THE STRENGTH OF SOME CONSEQUENCES OF RT^2_2 OVER RCA_0^*

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(Joint work with Leszek Kołodziejczyk and Katarzyna Kowalik)

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Reverse mathematics questions

- What is the exact strength needed to prove a mathematical theorem?

- Which are the appropriate axioms for mathematics?

Reverse mathematics questions

- What is the exact strength needed to prove a mathematical theorem?

Subsystems of Z_2

- Which are the appropriate axioms for mathematics?

$$\exists X \, \forall n \, (n \in X \leftrightarrow \varphi(n))$$

Subsystems

Π^1_1-CA	Π^1_1 comprehension
↓	
ATR_0	Σ^1_1-separation
↓	
ACA_0	Arithmetic comprehension
↓	
WKL_0	Σ^0_1-separation
↓	
RCA_0	Recursive (Δ^0_1) comprehension + $\mathsf{I}\Sigma^0_1$

Subsystems

ACA_0	
$\downarrow$	Arithmetic comprehension
WKL_0	
$\downarrow$	Σ^0_1-separation
RCA_0	
$\downarrow$	Δ^0_1-comprehension + $\mathsf{I}\Sigma^0_1$
RCA^*_0	
	Δ^0_1-comprehension + $\mathsf{I}\Delta^0_1$ + $\exp$

Why $I\Sigma^0_1$?

- RCA_0 (and WKL_0) is Π^0_2-conservative over PRA
- $I\Sigma^0_1$ proves the totality of primitive recursive functions
- unique notion of infinite set
- $I\Sigma^0_1$ is equivalent to bounded Σ^0_1-CA, $\forall k\, \exists X\, \forall m\, (m \in X \leftrightarrow \varphi(m) \wedge m < k)$

Why not $\mathsf{I}\Sigma^0_1$?

- which statements imply $\mathsf{I}\Sigma^0_1$?
- nice interplay with between first and second order arithmetic
- help to study conservativity issues
- study implications among theorems over a weaker base theory

RCA_0^* Δ^0_1-comprehension + Δ^0_1-induction + exp

RCA_0 Δ^0_1-comprehension + Σ^0_1-induction ($\mathsf{I}\Sigma^0_1$)

S. Simpson and R. Smith
Factorization of polynomials and Σ^0_1 induction.
Annals of Pure and Applied Logic, 1986

RT^2_2	For every $c\colon [\mathbb{N}]^2 \to 2$ there exists an unbounded homogeneous set
CAC	Each countable poset contains a $<$-unbounded chain or a $<$-unbounded antichain
ADS	Each linear order contains either an unbounded ascending or an unbounded descending chain
CRT^2_2	For every $c\colon [\mathbb{N}]^2 \to 2$ there exists an unbounded set $S \subseteq \mathbb{N}$ such that for each $x \in S$ there exists $y \in S$ such that $c(x, z) = c(x, y)$ holds for all $z \in S$ with $z \geq y$.

Σ_1^0-cut

Let $(\mathsf{M}, \mathcal{X}) \vDash \mathsf{RCA}_0^* + \neg \mathsf{I}\Sigma_1^0$.

Then there exists a $\underline{\Sigma_1^0 \text{ proper cut}}$ $\mathsf{I} \subsetneq \mathsf{M}$,

i.e. a subset which is

- closed under successor and closed downwards
- $\mathsf{I} \neq \mathsf{M}$
- is Σ_1^0-definable in M

Let $\mathrm{Cod}(\mathsf{M}/\mathsf{I}) = \{X \cap \mathsf{I} \mid X \text{ is M-finite}\}$

When I is closed under exp, $(\mathsf{I}, \mathrm{Cod}(\mathsf{M}/\mathsf{I})) \vDash \mathsf{WKL}_0^*$,

Main tool

Theorem

Let $(M, \mathcal{X}) \models \mathsf{RCA}_0^*$ and $I \subsetneq_e M$ be a Σ_1^0-cut.

