(z,\infty)$ be its Green's function. The level curve $\Gamma(K,R)$, $g_K(z,\infty)=R$, for R sufficiently large is a Jordan curve behaving asymptotically like the circle $|z|=\operatorname{cap}(K)e^R$.

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Let $D(K,R)$ be the domain bounded by K and $\Gamma(K,R)$ and $\mathcal{G}(K,R)$ be the class of locally rectifiable curves running in $D(K,R)$ from K to $\Gamma(K,R)$.

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Let $D(K,R)$ be the domain bounded by K and $\Gamma(K,R)$ and $\mathcal{G}(K,R)$ be the class of locally rectifiable curves running in $D(K,R)$ from K to $\Gamma(K,R)$.
Let $m(K,R)$ be the module of this class of curves. It is well known that $m(K,R)=2\pi/R$, the extremal metric being $R^{-1}|\operatorname{grad} g_K(z,\infty)|$.

Jenkins' Theorem, Proof

The transformation T induces a point transformation $\mathcal{T}$ on a subset $\widetilde{D}$ of $D := \widehat{\mathbb{C}} \setminus S$ obtained by deleting a finite number of open analytic arcs and points (i.e., $\widetilde{D}$ is the point set union of $L \in \mathcal{S}^{\perp}$) by taking $\mathcal{T}(P)$ for $P \in L$ to be the point on $T(L)$ with

$$g_S(\mathcal{T}(P), \infty) = g_S(P, \infty).$$

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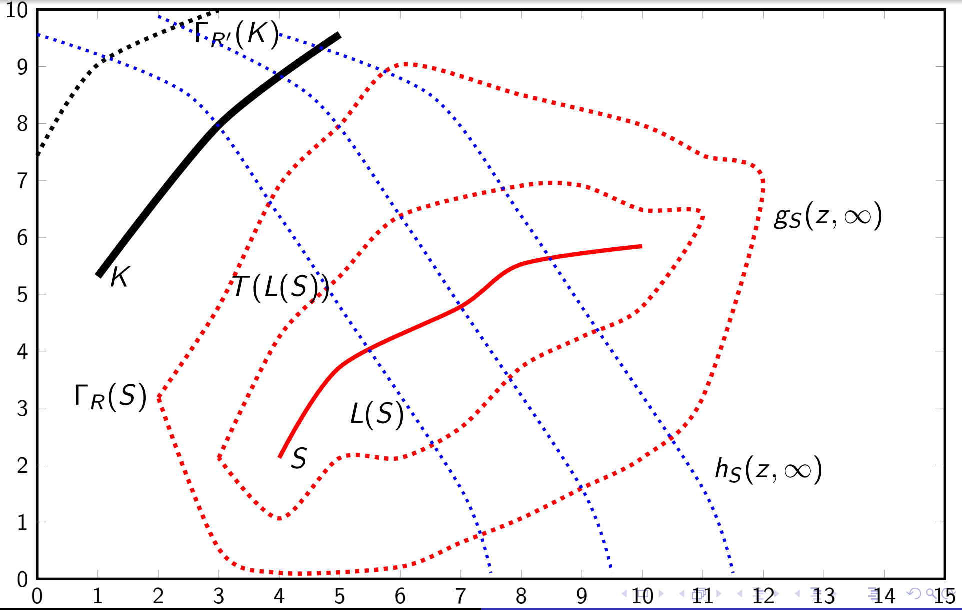
Under condition (a), $\mathcal{T}$ is an anticonformal mapping on $\widetilde{D}$ thus we can speak of its distortion $\tau(P) = \tau(z)$.

Jenkins' Theorem



Jenkins' Theorem, Proof

Now consider $\Gamma(S,R) := \{z : g_S(z,\infty) = R\}$ for R sufficiently large. There will be a level curve $\Gamma(K,R')$ lying inside $\Gamma(S,R)$ and touching it with

$$R' = R + \log(\operatorname{cap}(S)/\operatorname{cap}(K)) + o(1), \quad R \to \infty.$$

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Let $\rho(z)$ be the extremal metric for $m(K,R')$. Let

$$\rho_1(z) = \begin{cases} \rho(z), & z \in D(S,R) \cap D(K,R'), \\ 0, & z \in D(S,R) \setminus D(K,R'). \end{cases}$$

Jenkins' Theorem, Proof

Let

$$\rho_2(z) := \begin{cases} \frac{1}{2}(\rho_1(z) + \tau(z)\rho_1(\mathcal{T}z)), & z \in D(S,R) \cap \widetilde{D}, \\ 0, & z \in D(S,R) \setminus \widetilde{D}. \end{cases}$$

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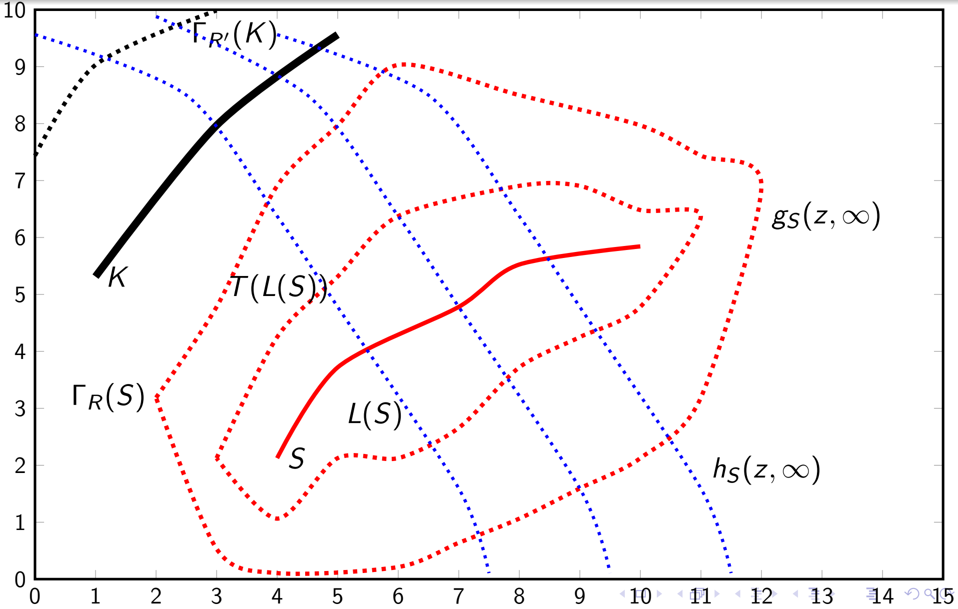
If $L(R)$ denotes the (open) arc on $L \in \mathcal{S}^\perp$ for which $0 < g_S(z,\infty) < R$ we have

$$\int_{L(R)} \rho_2(z)\,|dz| \geqslant 1.$$

Jenkins' Theorem, Proof



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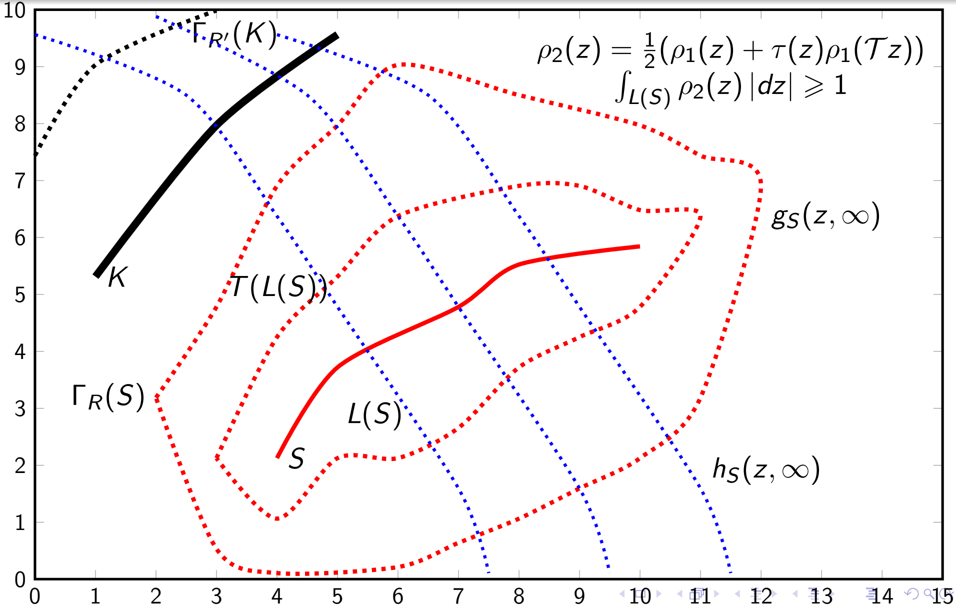

Jenkins' Theorem, Proof

Since the $L(R)$, $0 < g_S 0 < g_S(z, \infty) < R$, are precisely those curves in $\mathcal{G}(S, R)$ which have length 1 in the extremal metric for the module problem defining $m(S, R)$ we have

$$\iint_{D(S,R)} \rho_2^2(z)\, dA \geqslant m(S, R),$$

where dA denotes the element of area in the $\mathbb{C}$.

Jenkins' Theorem, Proof

Moreover

$$\iint_{D(S,R)} \rho_2^2(z)\,dA = \frac{1}{4}\iint_{D(S,R)} (\rho_1(z) + \tau(z)\rho_1(\mathcal{T}z))^2\,dA$$
$$\leqslant \frac{1}{2}\iint_{D(S,R)} (\rho_1(z))^2\,dA + \frac{1}{2}\iint_{D(S,R)} (\rho_1(\mathcal{T}z))^2(\tau(z))^2\,dA.$$

Jenkins' Theorem, Proof

This last term is just

$$\iint_{D(S,R)} (\rho_1(z))^2\, dA \leqslant m(K, R')$$

(actually since S has zero area). Thus

$$m(K, R') \geqslant m(S, R)$$

or

$$R \geqslant R' = R + \log(\operatorname{cap}(S)/\operatorname{cap}(K)) + o(1).$$

So finally

$$\operatorname{cap}(K) \geqslant \operatorname{cap}(S).$$

Теорема Дженкинса

[1] James A. Jenkins, “On certain problems of minimal capacity.”, *Illinois J. Math.*, **10** (1966), 460–465.

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Спасибо за внимание !