Then $(M, \mathcal{X}) \models \mathsf{RT}_2^2/\mathsf{CAC}/\mathsf{ADS}/\mathsf{CRT}_2^2$ if and only if $(I, \mathrm{Cod}(M/I)) \models \mathsf{RT}_2^2/\mathsf{CAC}/\mathsf{ADS}/\mathsf{CRT}_2^2$

Main tool

Theorem

Let $(\mathsf{M}, \mathcal{X}) \vDash \mathsf{RCA}_0^*$ and $\mathsf{I} \subsetneq_e \mathsf{M}$ be a Σ_1^0-cut.
Then $(\mathsf{M}, \mathcal{X}) \vDash \mathsf{RT}_2^2/\mathsf{CAC}/\mathsf{ADS}/\mathsf{CRT}_2^2$ if and only if $(\mathsf{I}, \mathrm{Cod}(\mathsf{M}/\mathsf{I})) \vDash \mathsf{RT}_2^2/\mathsf{CAC}/\mathsf{ADS}/\mathsf{CRT}_2^2$

Let $(\mathsf{M}, \mathcal{X}) \vDash \mathsf{RCA}_0^* + \neg \mathsf{I}\Sigma_1(\mathsf{A}) + \mathsf{CAC}$ and $\mathsf{I} \subsetneq_e \mathsf{M}$ be a $\Sigma_1(\mathsf{A})$-cut.
Then $(\mathsf{I}, \mathrm{Cod}(\mathsf{M}/\mathsf{I})) \vDash \mathsf{CAC}$.
Then $(\mathsf{M}, \Delta_1\text{-Def}(\mathsf{M}; \mathsf{A})) \vDash \mathsf{RCA}_0^* + \neg \mathsf{I}\Sigma_1^0 + \mathsf{CAC}$

Corollary

$\mathsf{RCA}_0^* + \mathsf{RT}_2^2/\mathsf{CAC}/\mathsf{ADS}/\mathsf{CRT}_2^2$ prove the following sentence:
"If $\neg \mathsf{I}\Sigma_1$, then any computable instance of $\mathsf{RT}_2^2/\mathsf{CAC}/\mathsf{ADS}/\mathsf{CRT}_2^2$ has a computable solution"

Implications

Theorem

$\mathsf{RCA}_0^* \vdash \mathsf{RT}^2_2 \to \mathsf{CAC} \to \mathsf{ADS}$ and $\mathsf{RCA}_0^* \vdash \mathsf{RT}^2_2 \to \mathsf{CRT}^2_2$.

These implications do not reverse, even over $\mathsf{RCA}_0^* + \neg \mathsf{I}\Sigma^0_1$

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$\mathsf{RCA}_0^* \vdash \mathsf{RT}_2^2 \to \mathsf{CAC} \to \mathsf{ADS}$ and $\mathsf{RCA}_0^* \vdash \mathsf{RT}_2^2 \to \mathsf{CRT}_2^2$.

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Question

Over RCA_0^*, does CAC or ADS imply CRT_2^2?

Theorem

CAC/ADS/CRT^2_2 are Π^0_3-conservative over RCA^*_0.

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CAC/ADS are not Π^0_4-conservative over RCA^*_0.

Conservation

Theorem

CAC/ADS/CRT^2_2 are Π^0_3-conservative over RCA^*_0.

Theorem

CAC/ADS are not Π^0_4-conservative over RCA^*_0.

CRT^2_2 is not Π^0_5-conservative over RCA^*_0.

Question

Is CRT^2_2 Π^0_4-conservative over RCA^*_0?

Closure property

$\mathsf{I}^0_1 = \{\mathsf{x} \in \mathbb{N} \mid$ every unbounded $\mathsf{S} \subseteq \mathbb{N}$ has a subset of cardinality $\mathsf{x}\}$

Closure property

$I_1^0 = \{x \in \mathbb{N} \mid$ every unbounded $S \subseteq \mathbb{N}$ has a subset of cardinality $x\}$

Theorem (Kołodziejczyk, Wong, Yokoyama)

RT_2^2 proves that I_1^0 is closed under exp

Closure property

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Theorem

CAC, and hence ADS, does not prove that I_1^0 is closed under exp

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Theorem

CAC, and hence ADS, does not prove that I_1^0 is closed under exp

Question

Does CRT_2^2 prove that I_1^0 is closed under exp?

Normal vs long version: CAC

CAC	Each countable poset $(\mathbb{N}, \prec)$ contains a $<$-cofinal chain or a $<$-cofinal antichain
long-CAC	Each countable poset $(\mathbb{N}, \prec)$ contains a chain of cardinality $\mathbb{N}$ or an antichain of cardinality $\mathbb{N}$

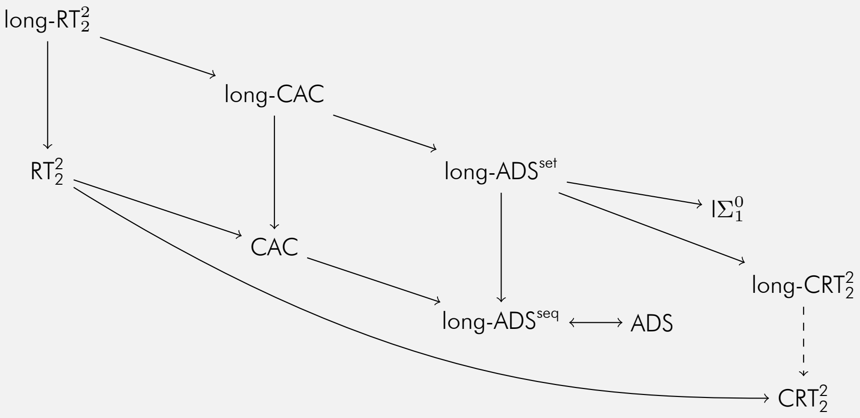
What is an ω chain?

ADS | Each countable linear order $(\mathbb{N}, \prec)$ contains an ω or an ω^* chain.

- a set $S \subseteq \mathbb{N}$ which is increasingly ordered with respect to both $<$ and $\prec$
- a sequence $(s_i)_{i \in \mathbb{N}}$ which is strictly $\prec$-increasing
- a set $S \subseteq \mathbb{N}$ such that each element of S has only finitely many predecessors

Normal vs long version: ADS

ADS	For each linear order $(\mathbb{N}, \prec)$ there exists a cofinal set $S \subseteq \mathbb{N}$ such that either for all $x, y \in S$ it holds that $x < y$ iff $x \prec y$ or for all $x, y \in S$ it holds that $x < y$ iff $x \succ y$
long-ADS^{set}	For each linear order $(\mathbb{N}, \prec)$ there exists a set $S \subseteq \mathbb{N}$ of universe cardinality such that either for all $x, y \in S$ it holds that $x < y$ iff $x \prec y$ or for all $x, y \in S$ it holds that $x < y$ iff $x \succ y$
long-ADS^{seq}	For each linear order $(\mathbb{N}, \prec)$ there exists a sequence $(s_i)_{i \in \mathbb{N}}$ which is either strictly $\prec$-increasing or strictly $\prec$-decreasing


long-RT^2_2
long-CAC
RT^2_2
long-ADS^{set}
$I\Sigma^0_1$
CAC
long-CRT^2_2
long-ADS^{seq}
ADS
CRT^2_2

Marta Fiori-Carones, Leszek Kołodziejczyk, Katarzyna W. Kowalik
Weaker cousins of Ramsey's theorem over a weak base theory
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Leszek Kołodziejczyk, Katarzyna W. Kowalik, and Keita Yokoyama
How strong is Ramsey's theorem if infinity can be weak?
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Leszek Kołodziejczyk, Tin Lok Wong, Keita Yokoyama
Ramsey's theorem for pairs, collection, and proof size
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Factorization of polynomials and Σ^0_1 induction
Annals of Pure and Applied Logic 1986